Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 3, 331–341

On some numerical iterative methods for Fredholm integral equations with deviating arguments

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. In this paper we develop iterative methods for nonlinear Fredholm integral equations of the second kind with deviating arguments, by applying Mann's iterative algorithm. This proves the existence and the uniqueness of the solution and gives a better error estimate than the classical Banach Fixed Point Theorem. The iterates are then approximated using a suitable quadrature formula. The paper concludes with numerical examples.

Mathematics Subject Classification (2010): 45B05, 47H10, 47N20, 65R20.

Keywords: Fredholm integral equations, deviating arguments, numerical approximations, Altman's algorithm, Mann's iterative algorithm.

1. Preliminaries

Integral equations arise in many fields of mathematics, engineering, physics, etc., as they provide a strong tool for modeling various applications, phenomena and processes occurring in actuarial sciences, statistical study of dynamic living population, elasticity theory, diffraction problems, quantum mechanics, etc. In addition, a large class of initial and boundary value problems can be reformulated as integral equations. Thus, many researchers aim to find efficient and rapidly convergent algorithms for the numerical solution of Fredholm integral equations (see e.g. [2], [10], [11], [9]).

In this paper, we consider a Fredholm integral equation of the type

$$x(t) \quad = \quad \int_a^b K\big(t, s, x(s), x(g(s))\big)\, ds + f(t), \quad t \in [a, b], \tag{1.1}$$

where $K \in C\left([a, b] \times [a, b] \times \mathbb{R}^2\right)$, $f \in C[a, b]$ and $g \in C\left([a, b], [a, b]\right)$.
Other assumptions will be made on K, g and f later on.

As is well known, the solvability of (1.1) is based on fixed point theory. We define the operator $F : C[a,b] \to C[a,b]$ by

$$Fx(t) \quad = \quad \int_a^b K\big(t, s, x(s), x(g(s))\big)\ ds + f(t). \tag{1.2}$$

Then finding a solution of the integral equation (1.1) is equivalent to finding a fixed point for the operator F:

$$x = Fx. \tag{1.3}$$

We recall the main results of fixed point theory on a Banach space.

Definition 1.1. *Let $(X, ||\cdot||)$ be a Banach space. A mapping $F : X \to X$ is called a* $\mathbf{q-contraction}$ *if there exists a constant $0 \le q < 1$ such that*

$$||Fx - Fy|| \le q||x - y||, \tag{1.4}$$

for all $x, y \in X$.

We have the classical result, the contraction principle on a Banach space.

Theorem 1.2. *Let $(X, ||\cdot||)$ be a Banach space and $F : X \to X$ be a $q-$contraction. Then*

(a) *F has exactly one fixed point $x^* \in X$;*
(b) *the sequence of successive approximations $x_{n+1} = Fx_n, n \in \mathbb{N}$, converges to the solution x^*, for any arbitrary choice of initial point $x_0 \in X$;*
(c) *the error estimates*

$$\begin{aligned} ||x_n - x^*|| &\le \frac{q^n}{1-q}\ ||x_1 - x_0||, \\ ||x_n - x^*|| &\le \frac{q}{1-q}\ ||x_n - x_{n-1}|| \end{aligned} \tag{1.5}$$

hold for every $n \in \mathbb{N}$.

This result can be improved, using Mann iteration (Altman's algorithm) instead of Picard iteration. We recall the main results (see [1], [4]).

Theorem 1.3. *Let $(X, ||\cdot||)$ be a Banach space and $F : X \to X$ be a $q-$contraction. Let $0 < \varepsilon_n \le 1$ be a sequence of numbers satisfying*

$$\sum_{n=0}^{\infty} \varepsilon_n = \infty. \tag{1.6}$$

Then

(a) *equation $x = Fx$ has exactly one solution $x^* \in X$;*
(b) *the sequence of successive approximations*

$$x_{n+1} = (1 - \varepsilon_n)x_n + \varepsilon_n F x_n, \quad n = 0, 1, \ldots \tag{1.7}$$

converges to the solution x^*, for any arbitrary choice of initial point $x_0 \in X$;

(c) *for every $n \in \mathbb{N}$, there holds the error estimate*

$$||x_n - x^*|| \quad \leq \quad \frac{e^{1-q}}{1-q}\, ||x_0 - Fx_0||\; e^{-(1-q)y_n}, \tag{1.8}$$

where $y_0 = 0$, $y_n = \sum_{i=0}^{n-1} \varepsilon_i$, *for* $n \geq 1$.

