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Existence and Ulam stability results for Hadamard partial fractional integral inclusions via Picard operators

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. In this paper, by using the weakly Picard operators theory, we investigate some existence and Ulam type stability results for a class of Hadamard partial fractional integral inclusions.

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1. Introduction

The fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [16, 27, 38]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [1, 3, 4], Kilbas *et al.* [22], Miller and Ross [24], the papers of Abbas *et al.* [2, 5, 6, 7], Vityuk and Golushkov [40], and the references therein. In [10], Butzer *et al.* investigate properties of the Hadamard fractional integral and the derivative. In [11], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators and in [28], Pooseh *et al.* obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [29] and the references therein.

The stability of functional equations was originally raised by Ulam [39] in 1940 and Hyers [17] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [30] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs of [18, 19]. Bota-Boriceanu and Petruşel [9], Petru *et al.* [25, 26], and Rus [31, 32] discussed the Ulam-Hyers stability for operatorial equations and inclusions. Castro and Ramos [12], and Jung [21] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative are proposed by Wang *et al.* [41, 42]. Some stability results for fractional integral equation are obtained by Wei *et al.* [43]. More details from historical point of view, and recent developments of such stabilities are reported in [20, 31, 43].

The theory of Picard operators was introduced by Ioan A. Rus (see [33, 34, 35] and their references) to study problems related to fixed point theory. This abstract approach was used later on by many mathematicians as a very powerful method in the study of integral equations and inequalities, ordinary and partial differential equations (existence, uniqueness, differentiability of the solutions, ...), see [35] and the references therein. The theory of Picard operators is also a very powerful tool in the study of Ulam-Hyers stability of functional equations. We only have to define a fixed point equation from the functional equation we want to study, then if the defined operator is c-weakly Picard we also have immediately the Ulam-Hyers stability of the desired equation. Of course it is not always possible to transform a functional equation or a differential equation into a fixed point problem and actually this point shows a weakness of this theory. The uniform approach with Picard operators to the discuss of the stability problems of Ulam-Hyers type is due to Rus [32].

In [2, 5, 6], Abbas *et al.* studied some Ulam stabilities for functional fractional partial differential and integral inclusions via Picard operators. In this paper, we discuss the Ulam-Hyers and the Ulam-Hyers-Rassias stability for the following new class of fractional partial integral inclusions of the form

$$u(x,y)-\mu(x,y)\in({}^{H}I^{r}_{\sigma}F)(x,y,u(x,y));\ (x,y)\in J:=[1,a]\times[1,b], \tag{1.1}$$

where $a,b>1,\ \sigma=(1,1),\ F:J\times E\to\mathcal{P}(E)$ is a set-valued function with nonempty values in a (real or complex) separable Banach space E, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, ${}^{H}I^{r}_{\sigma}F$ is the definite Hadamard integral for the set-valued function F of order $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, and $\mu:J\to E$ is a given continuous function.

This paper initiates the existence of the solution and the Ulam stability via Picard operators for such new class of fractional integral inclusions.

2. Basic concepts and auxiliary results

Let $L^1(J)$ be the space of Bochner-integrable functions $u:J\to E$ with the norm

$$\|u\|_{L^1}=\int_1^a\int_1^b\|u(x,y)\|_E dydx,$$

where $\|\cdot\|_E$ denotes a complete norm on E. By $L^\infty(J)$ we denote the Banach space of measurable functions $u : J \to E$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(x,y)\|_E \leq c,\ a.e.\ (x,y) \in J\}.$$

As usual, by $\mathcal{C} := C(J)$ we denote the Banach space of all continuous functions from J into E with the norm $\|.\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(x,y)\in J} \|u(x,y)\|_E.$$

Let (X,d) be a metric space induced from the normed space $(X,\|.\|)$. Denote
$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
$\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
$\mathcal{P}_{cp}(E) = \{Y \in \mathcal{P}(E) : Y\ \text{ compact}\}$ and
$\mathcal{P}_{cp,cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ compact and convex}\}$.

Definition 2.1. *A multivalued map $T : X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$, T is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. T is lower semi-continuous (l.s.c.) if the set $\{t \in X : T(t) \cap B \neq \emptyset\}$ is open for any open set B in X. T is said to be completely continuous if $T(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. T has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator T will be denoted by $Fix(T)$. The graph of T will be denoted by $Graph(T) := \{(u,v) \in X \times \mathcal{P}(X) : v \in T(u)\}$.*

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,\infty) \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b),\ d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a Hausdorff metric space.

