Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 239–250

# Weingarten tube-like surfaces in Euclidean 3-space

Adel H. Sorour

**Abstract.** In this paper, we study a special kind of tube surfaces, so-called tube-like surface in 3-dimensional Euclidean space $\mathbf{E}^3$. It is generated by sweeping a space curve along another central space curve. This study investigates a tube-like surface satisfying some equations in terms of the Gaussian curvature, the mean curvature, the second Gaussian curvature and the second mean curvature. Furthermore, some important theorems are obtained. Finally, an example of tube-like surface is used to demonstrate our theoretical results and graphed.

**Mathematics Subject Classification (2010):** 53A04, 56B34, 53B25.

**Keywords:** Tube-like surface, Gaussian curvature, mean curvature, second Gaussian curvature, second mean curvature, Weingarten surfaces, linear Weingarten surfaces, Euclidean 3-space.

## 1. Introduction

Weingarten surfaces are surfaces whose Gaussian and mean curvatures satisfy a functional relationship (of class $C^0$ at least). The class of Weingarten surfaces contains already mentioned surfaces of constant curvatures $K$ or $H$. Furthermore, a $C^r$-surface, $r > 3$, is Weingarten if and only if $K_s H_t - K_t H_s = 0$. On the other hand, let $A$ and $B$ be smooth functions on a surface $M(s,t)$ in Euclidean 3-space $\mathbf{E}^3$. The Jacobi function $\Phi(A,B)$ formed with $A$ and $B$ is defined by:

$$\Phi(A,B) = \det \begin{pmatrix} A_s & A_t \\ B_s & B_t \end{pmatrix},$$

where $A_s = \dfrac{\partial A}{\partial s}$ and $A_t = \dfrac{\partial A}{\partial t}$.

For the pair $(A,B)$ of curvatures $K$, $H$ and $K_{II}$ of $M$ in $\mathbf{E}^3$, if $M$ satisfies $\Phi(A,B) = 0$ and $aA + bB = c$, then we call $(A,B)$-Weingarten surface (*W-surface*) and $(A,B)$-linear Weingarten surface (*LW-surface*), respectively, where $a, b, c \in \mathbb{R}$, $(a,b,c) \neq (0,0,0)$.

The classification of the Weingarten surfaces in Euclidean space is almost completely open today. These surfaces were introduced by J. Weingarten [21, 22] in the context of the problem of finding all surfaces isometric to a given surface of revolution. Applications of Weingarten surfaces on computer aided design and shape investigation can seen in [19].

The authors in [9, 25] have investigated ruled Weingarten surfaces and ruled linear Weingarten surfaces in $\mathbf{E}^3$. Besides, a classification of ruled Weingarten surfaces and ruled linear Weingarten surfaces in a Minkowski 3-space $\mathbf{E}_1^3$ is given in [4, 7, 16]. Munteanu and Nistor [13] studied polynomial translation linear Weingarten surfaces in Euclidean 3-space. Also, Lopez [10, 11] studied cyclic linear Weingarten surface in Euclidean 3-space. In [12] Lopez classified all parabolic linear Weingarten surfaces in hyperbolic 3-space. Ro and Yoon [15] studied a tube of Weingarten types in Euclidean 3-space satisfying some equation in terms of the Gaussian curvature, mean curvature and second Gaussian curvature. Kim and Yoon [8] classified quadric surfaces in Euclidean 3-space in terms of the Gaussian curvature and the mean curvature. In addition to, Yoon and Jun [26] classified non-degenerate quadric surfaces in Euclidean 3-space in terms of the isometric immersion and the Gauss map. Furthermore in [1, 2], Weingarten timelike tube surfaces around spacelike and timelike curves were studied in Minkowski 3-space $\mathbf{E}_1^3$.

Several geometers [15, 1, 18] have studied tubes in Euclidean 3-space and Minkowski 3-space satisfying some equations in terms of the Gaussian curvature $K$, the mean curvature $H$ and the second Gaussian curvature $K_{II}$. Following the Jacobi function and the linear equation with respect to the Gaussian curvature $K$, the mean curvature $H$ and the second Gaussian curvature $K_{II}$ an interesting geometric question is raised: Classify all surfaces in Euclidean 3-space satisfying the conditions

$$\Phi(A, B) = 0, \tag{1.1}$$

$$a\,A + b\,B = c, \tag{1.2}$$

where $A, B \in \{K, H, K_{II}\}$, $A \neq B$ and $(a, b, c) \neq (0, 0, 0)$.