Remark 1.4. Theorem 1.3 still holds true if X is replaced by any closed convex subset $Y \subseteq X$.

Most of the times (for suitable choices of ε_n and q), the error estimate in (1.8) is better than the one in (1.5) and the iterative method (1.7) converges faster than the classical one.

For more considerations on iterative algorithms, see e.g. [4], [7], [8]. The aim of this paper is to apply Altman's Theorem 1.3 to Fredholm integral equations of the second kind with deviating arguments.

2. Existence and uniqueness of the solution

We want to apply Altman's iterative algorithm to the operator equation (1.3). To this end, we consider the space $X = C[a, b]$ equipped with the Chebyshev norm

$$||x|| := \max_{t\in[a,b]} |x(t)|, \quad x \in X \tag{2.1}$$

and the ball $B_R := \{x \in C[a, b] \;\big|\; ||x - f|| \leq R\}$, for some $R > 0$. Then $(X, ||\cdot||)$ is a Banach space and $B_R \subseteq X$ is a closed convex subset.

Theorem 2.1. *Let $F : (X, ||\cdot||) \to (X, ||\cdot||)$ be defined by* (1.2). *Assume that*

(i) *there exist constants $l_1, l_2 > 0$ such that*

$$|K(t, s, u_1, v_1) - K(t, s, u_2, v_2)| \leq l_1|u_1 - u_2| + l_2|v_1 - v_2|, \tag{2.2}$$

for all $t, s \in [a, b]$ and all $u_1, u_2, v_1, v_2 \in [R_1 - R, R_2 + R]$, where

$$R_1 := \min_{t\in[a,b]} f(t), \quad R_2 := \max_{t\in[a,b]} f(t);$$

(ii)

$$q \quad := \quad (b - a)(l_1 + l_2) \quad < \quad 1; \tag{2.3}$$

(iii)

$$M_K(b - a) \quad \leq \quad R, \tag{2.4}$$

where $M_K := \max |K(t, s, u, v)|$ over all $t, s \in [a, b]$ and all $u, v \in [R_1 - R, R_2 + R]$.

Then

(a) *equation* (1.3) *has exactly one solution $x^* \in X$;*

(b) *the sequence of successive approximations*

$$x_{n+1} = \left(1 - \frac{1}{n+1}\right) x_n + \frac{1}{n+1} Fx_n, \quad n = 0, 1, \ldots \tag{2.5}$$

converges to the solution x^*, for any arbitrary initial point $x_0 \in X$;

(c) *for every $n \in \mathbb{N}$, the error estimate*

$$||x_n - x^*|| \quad \leq \quad \frac{e^{1-q}}{1-q}\, ||x_0 - Fx_0||\; e^{-(1-q)y_n} \tag{2.6}$$

holds, where $y_0 = 0$, $y_n = \sum_{i=0}^{n-1} \frac{1}{i+1}$, *for* $n \geq 1$.

Proof. We want to use Theorem 1.3 for $\varepsilon_n = \dfrac{1}{n+1}$, which obviously satisfies the conditions of Theorem 1.3.
Let $t \in [a,b]$ be fixed. By (2.2), we have

$$\begin{aligned}
|(Fx - Fy)(t)| \quad &\leq \quad \int_a^b \Big|K\big(t,s,x(s),x(g(s))\big) - K\big(t,s,y(s),y(g(s))\big)\Big|\; ds \\
&\leq \quad l_1 \int_a^b |x(s) - y(s)|\; ds + l_2 \int_a^b |x(g(s)) - y(g(s))|\; ds \\
&\leq \quad l_1(b-a)||x-y|| + l_2(b-a) \max_{g(s)\in[a,b]} \Big|x(g(s)) - y(g(s))\Big| \\
&\leq \quad (b-a)(l_1 + l_2)||x-y||,
\end{aligned}$$

