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Hybrid differential equations with maxima via Picard operators theory

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. The aim of this paper is to discuss some basic problems (existence and uniqueness, data dependence) of the Cauchy problem for a hybrid differential equation with maxima using weakly Picard operators technique.

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1. Introduction

Recently, the interest in differential equations with "maxima" has increased exponentially. Such equations model real world problems whose present state depends significantly on its maximum value on a past time interval. For example, many problems in the control theory correspond to the maximal deviation of the regulated quantity. Some qualitative properties of the solutions of ordinary differential equations with "maxima" can be found in [1, 2, 5], [16, 17] and the references therein.

The main goal of the presented paper is to study a hybrid differential equation with maxima, using the theory of weakly Picard operators. The theory of Picard operators was introduced by I. A. Rus (see [12], [14] and their references) to study problems related to fixed point theory. This abstract approach is used by many mathematicians and it seemed to be a very useful and powerful method in the study of integral equations and inequalities, ordinary and partial differential equations (existence, uniqueness, differentiability of the solutions), etc.

In this paper we consider the following hybrid differential equation with maxima

$$x'(t) = f(t, x(t)) + g(t, \max_{a \leq \xi \leq t} x(\xi)), \tag{1.1}$$

with initial condition

$$x(a) = x_0, \tag{1.2}$$

where $t \in [a,b],\ a,b \in \mathbb{R}, x_0 \in \mathbb{R}^m,\ f,g : [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$.

We use the terminologies and notations from [12] and [14]. For the convenience of the reader we recall some of them.

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by $A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A^n \circ A,\ \ n \in \mathbb{N}$, the iterate operators of the operator A. We also have:

$$P(X) := \{Y \subseteq X \mid Y \neq \phi\},$$
$$F_A := \{x \in X \mid A(x) = x\},$$
$$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}.$$

Definition 1.1. *Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.*

Definition 1.2. *Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 1.3. *If A is weakly Picard operator then we consider the operator A^∞ defined by $A^\infty : X \to X,\ A^\infty(x) := \lim\limits_{n\to\infty} A^n(x)$.*

Obviously, $A^\infty(X) = F_A$. Moreover, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

2. Existence and uniqueness

We prove the existence and uniqueness for the solution of the problem (1.1)-(1.2) using the Perov's Theorem as in [7]. For standard techniques, when it is used the Banach contraction principle, see [13], [9] and [10].

Theorem 2.1. *(Perov's fixed point theorem) Let (X,d) with $d(x,y) \in \mathbb{R}^m$, be a complete generalized metric space and $A : X \to X$ an operator. We suppose that there exists a matrix $Q \in M_{m\times m}(\mathbb{R}_+)$, such that*

(i) *$d(A(x),A(y)) \leq Qd(x,y)$, for all $x,y \in X$;*
(ii) *$Q^n \to 0$, as $n \to \infty$.*

Then

(a) *$F_A = \{x^*\}$,*
(b) *$A^n(x) \to x^*$, as $n \to \infty$ and*
$$d(A^n(x),x^*) \leq (I-Q)^{-1}Q^n d(x_0,A(x_0)),\ \forall x_0,x \in X, \forall n \in \mathbb{N}^*;$$
(c) *$d(x,x^*) \leq (I-Q)^{-1}d(x,A(x)),\ \forall x \in X$.*

We consider on $\mathbb{R}^m$ the following vectorial norm

$$|x| := \begin{pmatrix} |x_1| \\ \vdots \\ |x_m| \end{pmatrix}.$$

We have the following result:

Theorem 2.2. *We assume that:*

(i) $f, g \in C([a,b] \times \mathbb{R}^m, \mathbb{R}^m)$;

(ii) there exist L_f *and* L_g *nonnegative matrices such that*

$$\begin{gathered} \left|f(t,u^1) - f(t,u^2)\right| \leq L_f \left|u^1 - u^2\right|, \\ \left|g(t,v^1) - g(t,v^2)\right| \leq L_g \left|v^1 - v^2\right|, \end{gathered}$$

$\forall\, t \in [a,b]$ *and* $u^1 = (u_1^1, \ldots, u_m^1)$, $u^2 = (u_1^2, \ldots, u_m^2)$, $v^1 = (v_1^1, \ldots, v_m^1)$, $v^2 = (v_1^2, \ldots, v_m^2) \in \mathbb{R}^m$;

(iii) the matrix

$$Q := (b-a)(L_f + L_g) \tag{2.1}$$

is convergent to 0, i.e. $Q^n \to 0$, *as* $n \to \infty$.

Then, the problem (1.1)*-*(1.2) *has a unique solution* $x^* \in C([a,b], \mathbb{R}^m)$.