In this paper, we investigate the tube-like surfaces in 3-dimensional Euclidean space satisfying the Jacobi condition and the linear equation with respect to their curvatures have been studied. Furthermore, we obtained some theorems.

## 2. Preliminaries

Let $\mathbf{E}^3$ be a Euclidean 3-space with the scalar product given by [5]

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbf{E}^3$. In particular, the norm of a vector $X \in \mathbf{E}^3$ is given by $\|X\| = \sqrt{\langle X, X\rangle}$. If $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ are arbitrary vectors in $\mathbf{E}^3$, the vector product of $X$ and $Y$ is given by

$$X \wedge Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1). \tag{2.1}$$

Let $M : \Phi = \Phi(s,t)$ be a surface in Euclidean 3-space. The unit normal vector field of $M$ can be defined by

$$N = \frac{\Phi_s \wedge \Phi_t}{\|\Phi_s \wedge \Phi_t\|}, \;\; \Phi_s = \frac{\partial \Phi}{\partial s}, \;\; \Phi_t = \frac{\partial \Phi}{\partial t}, \tag{2.2}$$

where $\wedge$ stands the vector product of $\mathbf{E}^3$. The first fundamental form $I$ of the surface $M$ is

$$I = Eds^2 + 2Fdsdt + Gdt^2, \tag{2.3}$$

with coefficients

$$E = \langle \Phi_s, \Phi_s \rangle, \quad F = \langle \Phi_s, \Phi_t \rangle, \quad G = \langle \Phi_t, \Phi_t \rangle. \tag{2.4}$$

The second fundamental form of the surface $M$ is given by

$$II = eds^2 + 2fdsdt + gdt^2. \tag{2.5}$$

From which the components of the second fundamental form $e, f$ and $g$ are expressed as

$$e = \langle \Phi_{ss}, N \rangle, \quad f = \langle \Phi_{st}, N \rangle, \quad g = \langle \Phi_{tt}, N \rangle. \tag{2.6}$$

Under this parametrization of the surface $M$, the Gaussian curvature $K$ and the mean curvature $H$ have the classical expressions, respectively [14]

$$K = \frac{eg - f^2}{EG - F^2}, \tag{2.7}$$

$$H = \frac{Eg + Ge - 2Ff}{2\,(EG - F^2)}. \tag{2.8}$$

From Brioschi's formula in a Euclidean 3-space, we are able to compute $K_{II}$ of a surface by replacing the components of the first fundamental form $E$, $F$ and $G$ by the components of the second fundamental form $e$, $f$ and $g$ respectively. Consequently, the second Gaussian curvature $K_{II}$ of a surface is defined by [3]

$$\mathbf{K}_{II} = \frac{1}{(eg - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}. \tag{2.9}$$

Having in mind the usual technique for computing the second mean curvature $H_{II}$ by using the normal variation of the area functional for the surfaces in $\mathbf{E}^3$ one gets [20]

$$H_{II} = H + \frac{1}{4}\Delta_{II} \ln(K),$$

where $H$ and $K$ denote the mean, respectively Gaussian curvatures of surface and $\Delta_{II}$ is the Laplacian for functions computed with respect to the second fundamental form $II$ as metric. The second mean curvature $H_{II}$ can be equivalently expressed as

$$H_{II} = H + \frac{1}{2\sqrt{\det(II)}} \sum_{i,j} \frac{\partial}{\partial u^i} \left[ \sqrt{\det(II)} h^{ij} \frac{\partial}{\partial u^j} (\ln \sqrt{K}) \right], \tag{2.10}$$

where $(h_{ij})$ denotes the associated matrix with its inverse $(h^{ij})$, the indices $i, j$ belong to $\{1, 2\}$ and the parameters $u^1, u^2$ are $s, t$ respectively.