since $\max\limits_{g(s)\in[a,b]} \Big|x(g(s)) - y(g(s))\Big| \leq \max\limits_{s\in[a,b]} |x(s) - y(s)|$. Hence,

$$||Fx - Fy|| \;=\; \max_{t\in[a,b]} |(Fx - Fy)(t)| \;\leq\; q||x-y||$$

and by (2.3), it follows that F is a $q-$contraction.

Next, for every fixed $t \in [a,b]$, we have

$$\begin{aligned}
|Fx(t) - f(t)| \quad &\leq \quad \int_a^b \Big|K\big(t,s,x(s),x(g(s))\big)\Big|\; ds \\
&\leq \quad M_K(b-a).
\end{aligned}$$

Thus, by (2.4), we have $F(B_R) \subseteq B_R$. Now our result follows from Theorem 1.3 and Remark 1.4. □

For more considerations on Mann iteration, see e.g. [4].

3. Numerical iterative methods

Altman's fixed point theorem provides iterative methods for solving equation (1.3). But, obviously, the iterates in (2.5) cannot be computed analytically, they have to be approximated numerically.

Consider a quadrature formula

$$\int_a^b \varphi(s)ds = \sum_{i=0}^{m} a_i \varphi(s_i) + R_\varphi, \tag{3.1}$$

with nodes $a = s_0 < s_1 < \cdots < s_m = b$, coefficients $a_i \in \mathbb{R}, i = 0, 1, \ldots, m$ and for which the remainder satisfies

$$|R_\varphi| \le M, \tag{3.2}$$

for some $M > 0$, with $M \to 0$ as $m \to \infty$.

Let $a = t_0 < t_1 < \cdots < t_m = b$ be the nodes and let $x_0 = \tilde{x}_0 \equiv f$ be the initial approximation. Then we use the iteration (2.5) and the numerical integration scheme (3.1) to approximate $x_n(t_k)$ and $x_n(g(t_k))$ with $\tilde{x}_n(t_k)$ and $\tilde{x}_n(g(t_k))$, respectively, for $k = \overline{0, m}$ and $n = 0, 1, \ldots$. For simplicity, we make the following notations:

$$\begin{aligned}
K_{k,i,n} &:= K\big(t_k, t_i, x_n(t_i), x_n(g(t_i))\big),\\
K_{g,k,i,n} &:= K\big(g(t_k), t_i, x_n(t_i), x_n(g(t_i))\big),\\
\tilde{K}_{k,i,n} &:= K\big(t_k, t_i, \tilde{x}_n(t_i), \tilde{x}_n(g(t_i))\big),\\
\tilde{K}_{g,k,i,n} &:= K\big(g(t_k), t_i, \tilde{x}_n(t_i), \tilde{x}_n(g(t_i))\big),\\
\tilde{x}_{n+1}(t_k) &:= \left[\left(1 - \frac{1}{n+1}\right)\tilde{x}_n(t_k) + \frac{1}{n+1}\left(\sum_{i=0}^{m} a_i \tilde{K}_{k,i,n} + f(t_k)\right)\right],\\
\tilde{x}_{n+1}(g(t_k)) &:= \left[\left(1 - \frac{1}{n+1}\right)\tilde{x}_n(g(t_k))\right.\\
&+ \left.\frac{1}{n+1}\left(\sum_{i=0}^{m} a_i \tilde{K}_{g,k,i,n} + f(g(t_k))\right)\right],\\
\tilde{R}_{n,k} &:= x_n(t_k) - \tilde{x}_n(t_k),\\
\tilde{R}_{g,n,k} &:= x_n(g(t_k)) - \tilde{x}_n(g(t_k)).
\end{aligned}$$