Proof. We consider the generalized Banach space $X = (C([a,b], \mathbb{R}^m), \|\cdot\|)$ where $\|\cdot\|$ is the norm,

$$\|x\| := \begin{pmatrix} \max\limits_{a \leq t \leq b} |x_1(t)| \\ \vdots \\ \max\limits_{a \leq t \leq b} |x_m(t)| \end{pmatrix}. \tag{2.2}$$

The problem (1.1)-(1.2), $x \in C^1([a,b], \mathbb{R}^m)$ is equivalent with the following fixed point equation

$$x(t) = x_0 + \int_a^t f(s, x(s))ds + \int_a^t g(s, \max_{a \leq \xi \leq s} x(\xi))ds,\ t \in [a,b]. \tag{2.3}$$

We consider the operator $A : X \to X$, where

$$A(x)(t) = x_0 + \int_a^t f(s, x(s))ds + \int_a^t g(s, \max_{a \leq \xi \leq s} x(\xi))ds. \tag{2.4}$$

It is easy to see that if $x^* \in F_A$ then x^* is a solution of (1.1)-(1.2).

Condition (ii) implies that

$$
\begin{aligned}
&|A(x)(t)-A(y)(t)| \\
&\leq \int_a^t |f(s,x(s))-f(s,y(s))|\,ds+\int_a^t \left|g(s,\max_{a\leq\xi\leq s} x(\xi))-g(s,\max_{a\leq\xi\leq s} y(\xi))\right| ds \\
&\leq (b-a)L_f \begin{pmatrix} \max\limits_{a\leq s\leq b} |x_1(s)-y_1(s)| \\ \vdots \\ \max\limits_{a\leq s\leq b} |x_m(s)-y_m(s)| \end{pmatrix} \\
&\quad + (b-a)L_g \begin{pmatrix} \max\limits_{a\leq s\leq b} \left|\max\limits_{a\leq\xi\leq s} x_1(s)-\max\limits_{a\leq\xi\leq s} y_1(s)\right| \\ \vdots \\ \max\limits_{a\leq s\leq b} \left|\max\limits_{a\leq\xi\leq s} x_m(s)-\max\limits_{a\leq\xi\leq s} y_m(s)\right| \end{pmatrix}.
\end{aligned}
$$

But

$$\max_{a\leq s\leq b} \left|\max_{a\leq\xi\leq s} x_i(s)-\max_{a\leq\xi\leq s} y_i(s)\right| \leq \max_{a\leq s\leq b} |x_i(s)-y_i(s)|\,.$$

So,

$$\|A(x)-A(y)\| \leq Q\,\|x-y\|\,.$$

Using (iii), we get that the operator $A : X \to X$ is a Q-contraction, so

$$F_A=(x_1^*,\ldots,x_m^*)=x^*$$

is the unique solution of (1.1)-(1.2). □

The equation (1.1) is equivalent with

$$x(t)=x(a)+\int_a^t f(s,x(s))ds+\int_a^t g(s,\max_{a\leq\xi\leq s} x(\xi))ds,\ t\in[a,b], \tag{2.5}$$

$x\in C([a,b],\mathbb{R}^m)$.

In what follows we consider the operator $B : X \to X$ defined by $B(x)(t) :=$the right hand side of (2.5). For $x_0 \in \mathbb{R}^m$, we consider

$$X_{x_0} := \{x\in C([a,b],\mathbb{R}^m)|\ x(a)=x_0\}.$$

It is clear that

$$X=\bigcup_{x_0\in\mathbb{R}^m} X_{x_0}$$

is a partition of X. We have

Lemma 2.3. *We suppose that the condition* (C_1) *is satisfied. Then*

(a) $A(X)\subset X_{x_0}$ *and* $A(X_{x_0})\subset X_{x_0}$;

(b) $A|_{X_{x_0}}=B|_{X_{x_0}}$.

Remark 2.4. From Theorem 2.2 we have that the operator A is PO. Because $A|_{X_{x_0}} = B|_{X_{x_0}}$, $X := C([a,b],\mathbb{R}^m) = \underset{x_0\in\mathbb{R}^m}{\cup} X_{x_0}, X_{x_0} \in I(B)$ it follows that the operator B is WPO and

$$F_B \cap X_{x_0} = \{x^*\},\ \forall x_0 \in \mathbb{R}^m,$$

where x^* is the unique solution of the problem (1.1)-(1.2).

3. Data dependence: comparison results

Now we consider the operators A and B on the ordered Banach space $(C([a,b],\mathbb{R}^m), \|\cdot\|, \leq)$ where the order relation on $\mathbb{R}^m$ is given by: $x \leq y \Leftrightarrow x_i \leq y_i$, $i = \overline{1,m}$.