Now, we can write the following important definition [23]:

**Definition 2.1.** *(1): A regular surface is flat (developable) if and only if its Gaussian curvature vanishes identically.*

*(2): A regular surface for which the mean curvature vanishes identically is called a minimal surface.*

*(3): A non-developable surface is called II-flat if the second Gaussian curvature vanishes identically.*

*(4): A non-developable surface is called II-minimal if the second mean curvature vanishes identically.*

**Remark 2.2.** [24] It is well known that: a minimal surface has a vanishing second Gaussian curvature but that a surface with the vanishing second Gaussian curvature need not to be minimal.

## 3. Tube-like surface in $\mathbf{E}^3$

The aim of this section, we will obtain the tube-like surface from the tube surface. Since the tube surfaces are special kinds of the canal surfaces in Euclidean 3-space. If we find the canal surface with taking variable radius $r(s)$ as constant, then the tube surface can be found, since the canal surface is a general case of the tube surface.

A canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(s)$ (center curve) of its center and a radius function $r(s)$. If the center curve $C(s)$ is a helix and the radius function $r(s)$ is a constant, then the surface is called helical canal surface. If the radius function $r(s)$ is a constant, this time the canal surface is called a tube [6]. Canal surface around the center curve $C(s)$ is parametrized as

$$K(s,t) = C(s) - r(s)r'(s)e_1(s) \mp r(s)\sqrt{1-r'^2(s)}\Big(\cos[t]e_2(s) + \sin[t]e_3(s)\Big), \; 0 \leqslant t \leqslant 2\pi,$$

where $s$ is arclength parameter and $e_1(s), e_2(s), e_3(s)$ Frenet vectors of $C(s)$. If the radius function $r(s) = r$ is a constant, then, the canal surface is called a tube (pipe) surface and it parametrized as

$$\mathrm{Tube}(s,t) = C(s) + r\Big(\cos[t]e_2(s) + \sin[t]e_3(s)\Big).$$

The aim of this work is to introduce a simple method for parametrization of tube-like surface in Euclidean 3-space. Given a space curve $\alpha(t) = \Big(x(t), y(t), z(t)\Big)$, at each point, there are three directions associated with it, the tangent, normal and binormal directions. The unit tangent vector is denoted by $e_1$, i.e., $e_1(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}$, the unit normal vector is denoted by $e_2$, i.e., $e_2(t) = \frac{e_1'(t)}{\|e_1'(t)\|}$, the unit binormal vector is denoted by $e_3$, i.e., $e_3(t) = e_1(t) \wedge e_2(t)$ (cross product). With $\alpha(t), e_1(t), e_2(t)$ and $e_3(t)$, a tube-like surface can be expressed as follows

$$M : \Phi(s,t) = \alpha(t) + r\Big(\cos[s]e_2(t) - \sin[s]e_3(t)\Big), \tag{3.1}$$

where $r$ is a parameter corresponding to the radius of the rotation (In general $r$ can be a function of $t$). For fixed $t$, when $s$ runs from 0 to $2\pi$, we have a circle around the point $\alpha(t)$ in the $e_1, e_2$ plane. As we change $t$, this circle moves along the space curve $\alpha$, and we will generate a tube-like surface along $\alpha$ (a special kind of tube surfaces defined by (3.1)). The Frenet-Serret equations, express the reat of change of the moving orthonormal tried $\{e_1(t), e_2(t), e_3(t)\}$ along the curve $\alpha$ are given by [17]

$$\begin{bmatrix} e_1'(t) \\ e_2'(t) \\ e_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}, \tag{3.2}$$

where the prime denotes the differentiation with respect to $t$ and we denote by $\kappa$, $\tau$ the curvature and the torsion of the curve $\alpha$. We can know that $e_1, e_2, e_3$ are mutually orthogonal vector fields satisfying equations

$$\begin{aligned} &\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1, \\ &\langle e_1, e_2\rangle = \langle e_2, e_3\rangle = \langle e_3, e_1\rangle = 0, \\ &\det(e_1, e_2, e_3) = 1. \end{aligned}$$