We have:

$$\begin{aligned}
x_{n+1}(t_k) &= \left(1 - \frac{1}{n+1}\right)x_n(t_k) + \frac{1}{n+1}\left(\int_a^b K\left(t_k, s, x_n(s), x_n(g(s))\right)ds + f(t_k)\right)\\
&= \left(1 - \frac{1}{n+1}\right)\left(\tilde{x}_n(t_k) + \tilde{R}_{n,k}\right) + \frac{1}{n+1}\left(\sum_{i=0}^{m} a_i K_{k,i,n} + R_K + f(t_k)\right) \qquad (3.3)\\
&= \left(1 - \frac{1}{n+1}\right)\left(\tilde{x}_n(t_k) + \tilde{R}_{n,k}\right)\\
&+ \frac{1}{n+1}\left(\sum_{i=0}^{m} a_i \tilde{K}_{k,i,n} + \sum_{i=0}^{m} a_i (K_{k,i,n} - \tilde{K}_{k,i,n}) + R_K + f(t_k)\right)\\
&= \left[\left(1 - \frac{1}{n+1}\right)\tilde{x}_n(t_k) + \frac{1}{n+1}\left(\sum_{i=0}^{m} a_i \tilde{K}_{k,i,n} + f(t_k)\right)\right] + \tilde{R}_{n+1,k} = \tilde{x}_{n+1}(t_k) + \tilde{R}_{n+1,k}.
\end{aligned}$$

Similarly, we derive

$$\begin{aligned}
\tilde{x}_{n+1}(g(t_k)) &= \left(1-\frac{1}{n+1}\right)\tilde{x}_n(g(t_k)) \\
&+ \frac{1}{n+1}\left(\sum_{i=0}^{m} a_i \tilde{K}_{g,k,i,n} + f(g(t_k))\right) + \tilde{R}_{g,n+1,k} \qquad (3.4)\\
&= \tilde{x}_{n+1}(g(t_k)) + \tilde{R}_{g,n+1,k}.
\end{aligned}$$

Let

$$\tilde{R}^{(n,m)} = \max_{t_k\in[a,b]} \left\{\left|x_n(t_k)-\tilde{x}_n(t_k)\right|, \left|x_n(g(t_k))-\tilde{x}_n(g(t_k))\right|\right\}. \qquad (3.5)$$

Suppose that for the quadrature formula (3.1), condition (3.2) ensures the fact that $\tilde{R}^{(n,m)}$ defined above depends only on m and that $\tilde{R}^{(n,m)} = \tilde{R}^{(m)} \to 0$, as $m \to \infty$. Then the exact solution x^* can be approximated by the iterates $\tilde{x}_n$ at the nodes t_k and $g(t_k)$ and we can give an error estimate for our numerical iterative method. To better illustrate the approximations, we consider below one of the most popular numerical integration schemes, the trapezoidal rule.

3.1. Approximation using the trapezoidal rule

As in [5], [6], consider the composite trapezoidal rule

$$\int_a^b \varphi(s)\, ds = \frac{b-a}{2m}\left[\varphi(a) + 2\sum_{j=1}^{m-1}\varphi(s_j) + \varphi(b)\right] + R_\varphi,$$

where the $m+1$ nodes are $s_j = a + \dfrac{b-a}{m} j,\ j=\overline{0,m}$ and the remainder is given by

$$R_\varphi = -\frac{(b-a)^3}{12m^2}\varphi''(\eta), \quad \eta \in (a,b).$$

We use it to approximate the integrals in (2.5), as in (3.3) and (3.4), with initial approximation $x_0 = \tilde{x}_0 \equiv f$. For the error, we need the second derivative $[K(t_k, s, x_n(s), x_n(g(s)))]_s''$. We have

$$\begin{aligned}
[K(t_k,s,u,v)]_s' &= \frac{\partial K}{\partial s} + \frac{\partial K}{\partial u}u' + \frac{\partial K}{\partial v}v' \\
[K(t_k,s,u,v)]_s'' &= \frac{\partial^2 K}{\partial s^2} + 2\frac{\partial^2 K}{\partial s \partial u}u' + 2\frac{\partial^2 K}{\partial s\partial v}v' + 2\frac{\partial^2 K}{\partial u \partial v}u'v' \\
&+ \frac{\partial^2 K}{\partial u^2}(u')^2 + \frac{\partial^2 K}{\partial v^2}(v')^2 + \frac{\partial K}{\partial u}u'' + \frac{\partial K}{\partial v}v''
\end{aligned}$$

i.e.