Notice that $A : X \to X$ is a selection for $T : X \to \mathcal{P}(X)$ if $A(u) \in T(u)$; for each $u \in X$. For each $u \in \mathcal{C}$, define the set of selections of the multivalued $F : J \times \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$S_{F,u} = \{v :\in L^1(J) : v(x,y) \in F(x,y,u(x,y));\ (x,y) \in J\}.$$

Definition 2.2. *A multivalued map $G : J \to \mathcal{P}_{cl}(E)$, is said to be measurable if for every $v \in E$ the function $(x,y) \to d(v, G(x,y)) = \inf\{d(v,z) : z \in G(x,y)\}$ is measurable.*

In what follows we will give some basic definitions and results on Picard operator theory [35]. Let (X,d) be a metric space and $A : X \to X$ be an operator. We denote by F_A the set of the fixed points of A. We also denote $A^0 := 1_X,\ A^1 := A, \ldots, A^{n+1} := A^n \circ A;\ n \in \mathbb{N}$ the iterate operators of the operator A.

Definition 2.3. *The operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;
(ii) *The sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 2.4. *The operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 2.5. *If A is weakly Picard operator then we consider the operator A^∞ defined by*

$$A^\infty : X \to X;\ A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

Remark 2.6. *It is clear that $A^\infty(X) = F_A$.*

Definition 2.7. *Let A be a weakly Picard operator and $c > 0$. The operator A is c-weakly Picard operator if*

$$d(x, A^\infty(x)) \leq c\ d(x, A(x));\ x \in X.$$

In the multivalued case we have the following concepts (see [36]).

Definition 2.8. *Let (X, d) be a metric space, and $F : X \to \mathcal{P}_{cl}(X)$ be a multivalued operator. By definition, F is a multivalued weakly Picard operator (MWPO), if for each $u \in X$ and each $v \in F(x)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that*

(i) $u_0 = u,\ u_1 = v$;
(ii) $u_{n+1} \in F(u_n)$, *for each* $n \in \mathbb{N}$;
(iii) *the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.*

Remark 2.9. *A sequence $(u_n)_{n\in\mathbb{N}}$ satisfying condition (i) and (ii) in the above Definition is called a sequence of successive approximations of F starting from $(x, y) \in Graph(F)$.*

If $F : X \to \mathcal{P}_{cl}(X)$ is a (MWPO) then we define $F_1 : Graph(F) \to \mathcal{P}(Fix(F))$ by the formula $F_1(x, y) := \{u \in Fix(F)$: there exists a sequence of successive approximations of F starting from (x, y) that converges to $u\}$.

Definition 2.10. *Let (X, d) be a metric space and let $\Psi : [0, \infty) \to [0, \infty)$ be an increasing function which is continuous at 0 and $\Psi(0) = 0$. Then $F : X \to \mathcal{P}_{cl}(X)$ is said to be a multivalued $\Psi-$weakly Picard operator ($\Psi-MWPO$) if it is a multivalued weakly Picard operator and there exists a selection $A^\infty : Graph(F) \to Fix(F)$ of F^∞ such that*

$$d(u, A^\infty(u, v)) \leq \Psi(d(u, v));\ for\ all\ (u, v) \in Graph(F).$$

If there exists $c > 0$ such that $\Psi(t) = ct$, for each $t \in [0.\infty)$, then F is called a multivalued c-weakly Picard operator ($c - MWPO$).

Let us recall the notion of comparison function.

Definition 2.11. *A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a comparison function (see [35]) if it is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0$.*

As a consequence, we have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous at 0.

Definition 2.12. *A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a strict comparison function (see [35]) if it is strictly increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, for each $t > 0$.*

Example 2.13. *The mappings* $\varphi_1, \varphi_2 : [0,\infty) \to [0,\infty)$ *given by* $\varphi_1(t) = ct;\ c \in [0,1)$, *and* $\varphi_2(t) = \frac{t}{1+t};\ t \in [0,\infty)$, *are strict comparison functions.*

Definition 2.14. *A multivalued operator* $N : X \to \mathcal{P}_{cl}(X)$ *is called*

a) γ*-Lipschitz if and only if there exists* $\gamma \geq 0$ *such that*

$$H_d(N(u), N(v)) \leq \gamma d(u,v);\ \textit{for each}\ \ u,\ v \in X,$$

b) *a multivalued* $\gamma-$*contraction if and only if it is* γ*-Lipschitz with* $\gamma \in [0,1)$,

c) *a multivalued* $\varphi-$*contraction if and only if there exists a strict comparison function* $\varphi : [0,\infty) \to [0,\infty)$ *such that*

$$H_d(N(u), N(v)) \leq \varphi(d(u,v));\ \textit{for each}\ \ u,\ v \in X.$$

Now, we introduce notations and definitions concerning to partial Hadamard integral of fractional order.