Calculating the partial derivative of (3.1) with respect to $s$ and $t$ respectively, we get

$$\begin{aligned} \Phi_s &= -r\Big[\sin[s]e_2 + \cos[s]e_3\Big], \\ \Phi_t &= Qe_1 + r\tau\Big[\sin[s]e_2 + \cos[s]e_3\Big], \end{aligned} \tag{3.3}$$

where $Q = 1 - r\kappa\cos[s]$. From which, the components of the first fundamental form are

$$E = r^2, \quad F = -r^2\tau, \quad G = Q^2 + r^2\tau^2. \tag{3.4}$$

Using equations (2.1) and (2.2) the unit normal vector on $\Phi$ takes the form

$$N = -\cos[s]e_2 + \sin[s]e_3. \tag{3.5}$$

The second order partial differentials of $M$ are found

$$\begin{aligned} \Phi_{ss} &= -r\Big[\cos[s]e_2 - \sin[s]e_3\Big], \\ \Phi_{st} &= r\Big[\kappa\sin[s]e_1 + \tau(\cos[s]e_2 - \sin[s]e_3)\Big], \\ \Phi_{tt} &= -r(\kappa\tau\sin[s] + \kappa'\cos[s])e_1 + (\kappa - r(\kappa^2 + \tau^2)\cos[s] \\ &\quad + r\tau'\sin[s])e_2 + r(\tau^2\sin[s] + \tau'\cos[s])e_3. \end{aligned}$$

From the equation (3.5) and the last equations, we find the second fundamental form coefficients as follows

$$e = r, \quad f = -r\tau, \quad g = -Q\kappa\cos[s] + r\tau^2, \tag{3.6}$$

**Theorem 3.1.** *$M$ is a regular tube-like surface if and only if* $1 - r\kappa\cos[s] \neq 0$.

*Proof.* For a regular surface, $EG - F^2 \neq 0$. From (3.6), we get

$$EG - F^2 = r^2\Big(1 - r\kappa\cos[s]\Big)^2,$$

where $EG - F^2 \neq 0$ and $r > 0$, $M$ is a regular tube-like surface if and only if

$$1 - r\kappa\cos[s] \neq 0.$$

Based on the above calculations, the Gaussian curvature $K$ and the mean curvature $H$ of (3.1) are given by

$$K = -\frac{\kappa\cos[s]}{rQ}, \tag{3.7}$$

$$H = \frac{1-2r\kappa\cos[s]}{2rQ}. \tag{3.8}$$

If the second fundamental form of $\Phi$ is non-degenerate, i.e., $eg - f^2 \neq 0$. In this case, we can define formally the second Gaussian $K_{II}$ and second mean $H_{II}$ curvatures on $\Phi(s,t)$ as follows

$$K_{II} = \frac{1}{4rQ^4\cos^2[s]}\Big[1+\cos^2[s]-6r\kappa\cos^3[s]+4r^2\kappa^2\cos^4[s]\Big], \tag{3.9}$$

$$H_{II} = \frac{-1}{64rQ^3\kappa^3\cos^2[s]}\Big[A_0+\sum_{i=1}^{6}A_i\cos[is]+\sum_{j=1}^{3}B_j\sin[js]\Big],$$

where the coefficients $A_i$ and $B_j$ are

$$\begin{aligned}
A_0 &= -r\Big[\kappa^2[33\kappa^2+20\kappa^2(r^2\kappa^2-\tau^2)]-4(3\kappa'^2-2\kappa\kappa'')\Big],\\
A_1 &= 2\kappa\Big[\kappa^2[5-4r^2(3\tau^2-11\kappa^2)]-6r^2(3\kappa'^2-\kappa\kappa'')\Big],\\
A_2 &= -2r\Big[\kappa^2[3\kappa^2(8+5r^2\kappa^2)+2\tau^2]-2(3\kappa'^2-2\kappa\kappa'')\Big],\\
A_3 &= 2\kappa\Big[\kappa^2[3+r^2(23\kappa^2+4\tau^2)]-2r^2(3\kappa'^2-\kappa\kappa'')\Big],\\
A_4 &= -3r\kappa^4\Big[5+4r^2\kappa^2\Big],\quad A_5=10r^2\kappa^5,\quad A_6=-2r^3\kappa^6
\end{aligned}$$

and

$$B_1 = 4r^2\kappa^2\Big[4\kappa'\tau-\kappa\tau'\Big],\quad B_2=-8r\kappa\Big[\kappa'\tau-\kappa\tau'\Big],\quad B_3=4r^2\kappa^2\Big[4\kappa'\tau-\kappa\tau'\Big].$$

Under the previous calculations, one can formulate the following theorems:

**Theorem 3.2.** *If the Gaussian curvature $K$ is zero, then $M$ is generated by a moving sphere with the radius $r = 1$.*