In order to establish the Čaplygin type inequalities we need the following abstract result.

Lemma 3.1. *(see [14]) Let $(X,d,\leq)$ be an ordered metric space and $A : X \to X$ an operator. Suppose that A is increasing and WPO. Then the operator A^∞ is increasing.*

We have the following result

Theorem 3.2. *Suppose that:*

(a) the conditions of Theorem 2.2 are satisfied;

(b) $f(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m, g(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ are increasing, $\forall t \in [a,b]$.

Let x^* be a solution of equation (1.1) and y^* a solution of the inequality

$$y'(t) \leq f(t,y(t)) + g(t, \max_{a\leq\xi\leq t} y(\xi)),\ t \in [a,b].$$

Then $y^*(a) \leq x^*(a)$ implies that $y \leq x$.

Proof. From Remark 2.4 we have that B is WPO. On the other hand, from the condition (b) and Lemma 3.1 we get that the operator B^∞ is increasing. If $x_0 \in \mathbb{R}^m$, then we denote by $\widetilde{x}_0$ the following function

$$\widetilde{x}_0 : [a,b] \to \mathbb{R}^m, \widetilde{x}_0(t) = x_0,\ \forall t \in [a,b].$$

Hence $y^* \leq B(y^*) \leq B^2(y^*) \leq \ldots \leq B^\infty(y^*) = B^\infty(\widetilde{y^*(a)}) \leq B^\infty(\widetilde{x^*(a)}) = x^*$. □

In order to study the monotony of the solution of the problem (1.1)-(1.2) with respect to x_0, f, g we need the following result from WPOs theory.

Lemma 3.3. *(Abstract comparison lemma, [15]) Let $(X,d,\leq)$ be an ordered metric space and $A,B,C : X \to X$ be such that:*

(i) the operator A,B,C are WPOs;

(ii) $A \leq B \leq C$;

(iii) the operator B is increasing.

Then $x \leq y \leq z$ imply that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

From this abstract result we obtain the following result:

Theorem 3.4. *Let $f^j, g^j \in C([a,b]\times\mathbb{R}^m, \mathbb{R}^m), j = \overline{1,3}$, and suppose that the conditions from Theorem 2.2 hold. Furthermore suppose that:*

(i) $f^1 \le f^2 \le f^3, g^1 \le g^2 \le g^3$;

(ii) $f^2(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m, g^2(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ are increasing.

Let x^{*j} be a solution of the equation

$$x^{j\prime}(t) = f^j(t, x(t)) + g^j(t, \max_{a\le\xi\le t} x(\xi)),\ t \in [a,b] \text{ and } j = \overline{1,3}.$$

Then $x^{*1}(a) \le x^{*2}(a) \le x^{*3}(a)$, implies $x^{*1} \le x^{*2} \le x^{*3}$, i.e. the unique solution of the problem (1.1)-(1.2) is increasing with respect to x_0, f and g.

Proof. From Remark 2.4, the operators $B_j, j = \overline{1,3}$, are WPOs. From the condition (ii) the operator B_2 is monotone increasing. From the condition (i) it follows that $B_1 \le B_2 \le B_3$. Let $\widetilde{x}^j(a) \in (C[a,b], \mathbb{R}^m)$ be defined by $\widetilde{x}^j(a) = x^j(a),\ \forall t \in [a,b]$. We notice that

$$\widetilde{x}^1(a)(t) \le \widetilde{x}^2(a)(t) \le \widetilde{x}^3(a)(t),\ \forall t \in [a,b].$$

From Lemma 3.3 we have that $B_1^\infty(\widetilde{x}^{*1}(a)) \le B_2^\infty(\widetilde{x}^{*2}(a)) \le B_3^\infty(\widetilde{x}^{*3}(a))$.
But $x^{*j} = B_j^\infty(\widetilde{x}^{*j}(a))$, so $x^{*1} \le x^{*2} \le x^{*3}$. □

4. Data dependence: continuity

In this section we prove the continuous dependence of the solution for equation (1.1) and suppose the conditions of Theorem 2.2 are satisfied.

Theorem 4.1. *Let $x_0^j, f^j, g^j, j = 1,2$ satisfy the conditions from Theorem 2.2. Furthermore we suppose there exist $\eta^1, \eta^2, \eta^3 \in \mathbb{R}^m_+$, such that*

(i) $\left|x_0^j - x_0^j\right| \le \eta^1$;

(ii) $\left|f^1(t,u) - f^2(t,u)\right| \le \eta^2, \left|g^1(t,v) - g^2(t,v)\right| \le \eta^3,\ \forall t \in C[a,b], u, v \in \mathbb{R}^m$.