Definition 2.15. [15, 22] *The Hadamard fractional integral of order* $q > 0$ *for a function* $g \in L^1([1,a],\mathbb{R})$, *is defined as*

$$({}^H I_1^r g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

where $\Gamma(\cdot)$ *is the Euler gamma function.*

Definition 2.16. *Let* $r_1,\ r_2 \geq 0,\ \sigma = (1,1)$ *and* $r = (r_1, r_2)$. *For* $w \in L^1(J,\mathbb{R})$, *define the Hadamard partial fractional integral of order* r *by the expression*

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st} dtds.$$

Definition 2.17. *Let* $F : J \times E \to \mathcal{P}(E)$ *be a set-valued function with nonempty values in* E. $({}^H I_\sigma^r F)(x,y,u(x,y))$ *is the definite Hadamard integral for the set-valued functions* F *of order* $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$ *which is defined as*

$${}^H I_\sigma^r F(x,y,u(x,y)) = \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds : f \in S_{F,u} \right\}.$$

Remark 2.18. *Solutions of the inclusion (*1.1*) are solutions of the fixed point inclusion* $u \in N(u)$ *where* $N : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ *is the multivalued operator defined by*

$$(Nu)(x,y) = \left\{\mu(x,y) + ({}^H I_\sigma^r f)(x,y) :\ f \in S_{F,u}\right\};\ (x,y) \in J.$$

Let us give the definition of Ulam-Hyers stability of the fixed point inclusion $u \in N(u)$, see for instance [2]. Let ϵ be a positive real number and $\Phi : J \to [0,\infty)$ be a continuous function.

Definition 2.19. *The fixed point inclusion* $u \in N(u)$ *is said to be Ulam-Hyers stable if there exists a real number* $c_N > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $u \in \mathcal{C}$ *of the inequality* $H_d(u(x,y), (Nu)(x,y)) \leq \epsilon;\ (x,y) \in J$, *there exists a solution* $v \in \mathcal{C}$ *of the inclusion* $u \in N(u)$ *with*

$$\|u(x,y) - v(x,y)\|_E \leq \epsilon c_N;\ (x,y) \in J.$$

Definition 2.20. *The fixed point inclusion $u \in N(u)$ is said to be generalized Ulam-Hyers stable if there exists an increasing function $\theta_N \in C([0,\infty),[0,\infty))$, $\theta_N(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y),(Nu)(x,y)) \leq \epsilon$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with*

$$\|u(x,y) - v(x,y)\|_E \leq \theta_N(\epsilon);\ (x,y) \in J.$$

Definition 2.21. *The fixed point inclusion $u \in N(u)$ is said to be Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y),(Nu)(x,y)) \leq \epsilon\Phi(x,y)$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with*

$$\|u(x,y) - v(x,y)\|_E \leq \epsilon c_{N,\Phi}\Phi(x,y);\ (x,y) \in J.$$

Definition 2.22. *The fixed point inclusion $u \in N(u)$ is said to be generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y),(Nu)(x,y)) \leq \Phi(x,y)$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with*

$$\|u(x,y) - v(x,y)\|_E \leq c_{N,\Phi}\Phi(x,y);\ (x,y) \in J.$$

Remark 2.23. *It is clear that*

(i) *Definition* 2.19 $\Rightarrow$ *Definition* 2.20*,*
(ii) *Definition* 2.21 $\Rightarrow$ *Definition* 2.22*,*
(iii) *Definition* 2.21 *for* $\Phi(x,y) = 1 \ \Rightarrow$ *Definition* 2.19.

The following result, a generalization of Covitz-Nadler fixed point principle (see [14]), is known in the literature as Węgrzyk's fixed point theorem.