*Proof.* At $\kappa = 0$, from the equation (3.7) $\cos[s] = 0$, i.e., $s = \frac{\pi}{2}(2n+1)$, $n = 0, \pm1, \pm2, \pm3, ...$, and the unit normal vector on $M$ takes the form

$$\begin{aligned}
N(s,t) &= -\cos[s]e_2(t)+\sin[s]e_3(t)\\
&= \pm e_3(t).
\end{aligned}$$

Again, when $\cos[s] = 0$, i.e., $s = \frac{\pi}{2}(2n+1)$, $n = 0, \pm1, \pm2, \pm3, ...$, implies that

$$\begin{aligned}
\Phi(s,t)-\alpha(t) &= r\Big(\cos[s]e_2(t)-\sin[s]e_3(t)\Big)\\
N(s,t) &= -\Big(\cos[s]e_2(t)-\sin[s]e_3(t)\Big)\\
\pm e_3(t) &= \mp re_3(t).
\end{aligned}$$

From the last equation, we get $r = 1$.

**Theorem 3.3.** *The surface* (3.1) *is a developable surface if and only if it is an open part of a circular-like cylinder.*

**Theorem 3.4.** *There are no minimal tube-like surfaces in Euclidean 3-space* $\mathbf{E}^3$.

**Theorem 3.5.** *Let $M$ be a tube-like surface with non-degenerate second fundamental form in the Euclidean 3-space $\mathbf{E}^3$, then $M$ is not II-flat as well as not II-minimal.*

### 3.1. Weingarten tube-like surfaces

In the following, we study the tube-like surface $\Phi$ in $\mathbf{E}^3$ satisfying the Jacobi equation $\Phi(X,Y)=0$, $X\neq Y$, of the curvatures $K$, $H$ and $K_{II}$ of $\Phi$ and we formulate the main results in the next theorems.

**Theorem 3.6.** *Let $M$ be a tube-like surface in $\mathbf{E}^3$ defined by (3.1). Then $M$ is a $(K,H)$-Weingarten surface.*

*Proof.* Let $M$ be a tube-like surface in $\mathbf{E}^3$. Differentiating $K$ and $H$ with respect to $s$ and $t$ respectively, then we obtain

$$K_s=\frac{\kappa\sin[s]}{rQ^2},\quad K_t=-\frac{\kappa'\cos[s]}{rQ^2}, \tag{3.10}$$

$$H_s=\frac{\kappa\sin[s]}{rQ^2},\quad H_t=-\frac{\kappa'\cos[s]}{rQ^2}. \tag{3.11}$$

By using (3.10) and (3.11), $M$ satisfies identically the Jacobi equation

$$\Phi(K,H)=K_sH_t-K_tH_s=0.$$

Therefore $M$ is a *Weingarten surface.*

**Theorem 3.7.** *Let $M$ be a tube-like surface in the Euclidean 3-space $\mathbf{E}^3$ parametrized by (3.1) with non-degenerate second fundamental form. If $M$ is a $(K,K_{II})$-Weingarten surface, then $\kappa'=0$. Then, the curvature of $\alpha(t)$ is a non-zero constant.*

*Proof.* Let $M$ be a tube-like surface in $\mathbf{E}^3$ parametrized by (3.1). If we take derivative of $K_{II}$ given by (3.9) with respect to $s$ and $t$ respectively, then we have

$$\begin{cases}(K_{II})_s=\frac{1}{2rQ^3\cos^3[s]}\Big[1-r\kappa(2\sin^2[s]+r\kappa\cos^3[s])\cos[s]\Big]\sin[s],\\[2ex](K_{II})_t=\frac{\kappa'}{2Q^3\cos[s]}\Big[1-2\cos^2[s]+r\kappa\cos^3[s]\Big].\end{cases} \tag{3.12}$$

We consider a tube-like surface (3.1) in $\mathbf{E}^3$ satisfying the Jacobi equation

$$\Phi(K,K_{II})=K_s(K_{II})_t-K_t(K_{II})_s=0, \tag{3.13}$$

with respect to the Gaussian curvature $K$ and the second Gaussian curvature $K_{II}$. Then, substituting from (3.10) and (3.12) into (3.13), we get

$$\kappa'\sin[s]=0.$$

Since this polynomial is equal to zero for every $s$, its coefficient must be zero. Therefore, we conclude that $\kappa'=0$.