$$\begin{aligned}
\left[K(t_k,s,x_n(s),x_n(g(s)))\right]_s'' &= \frac{\partial^2 K}{\partial s^2}+2\frac{\partial^2 K}{\partial s\partial u}x_n'(s)+2\frac{\partial^2 K}{\partial s\partial v}x_n'(g(s))g'(s)\\
&+ 2\frac{\partial^2 K}{\partial u\partial v}x_n'(s)x_n'(g(s))g'(s)+\frac{\partial^2 K}{\partial u^2}(x_n'(s))^2\\
&+ \frac{\partial^2 K}{\partial v^2}\left[x_n'(g(s))g'(s)\right]^2+\frac{\partial K}{\partial u}x_n''(s) \qquad (3.6)\\
&+ \frac{\partial K}{\partial v}\left(x_n''(g(s))(g'(s))^2+x_n'(g(s))g''(s)\right)
\end{aligned}$$

For any $t\in[a,b]$,

$$\begin{aligned}
x_n(t) &= \left(1-\frac{1}{n}\right)x_{n-1}(t)\\
&+ \frac{1}{n}\left(\int_a^b K\left(t,s,x_{n-1}(s),x_{n-1}(g(s)\right)\,ds+f(t))\right),\\
x_n'(t) &= \left(1-\frac{1}{n}\right)x_{n-1}'(t)\\
&+ \frac{1}{n}\left(\int_a^b \frac{\partial K}{\partial t}\left(t,s,x_{n-1}(s),x_{n-1}(g(s)\right)\,ds+f'(t))\right),\\
x_n''(t) &= \left(1-\frac{1}{n}\right)x_{n-1}''(t)\\
&+ \frac{1}{n}\left(\int_a^b \frac{\partial^2 K}{\partial t^2}\left(t,s,x_{n-1}(s),x_{n-1}(g(s)\right)\,ds+f''(t))\right).
\end{aligned}$$

It is clear from our work so far, that if the functions K,g and f are C^2 functions with bounded second order derivatives, then for $\tilde{R}^{(n,m)}$ defined in (3.5), we have

$$\tilde{R}^{(n,m)} \le \frac{(b-a)^3}{12m^2}M_0, \qquad (3.7)$$

where M_0 depends on a,b,l_1,l_2 and the functions K,g and f, but *not* on n or m.

We can now give an error estimate for our approximation.

Theorem 3.1. *Assume the conditions of Theorem* 2.1 *hold. Further, assume that K,g and f are C^2 functions with bounded second order derivatives. Then for the true solution x^* of* (1.3) *and the approximations $\tilde{x}_n$ given by* (3.3) – (3.4), *the error estimate*

$$||x^*-\tilde{x}_n|| \le \frac{e^{1-q}}{1-q}\,||x_0-Fx_0||\,e^{-(1-q)y_n}+\frac{(b-a)^3}{12m^2}M_0 \qquad (3.8)$$

holds for every $n\in\mathbb{N}$, where $y_0=0$, $y_n=\sum_{i=0}^{n-1}\frac{1}{i+1}$, for $n\ge 1$, $||x^*-\tilde{x}_n||$ denotes $\max_{t_k\in[a,b]}\left\{\left|x^*(t_k)-\tilde{x}_n(t_k)\right|,\left|x^*(g(t_k))-\tilde{x}_n(g(t_k))\right|\right\}$ and M_0 is described in (3.7).