Lemma 2.24. [44] *Let (X,d) be a complete metric space. If $A : X \to \mathcal{P}_{cl}(X)$ is a $\varphi-$contraction, then $Fix(A)$ is nonempty and for any $u_0 \in X$, there exists a sequence of successive approximations of A starting from u_0 which converges to a fixed point of A.*

Also, the following result is known in the literature as Węgrzyk's theorem.

Lemma 2.25. [44] *Let (X,d) be a Banach space. If an operator $A : X \to \mathcal{P}_{cl}(X)$ is a $\varphi-$contraction, then A is a $(MWPO)$.*

Now we present an important characterization Lemma from the point of view of Ulam-Hyers stability.

Lemma 2.26. [26] *Let (X,d) be a metric space. If $A : X \to \mathcal{P}_{cp}(X)$ is a $(\Psi-MWPO)$, then the fixed point inclusion $u \in A(u)$ is generalized Ulam-Hyers stable. In particular, if A is $(c-MWPO)$, then the fixed point inclusion $u \in A(u)$ is Ulam-Hyers stable.*

Another Ulam-Hyers stability result, more efficient for applications, was proved in [23].

Theorem 2.27. [23] *Let (X,d) be a complete metric space and $A : X \to \mathcal{P}_{cp}(X)$ be a multivalued $\varphi-$contraction. Then:*

(i) Existence of the fixed point: *A is a $(MWPO)$;*

(ii) Ulam-Hyers stability for the fixed point inclusion: *If additionally $\varphi(ct) \le c\varphi(t)$ for every $t \in [0,\infty)$ (where $c > 1$), then A is a $(\Psi-MWPO)$, with $\Psi(t) := t + \sum_{n=1}^{\infty} \varphi^n(t)$, for each $t \in [0,\infty)$;*

(iii) Data dependence of the fixed point set: *Let $S : X \to \mathcal{P}_{cl}(X)$ be a multivalued $\varphi-$contraction and $\eta > 0$ be such that $H_d(S(x), A(x)) \le \eta$, for each $x \in X$. Suppose that $\varphi(ct) \le c\varphi(t)$ for every $t \in [0,\infty)$ (where $c > 1$). Then,*

$$H_d(Fix(S), Fix(F)) \le \Psi(\eta).$$

3. Existence and Ulam-Hyers stability results

In this section, we present conditions for the existence and the Ulam stability of the Hadamard integral inclusion (1.1).

Theorem 3.1. *Assume that the multifunction $F : J \times E \to \mathcal{P}_{cp}(E)$ satisfies the following hypotheses:*

(H_1) *$(x,y) \longmapsto F(x,y,u)$ is jointly measurable for each $u \in E$;*

(H_2) *$u \longmapsto F(x,y,u)$ is lower semicontinuous for almost all $(x,y) \in J$;*

(H_3) *There exists $p \in L^\infty(J,[0,\infty))$ and a strict comparison function $\varphi : [0,\infty) \to [0,\infty)$ such that for each $(x,y) \in J$ and each $u, v \in E$, we have*

$$H_d(F(x,y,u(x,y)), F(x,y,\overline{u})\| \le p(x,y)\varphi(\|u-\overline{u}\|_E), \tag{3.1}$$

and

$$\frac{(\log a)^{r_1}(\log b)^{r_2}\|p\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \le 1; \tag{3.2}$$

(H_4) *There exists an integrable function $q : [1,b] \to [0,\infty)$ such that for each $x \in [1,a]$ and $u \in E$, we have $F(x,y,u) \subset q(y)B(0,1)$, a.e. $y \in [1,b]$, where $B(0,1) = \{u \in E : \|u\|_E < 1\}$.*

Then the following conclusions hold:

(a) *The integral inclusion (1.1) has least one solution and N is a $(MWPO)$.*

(b) *If additionally $\varphi(ct) \le c\varphi(t)$ for every $t \in [0,\infty)$ (where $c > 1$), then the integral inclusion (1.1) is generalized Ulam-Hyers stable, and N is a $(\Psi-MWPO)$, with the function Ψ defined by $\Psi(t) := t + \sum_{n=1}^{\infty} \varphi^n(t)$, for each $t \in [0,\infty)$. Moreover, in this case the continuous data dependence of the solution set of the integral inclusion (3.1) holds.*

Remark 3.2. *For each $u \in \mathcal{C}$, the set $S_{F,u}$ is nonempty since by (H_1), F has a measurable selection (see [13], Theorem III.6).*

Proof. We shall show that N defined in Remark 2.18 satisfies the assumptions of Theorem 2.27. The proof will be given in two steps.