Then

$$\left\|x^*(t; x_0^1, f^1, g^1) - x^*(t; x_0^2, f^2, g^2)\right\| \le (I-Q)^{-1}(\eta^1 + (b-a)(\eta^2+\eta^3)),$$

where $x^*(t; x_0^j, f^j, g^j)$ are the solutions of the problem (1.1)-(1.2) with respect to $x_0^j, f^j, g^j, j = 1,2$.

Proof. Consider the operator $A_{x_0^j, f^j, g^j}, j = 1,2$. From Theorem 2.2 it follows that

$$\left\|A_{x_0^1, f^1, g^1}(x) - A_{x_0^1, f^1, g^1}(y)\right\| \le Q\left\|x-y\right\|, \forall x, y \in X.$$

Additionally

$$\left\|A_{x_0^1, f^1, g^1}(x) - A_{x_0^2, f^2, g^2}(x)\right\| \le \eta^1 + (b-a)(\eta^2+\eta^3).$$

Then

$$
\begin{aligned}
&\left\|x^*(t;x_0^1,f^1,g^1)-x^*(t;x_0^2,f^2,g^2)\right\| \\
&= \left\|A_{x_0^1,f^1,g^1}(x^*(t;x_0^1,f^1,g^1)) - A_{x_0^2,f^2,g^2}(x^*(t;x_0^2,f^2,g^2))\right\| \\
&\leq \left\|A_{x_0^1,f^1,g^1}(x^*(t;x_0^1,f^1,g^1)) - A_{x_0^1,f^1,g^1}(x^*(t;x_0^2,f^2,g^2))\right\| \\
&\quad + \left\|A_{x_0^1,f^1,g^1}(x^*(t;x_0^2,f^2,g^2)) - A_{x_0^2,f^2,g^2}(x^*(t;x_0^2,f^2,g^2))\right\| \\
&\leq Q\left\|x^*(t;x_0^1,f^1,g^1)-x^*(t;x_0^2,f^2,g^2)\right\| + \eta^1 + (b-a)(\eta^2+\eta^3).
\end{aligned}
$$

Since $Q^n \to 0$ as $n \to \infty$, implies that $(I-Q)^{-1} \in M_{mm}(\mathbb{R}_+)$ and we finally obtain

$$\left\|x^*(t;x_0^1,f^1,g^1)-x^*(t;x_0^2,f^2,g^2)\right\| \leq (I-Q)^{-1}(\eta^1 + (b-a)(\eta^2+\eta^3)). \qquad \square$$

5. Remarks

In this section we emphasize some special cases of (1.1).

Let $\tau > 0$ be a given number and we define the operator $G : C([-\tau,\infty),\mathbb{R}^m) \to \mathbb{R}^m$ such that for any function $x \in C([-\tau,\infty),\mathbb{R}^m)$ and any point $t \in \mathbb{R}_+$ there exists a point $\xi \in [t-\tau,t]$ such that $G(x)(t) = a(t)x(\xi)$ where $a \in C(\mathbb{R}_+,\mathbb{R})$.

Consider the nonlinear delay functional differential equation

$$x'(t) = f(t,x(t)) + g(t,G(x)(t)) \tag{5.1}$$

for $t \geq t_0$ with initial condition

$$x(t+t_0) = \varphi(t),\ t \in [-\tau,0],$$

where $x \in \mathbb{R}^m,\ f : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m,\ t_0 \in \mathbb{R}_+,\ \varphi : [-\tau,0] \to \mathbb{R}^m$.

Particular cases of (1.1):

- (i) For $G(x)(t) = x(t-\tau),\ t \in \mathbb{R}_+$, then (5.1) reduces to a delay differential equation (see [6], [12], [14], [15]);
- (ii) For $G(x)(t) = \max\limits_{s\in[t-\tau,t]} x(s),\ t \in \mathbb{R}_+$, then (5.1) reduces to a differential equation with maxima (see [16], [17], [9], [10], [1]);
- (iii) For $G(x)(t) = \int_{t-\tau}^t x(s)ds,\ t \in \mathbb{R}_+, \tau > 0$, then (5.1) reduces to a differential equation with distributed delay (see [11], [4]);
- (iv) For $g(t,G(x)(t)) = h(x)(t)$, where $h : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is an abstract Volterra operator, then (5.1) reduces to a differential equation with abstract Volterra operator (see [8]);
- (v) If $x'(t) - f(t,x(t)) := \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right],\ G(x)(t) = x(t),\ t \geq t_0$, then (5.1) reduces to a quadratic differential equation (see [3]).

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