Proof. Since

$$\begin{aligned}\left|x^*(t_k) - \tilde{x}_n(t_k)\right| &\le \left|x^*(t_k) - x_n(t_k)\right| + \left|x_n(t_k) - \tilde{x}_n(t_k)\right|,\\ \left|x^*(g(t_k)) - \tilde{x}_n(g(t_k))\right| &\le \left|x^*(g(t_k)) - x_n(g(t_k))\right| + \left|x_n(g(t_k)) - \tilde{x}_n(g(t_k))\right|,\end{aligned}$$

the assertion follows from (3.7) and Theorem 2.1. □

4. Numerical examples

Example 4.1. Consider the nonlinear Fredholm integral equation

$$\begin{aligned} x(t) &= \frac{3}{64}\int_0^{\pi} x(s)\left(\frac{1}{2}\cos t\cos\frac{s}{2} + x\left(\frac{s}{2}\right)\sin t\right) ds \\ &+ \frac{1}{64}(31\sin t - \cos t), \end{aligned} \tag{4.1}$$

for $t \in [0,\pi]$.

The exact solution of (4.1) is $x^*(t) = \dfrac{1}{2}\sin t$.
Here,

$$\begin{aligned} K(t,s,u,v) &= \frac{3}{64}u\left(\frac{1}{2}\cos t\cos\frac{s}{2} + v\sin t\right),\\ g(t) &= \frac{t}{2},\\ f(t) &= \frac{1}{64}(31\sin t - \cos t). \end{aligned}$$

Let $R = 1$. We have $R_1 = -\dfrac{1}{64}$ and $R_2 = \dfrac{\sqrt{962}}{64}$.
Then, on $[a,b]\times[a,b]\times[R_1 - R, R_2 + R]^2 = [0,\pi]\times[0,\pi]\times[-65/64, 1+\sqrt{962}/64]^2$, we have

$$M_K \le \frac{3}{64}(R_2+R)\left(\frac{1}{2} + R_2 + R\right)$$

and, so,

$$M_K(b-a) \le 0.434 < 1 = R.$$

Also, on $[0,\pi]\times[0,\pi]\times[-65/64, 1+\sqrt{962}/64]^2$,

$$\frac{\partial K}{\partial u} = \frac{3}{64}\left(\frac{1}{2}\cos t\cos\frac{s}{2} + v\sin t\right),$$

so $l_1 \le \dfrac{3}{64}\left(\dfrac{1}{2} + R_2 + R\right)$ and

$$\frac{\partial K}{\partial v} = \frac{3}{64}u\sin t,$$

so $l_2 \le \dfrac{3}{64}(R_2 + R)$. Hence,

$$q = (b-a)(l_1 + l_2) \approx 0.551 < 1.$$

Thus, conditions (2.2), (2.3) and (2.4) hold, which means the hypotheses of Theorem 3.1 are satisfied. Also, for $R = 1$, we have that $x^* \in B_R$.

We consider the trapezoidal rule with $m = 12$, $m = 16$ and $m = 24$, with the corresponding nodes $t_k = \frac{\pi}{m}k, k = \overline{0,m}$. Table 1 contains the errors

$$||x^* - \tilde{x}_n|| = \max_{t_k \in [a,b]} \{ |x^*(t_k) - \tilde{x}_n(t_k)|, |x^*(g(t_k)) - \tilde{x}_n(g(t_k))| \},$$

with initial approximation $x_0(t) = f(t) = \frac{1}{64}(31 \sin t - \cos t)$.

Table 1. Error estimates $||x^* - \tilde{x}_n||$ for Example 4.1

$n \setminus m$	12	16	24
1	$1.942720e - 00$	$1.354476e - 00$	$4.983236e - 01$
2	$8.223781e - 01$	$4.405026e - 01$	$6.338715e - 02$
3	$3.015578e - 01$	$9.174332e - 02$	$7.990126e - 03$
4	$7.997435e - 02$	$1.989751e - 02$	$8.986247e - 04$
5	$1.963239e - 02$	$7.428768e - 03$	$1.422981e - 04$
10	$9.795423e - 04$	$8.012446e - 05$	$3.116458e - 06$

Example 4.2. Next, consider the nonlinear two-point boundary-value problem

$$x''(t) - e^{x(t)} = 0, \quad t \in [0,1]; \quad x(0) = x(1) = 0, \tag{4.2}$$

which is used in magnetohydrodynamics (see [3]). The unique solution of (4.2) is given by

$$x^*(t) \quad = \quad -\ln(2) + 2\ln\left(\frac{c}{\cos\left(c\left(t - 1/2\right)/2\right)}\right),$$

where c is the only solution of $c/\cos(c/4) = \sqrt{2}$.