Step 1. $N(u) \in P_{cp}(\mathcal{C})$ *for each* $u \in \mathcal{C}$.

From the continuity of μ and Theorem 2 in Rybiński [37] we have that for each $u \in \mathcal{C}$

there exists $f \in S_{F,u}$, for all $(x,y) \in J$, such that $f(x,y)$ is integrable with respect to y and continuous with respect to x. Then the function $v(x,y) = \mu(x,y) +^H I_\sigma^r f(x,y)$ has the property $v \in N(u)$. Moreover, from (H_1) and (H_4), via Theorem 8.6.3. in Aubin and Frankowska [8], we get that $N(u)$ is a compact set, for each $u \in \mathcal{C}$.

Step 2. $H_d(N(u), N(\overline{u})) \leq \varphi(\|u - \overline{u}\|_\infty)$ *for each* $u, \overline{u} \in \mathcal{C}$.

Let $u, \overline{u} \in \mathcal{C}$ and $h \in N(u)$. Then, there exists $f(x,y) \in F(x,y,u(x,y))$ such that for each $(x,y) \in J$, we have

$$h(x,y) = \mu(x,y) +^H I_\sigma^r f(x,y).$$

From (H_3) it follows that

$$H_d(F(x,y,u(x,y)), F(x,y,\overline{u}(x,y))) \leq p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E).$$

Hence, there exists $w(x,y) \in F(x,y,\overline{u}(x,y)$ such that

$$\|f(x,y) - w(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E);\ (x,y) \in J.$$

Consider $U : J \to \mathcal{P}(E)$ given by

$$U(x,y) = \{w \in E : \|f(x,y) - w(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E)\}.$$

Since the multivalued operator $u(x,y) = U(x,y) \cap F(x,y,\overline{u}(x,y))$ is measurable (see Proposition III.4 in [13]), there exists a function $\overline{f}(x,y)$ which is a measurable selection for u. So, $\overline{f}(x,y) \in F(x,y,\overline{u}(x,y))$, and for each $(x,y) \in J$,

$$\|f(x,y) - \overline{f}(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E).$$

Let us define for each $(x,y) \in J$,

$$\overline{h}(x,y) = \mu(x,y) +^H I_\sigma^r \overline{f}(x,y).$$

Then for each $(x,y) \in J$, we have

$$\begin{aligned}
\|h(x,y) - \overline{h}(x,y)\|_E &\leq {}^H I_\sigma^r \|f(x,y) - \overline{f}(x,y)\|_E \\
&\leq {}^H I_\sigma^r (p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E)) \\
&\leq \|p\|_{L^\infty}\varphi(\|u - \overline{u}\|_\infty)\left(\int_1^x \int_1^y \frac{\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}}{st\Gamma(r_1)\Gamma(r_2)}dtds\right) \\
&\leq \frac{(\log a)^{r_1}(\log b)^{r_2}\|p\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}\varphi(\|u - \overline{u}\|_\infty).
\end{aligned}$$

Thus, by (3.2), we get

$$\|h - \overline{h}\|_\infty \leq \varphi(\|u - \overline{u}\|_\infty).$$

By an analogous relation, obtained by interchanging the roles of u and $\overline{u}$, it follows that

$$H_d(N(u), N(\overline{u})) \leq \varphi(\|u - \overline{u}\|_\infty).$$

Hence, N is a $\varphi-$contraction.

(a) By Lemma 2.24, N has a fixed point witch is a solution of the inclusion (1.1) on J, and by [Theorem 2.27,(i)], N is a $(MWPO)$.

(b) We will prove that the fixed point inclusion problem (1.1) is generalized Ulam-Hyers stable. Indeed, let $\epsilon > 0$ and $v \in \mathcal{C}$ for which there exists $u \in \mathcal{C}$ such that

$$u(x,y) \in \mu(x,y) + ({}^H I_\sigma^r F)(x,y,v(x,y)); \quad \text{if } (x,y) \in J,$$

and

$$\|u - v\|_\infty \le \epsilon.$$

Then $H_d(v, N(v)) \le \epsilon$. Moreover, by the above proof we have that N is a multivalued $\varphi-$contraction and using [Theorem 2.27,(i)-(ii)], we obtain that N is a is a $(\Psi - MWPO)$. Then, by Lemma 2.26 we obtain that the fixed point problem $u \in N(u)$ is generalized Ulam-Hyers stable. Thus, the integral inclusion (1.1) is generalized Ulam-Hyers stable.
Concerning the conclusion of the theorem, we apply [Theorem 2.27,(iii)].