**Theorem 3.8.** *Let $M$ be a tube-like surface in the Euclidean 3-space $\mathbf{E}^3$ parametrized by (3.1) with non-degenerate second fundamental form. If $M$ is a $(H,K_{II})$-Weingarten surface, then $\kappa'=0$. Then, the curvature of $\alpha(t)$ is a non-zero constant.*

*Proof.* We assume that a tube-like surface parametrized by (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$ is $(H, K_{II})$-Weingarten surface. Then, it satisfies the Jacobi equation

$$\Phi(H, K_{II}) = H_s(K_{II})_t - H_t(K_{II})_s = 0, \tag{3.14}$$

which implies

$$\kappa' \sin[s] = 0. \tag{3.15}$$

From (3.15),*one can get* $\kappa' = 0$. Thus, the curvature of $\alpha(t)$ is a non-zero constant.

## 4. Linear Weingarten tube-like surfaces

Now, to examine the linear Weingarten property of the tube-like surface $\Phi$ defined along the space curve $\alpha(t)$. Let us analyze the following theorems.

**Theorem 4.1.** *Suppose that a tube-like surface defined by (*3.1*) in* $\mathbf{E}^3$ *is a linear Weingarten surface satisfying* $aK+bH = c$*. Then* $\kappa = 0$*.* $M$ *is an open part of a circular-like cylinder.*

*Proof.* Consider the parametrization (3.1) with $K$ and $H$ given by (3.7) and (3.8) respectively, we have

$$aK + bH = c,$$

implies

$$2\kappa\Big[a + br - cr^2\Big]\cos[s] - b + 2cr = 0. \tag{4.1}$$

Since $\cos[s]$ and 1 are linearly independent, we have

$$2\kappa\Big[a + br - cr^2\Big] = 0, \qquad b = 2cr,$$

which imply

$$\kappa(a + cr^2) = 0.$$

If $a + cr^2 \neq 0$, then $\kappa = 0$. Thus, $M$ is an open part of a circular-like cylinder.

**Theorem 4.2.** *Let* $(A, B) \in \{(K, K_{II}), (H, K_{II})\}$*. Then, there are no* $(A, B)$*-linear Weingarten tube-like surfaces in Euclidean 3-space* $\mathbf{E}^3$*.*

*Proof.* Firstly, we suppose that a tube-like surface (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$ satisfies the equation

$$aK + bK_{II} = c. \tag{4.2}$$

By using (3.7) and (3.9), the equation (4.2) takes the form

$$\begin{aligned}\tfrac{1}{4rQ^2}\Big[&4r\kappa^2(a + br - cr^2)\cos^4[s] - 2\kappa(2a + 3br - 4cr^2)\cos^3[s]\\ &+(b - 4cr)\cos^2[s] + b\Big] = 0.\end{aligned} \tag{4.3}$$

Since the identity holds for every $s$, all the coefficients must be zero. Therefore, we obtain

$$\begin{cases} 4r\kappa^2(a+br-cr^2)=0, \\ 2\kappa(2a+3br-4cr^2)=0, \\ b-4cr=0, \\ b=0. \end{cases}$$

Thus, we get $b=0,\ c=0$ and $\kappa=0$. In this case, the second fundamental form of $M$ is degenerate. Thus, this completes proof.

Secondly, let a tube-like surface (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$ satisfy the relation

$$aH+bK_{II}=c. \tag{4.4}$$

From equations. (3.8), (3.9) and (4.4), we get

$$\frac{1}{4rQ^2}\Big[4r^2\kappa^2(a+b-cr)\cos^4[s]-2r\kappa(3a+3br-4cr)\cos^3[s]$$
$$+(2a+b-4cr)\cos^2[s]+b\Big]=0.$$

From which, one can obtain $b=0,\ c=0$ and $\kappa=0$. Also, the second fundamental form of tube-like is degenerate. Then, there are no $(H,K_{II})$-linear Weingarten tube-like surfaces in $\mathbf{E}^3$.

## 5. Applications

Here, we consider an example to illustrate the main results that we have presented in our paper.