Problem (4.2) can be reformulated as the Fredholm integral equation

$$x(t) \quad = \quad \int_0^1 k(t,s)e^{x(s)}\, ds, \quad t \in [0,1], \tag{4.3}$$

where the kernel

$$k(t,s) \quad = \quad -\min\{t,s\}(1 - \max\{t,s\}) \quad = \quad \begin{cases} -s(1-t), & s \le t, \\ -t(1-s), & s > t \end{cases} \tag{4.4}$$

is Green's function for the homogeneous problem

$$x''(t) = 0, \quad t \in [0,1]; \quad x(0) = x(1) = 0.$$

We have

$$\begin{aligned} K(t,s,u,v) &= k(t,s)e^u, \\ g(t) &= f(t) \equiv 0. \end{aligned}$$

Again, we take $R = 1$. In this case, $R_1 = R_2 = 0$ and $\max|K| = \max\left|\frac{\partial K}{\partial u}\right| = \frac{1}{4} \cdot e$, for $(t, s, u, v) \in [0,1]^2 \times [-1,1]^2$. Thus,

$$\begin{aligned} q &= (b-a)(l_1+l_2) = l_1 = \frac{1}{4} \cdot e < 1, \\ M_K(b-a) &= M_K = \frac{1}{4} \cdot e < 1 = R, \end{aligned}$$

so the hypotheses of Theorem 3.1 are satisfied.

As before, we use the trapezoidal rule with $m = 12$, $m = 16$ and $m = 24$ and nodes $t_k = \frac{1}{m}k, k = \overline{0,m}$. The errors $||x^* - \tilde{x}_n|| = \max\limits_{k=\overline{0,m}} |\tilde{x}_n(t_k) - x^*(t_k)|$ are given in Table 2, with initial approximation $x_0 \equiv 0$.

Table 2. Error estimates $||x^* - \tilde{x}_n||$ for Example 4.2

$n \backslash m$	12	16	24
1	$1.080564e-02$	$1.080564e-02$	$1.080564e-02$
2	$1.094821e-03$	$1.066866e-03$	$1.023419e-03$
3	$4.890231e-04$	$4.178235e-04$	$6.098823e-05$
4	$5.712236e-05$	$5.014429e-05$	$2.082737e-05$
5	$2.034852e-05$	$9.640748e-06$	$6.161384e-06$
10	$2.026459e-07$	$1.678721e-07$	$8.890239e-08$

5. Conclusions and future work

We have developed a numerical iterative method for approximating solutions of Fredholm integral equations of the second kind, with deviating arguments, using a combination of successive approximations (Mann iteration) for fixed points of integral operators and quadrature formulas (the trapezoidal rule). Compared to other recent numerical methods for solving these integral equations – such as collocation, Galerkin, Nyström or other projection methods, wavelets-based approximations methods, Adomian decomposition, etc – the present method has two major advantages, the relative simplicity in proving the convergence of the approximate solutions to the exact solution (using fixed point theory) and the low cost of implementation (as it uses a well known quadrature formula, which is already implemented in most mathematical software). Yet, as the examples show, it gives a good approximation even with a relatively small number of iterations and of quadrature nodes. In the examples chosen, the numerical results are quite good and the errors decrease rapidly as n (the number of iterations) and/or m (the number of quadrature nodes) increase.

As for future work, similar ideas to the ones described in this paper can be applied to other types of integral equations, integral equations with more complicated kernels, or kernels satisfying other conditions than the ones considered in this work. Also, other fixed point successive approximations can be considered, which, under certain conditions, may converge faster. Last, but not least, more accurate numerical integration schemes can be employed in order to increase the speed of convergence of the method.

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