4. An example

Let $E = l^1 = \left\{ w = (w_1, w_2, \ldots, w_n, \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty \right\}$, be the Banach space with norm

$$\|w\|_E = \sum_{n=1}^{\infty} |w_n|,$$

and consider the following partial functional fractional order integral inclusion of the form

$$u(x,y) \in \mu(x,y) + ({}^H I_\sigma^r F)(x,y,u(x,y)); \text{ a.e. } (x,y) \in [1,e] \times [1,e], \tag{4.1}$$

where $r = (r_1, r_2),\ r_1, r_2 \in (0, \infty)$,

$$u = (u_1, u_2, \ldots, u_n, \ldots),\ \mu(x,y) = (x + e^{-y}, 0, \ldots, 0, \ldots),$$

and

$$F(x,y,u(x,y))$$
$$= \{v \in C([1,e] \times [1,e], \mathbb{R}) : \|f_1(x,y,u(x,y))\|_E \le \|v\|_E \le \|f_2(x,y,u(x,y))\|_E\};$$

$(x,y) \in [1,e] \times [1,e]$, where $f_1, f_2 : [1,e] \times [1,e] \times E \to E$,

$$f_k = (f_{k,1}, f_{k,2}, \ldots, f_{k,n}, \ldots);\ k \in \{1,2\},\ n \in \mathbb{N},$$

$$f_{1,n}(x,y,u_n(x,y)) = \frac{xy^2 u_n}{(1 + \|u_n\|_E) e^{10+x+y}};\ n \in \mathbb{N},$$

and

$$f_{2,n}(x,y,u_n(x,y)) = \frac{xy^2 u_n}{e^{10+x+y}};\ n \in \mathbb{N}.$$

We assume that F is closed and convex valued. We can see that the solutions of the inclusion(4.1) are solutions of the fixed point inclusion $u \in A(u)$ where $A : C([1,e] \times [1,e], \mathbb{R}) \to \mathcal{P}(C([1,e] \times [1,e], \mathbb{R}))$ is the multifunction operator defined by

$$(Au)(x,y) = \left\{ \mu(x,y) + ({}^H I_\sigma^r f)(x,y);\ f \in S_{F,u} \right\};\ (x,y) \in [1,e] \times [1,e].$$

For each $(x,y) \in [1,e] \times [1,e]$ and all $z_1, z_2 \in E$, we have

$$\|f_2(x,y,z_2) - f_1(x,y,z_1)\|_E \leq xy^2 e^{-10-x-y}\|z_2 - z_1\|_E.$$

Thus, the hypotheses $(H_1)-(H_3)$ are satisfied with $p(x,y) = xy^2e^{-10-x-y}$. We shall show that condition (3.2) holds with $a = b = e$. Indeed, $\|p\|_{L^\infty} = e^{-9}$, $\Gamma(1+r_i) > \frac{1}{2}$; $i = 1,2$. A simple computation shows that

$$\zeta := \frac{(\log a)^{r_1}(\log b)^{r_2}\|p\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 4e^{-9} < 1.$$

The condition (H_4) is satisfied with $q(y) = \frac{y^2e^{-10-y}}{\|F\|_{\mathcal{P}}}$; $y \in [1,e]$, where

$$\|F\|_{\mathcal{P}} = \sup\{\|f\|_{\mathcal{C}} : f \in S_{F,u}\}; \quad \text{for all}\, u \in \mathcal{C}.$$

Consequently, by Theorem 3.1 we concluded that:

(a) The integral inclusion (4.1) has least one solution and A is a $(MWPO)$.

(b) The function $\varphi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) = \zeta t$ satisfies $\varphi(\zeta t) \leq \zeta\varphi(t)$ for every $t \in [0,\infty)$. Then the integral inclusion (4.1) is generalized Ulam-Hyers stable, and A is a ($\Psi-$MWPO), with the function Ψ defined by $\Psi(t) := t + (1-\zeta t)^{-1}$, for each $t \in [0,\zeta^{-1})$. Moreover, the continuous data dependence of the solution set of the integral inclusion (3.1) holds.

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