**Example 5.1.** Let us consider a surface

$$\Phi(s,t)=\alpha(t)+r\Big(\cos[s]e_2(t)-\sin[s]e_3(t)\Big), \tag{5.1}$$

where $\alpha(t)$ is

$$\alpha(t)=(\cos[t],\sin[t],0),$$

and the Frenet's frame is

$$e_1(t)=(-\sin[t],\cos[t],0),\ \ e_2(t)=-(\cos[t],\sin[t],0),\ \ e_3(t)=(0,0,1).$$

Thus, we obtained tube-like surface as follows

$$\Phi(s,t)=\Big((1-r\cos[s])\cos[t],(1-r\cos[s])\sin[t],-r\sin[s]\Big). \tag{5.2}$$

The components of the first and second fundamental forms of the surface (5.2) are given by, respectively

$$\begin{aligned} &E=r^2, \quad F=0, \quad G=(1-r\cos[s])^2, \\ &e=r, \quad f=0, \quad g=-(1-r\cos[s])\cos[s]. \end{aligned}$$

The unit normal vector of the surface (5.2) takes the form

$$N=-\cos[s]e_2(t)+\sin[s]e_3(t). \tag{5.3}$$

For this surface, the Gaussian curvature $K$ and the mean curvature $H$ are defined by, respectively

$$K = -\frac{\cos[s]}{r(1 - r\cos[s])}, \tag{5.4}$$

$$H = \frac{1 - 2r\cos[s]}{2r(1 - r\cos[s])}. \tag{5.5}$$

As $cos[s] = 0$, Eqs. (5.4) and (5.5) lead to

$$K = 0, \quad H = \frac{1}{2r},$$

i.e., the surface (5.2) is a developable and not minimal.

Since $eg - f^2 \neq 0$, then we can get the second Gaussian curvature $K_{II}$ and the second mean curvature $H_{II}$ on $\Phi(s, t)$ as follows

$$K_{II} = \frac{1 + \cos^2[s] - 6r\cos^3[s] + 4r^2\cos^4[s]}{4r(1 - r\cos[s])^2\cos^2[s]}, \tag{5.6}$$

$$H_{II} = \frac{-1 + 2r\cos[s] + 3\cos^2[s] - 12r\cos^3[s] + 8r^2\cos^4[s]}{8r(1 - r\cos[s])^2\cos^2[s]}. \tag{5.7}$$

From aforementioned data, one can deduce that the Weingarten and linear Weingarten on $\Phi$ corresponding to the induced metric form satisfies the above theorems.

One can see the graph of $\Phi(s, t)$ in Figure 1.

Under the previous, we consider the following remark:

**Remark 5.1. (1):** It easily seen that, the vector $e_3(t) = (0, 0, 1)$ is a constant vector, then the surface (5.1) is a circular-like cylinder surface.

**(2)**: The tube-like surface defined by (5.2) is a torus.

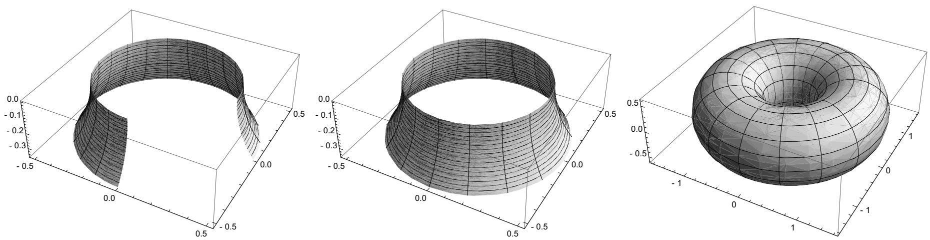


Figure 1. Some tube-like surfaces generated by circle with $r = \frac{1}{2}$,
Left (half circular-like cylinder): $s, t \in [0, \frac{6}{5}\pi]$,
Middle (circular-like cylinder): $s, t \in [0, \frac{3}{2}\pi]$ and
Right (torus): $s, t \in [0, 2\pi]$.

**Acknowledgements.** The author would like to thank the referees for the helpful suggestions.

## References


[1] Abdel-Aziz, H.S., Khalifa Saad, M., *Weingarten timelike tube surfaces around a spacelike curve,* Int. J. of Math. Analysis, **25(5)**(2011), 1225-1236.

[2] Abdel-Aziz, H.S., Khalifa Saad, M., Kiziltug, S., *Parallel Surfaces of Weingarten Type in Minkowski 3-Space,* Int. Math. Forum, **4(7)**(2012), 2293-2302.

[3] Baikoussis, C., Koufogiorgos, T., *On the inner curvature of the second fundamental form of helicoidal surfaces*, Arch. Math., **68(2)**(1997), 169–176.

[4] Dillen, F., Kühnel, W., *Ruled Weingarten surfaces in a Minkowski 3-space*, Manuscripta Math. **98**(1999), 307–320.

[5] do Carmo, M., *Differential geometry of curves and surfaces,* Prentice-Hall, Englewood Cliffs, 1976.

[6] Doğan, F., *A note on tubes,* Int. J. of Phys. and Math. Sciences, **3(1)**(2013), 98-105.

[7] Kimm, H.Y., Yoon, D.W., *Classification of ruled surfaces in a Minkowski 3-space*, J. Geom. Phys., **49**(2004), 89–100.

[8] Kim, Y.H., Yoon, D.W., *Weingarten quadric surfaces in a Euclidean 3-space*, Turk. J. Math., **35**(2011), 479–485.

[9] Kühnel, W., *Ruled W-surfaces*, Arch. Math. **62**(1994), 475–480.

[10] Lopez, R., *On linear Weingarten surfaces*, International J. Math., **19**(2008), 439–448.

[11] Lopez, R., *Special Weingarten surfaces foliated by circles*, Monatsh. Math. **154**(2008), 289–302.

[12] Lopez, R., *Parabolic Weingarten surfaces in hyperbolic space*, Publ. Math. Debrecen, **74**(2009), 59–80.

[13] Munteanu, M.I., Nistor, A.I., *Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space*, Differential geometry, 316-320, Worled Sci. Publ., Hackensack, NJ, 2009.

[14] O'Neill, B., *Elementary Differential Geometry*, Academic press, New York, 1966.

[15] Ro, J., Yoon, D.W., *Tubes of Weingarten types in a Euclidean 3-space*, J. Chungcheong Math. Soc., **22(3)**(2009), 359–366.

[16] Sodsiri, W., *Ruled surfaces of Weingarten type in Minkowski 3-space*, PhD. thesis, Katholieke Universiteit Leuven, Belgium, 2005.

[17] Struik, D.J., *Lectures on classical differential geometry*, Addison-Wesley Publishing Company, Inc., 1961.

[18] Tuncer, Y., Yoon, D.W., Karacan, M.K., *Weingarten and linear Weingarten type tubular surfaces in* $\mathbf{E}^3$, Mathematical Problems in Engineering, Article ID 191849, 2011.

[19] Van-Brunt, Grant, K., *Potential applications of Weingarten surfaces in CAGD, Part I: Weingarten surfaces and surface shape investigation*, Comput. Aided Geom. Des., **13**(1996), 569–582.

[20] Verpoort, S., *The Geometry of the second fundamental form: Curvature properties and variational aspects*, PhD. Thesis, Katholieke Universiteit Leuven, Belgium, 2008.

[21] Weingarten, J.,*Ueber eine Klasse auf einander abwickelbarer Flaachen*, J. Reine Angew. Math., **59**(1861), 382–393.

[22] Weingarten, J.,*Ueber die Flachen derer Normalen eine gegebene Flache beruhren*, J. Reine Angew. Math., **62**(1863), 61–63.

[23] Yilmaz, T., Murat, K.K., *Differential Geometry of three dimensions*, Syndic of Cambridge University Press, 1981.

[24] Yoon, D.W., *On non-developable ruled surfaces in Euclidean 3-spaces*, Indian J. Pure Appl. Math., **38**(2007), 281-289.

[25] Yoon, D.W., *Some properties of the helicoid as ruled surfaces*, JP J. Geom. Topology, **2(2)**(2002), 141–147.

[26] Yoon, D.W., Jun, J.S., *Non-degenerate quadric surfaces in Euclidean 3-space*, Int. J. Math. Anal., **6(52)**(2012), 2555–2562.

Adel H. Sorour
Department of Mathematics, Faculty of Science
Sohag University, 82524, Sohag, Egypt
e-mail: `adel7374@yahoo.com`