

Local C -semigroups and complete second order abstract Cauchy problems

Chung-Cheng Kuo

Abstract. Let $C : X \rightarrow X$ be an injective bounded linear operator on a Banach space X over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ and $0 < T_0 \leq \infty$. Under some suitable assumptions, we deduce some relationship between the generation of a local (or an exponentially bounded) $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ -semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and one of the following cases: (i) the well-posedness of a complete second-order abstract Cauchy problem $\text{ACP}(A, B, f, x, y): w''(t) = Aw'(t) + Bw(t) + f(t)$ for a.e. $t \in (0, T_0)$ with $w(0) = x$ and $w'(0) = y$; (ii) a Miyadera-Feller-Phillips-Hille-Yosida type condition; (iii) B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C -cosine function on X for which A may not be bounded; (iv) A is a subgenerator (resp., the generator) of a local C -semigroup on X for which B may not be bounded.

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1. Introduction

Let X be a Banach space over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ with norm $\|\cdot\|$, and let $L(X)$ denote the family of all bounded linear operators from X into itself. For each $0 < T_0 \leq \infty$, we consider the following two abstract Cauchy problems:

$$\text{ACP}(A, f, x) \quad \begin{cases} u'(t) = Au(t) + f(t) & \text{for a.e. } t \in (0, T_0) \\ u(0) = x \end{cases}$$

and

$$\text{ACP}(A, B, f, x, y) \quad \begin{cases} w''(t) = Aw'(t) + Bw(t) + f(t) & \text{for a.e. } t \in (0, T_0) \\ w(0) = x, w'(0) = y, \end{cases}$$

where $x, y \in X$, $A : D(A) \subset X \rightarrow X$ and $B : D(B) \subset X \rightarrow X$ are closed linear operators, and $f \in L^1_{loc}([0, T_0], X)$ (the family of all locally Bochner integrable functions from $[0, T_0]$ into X). A function u is called a (strong) solution of $ACP(A, f, x)$ if $u \in C([0, T_0], X)$ satisfies $ACP(A, f, x)$ (that is $u(0) = x$ and for a.e. $t \in (0, T_0)$, $u(t)$ is differentiable and $u(t) \in D(A)$, and $u'(t) = Au(t) + f(t)$ for a.e. $t \in (0, T_0)$). For each $\alpha > 0$ and each injection $C \in L(X)$, a subfamily $S(\cdot) = \{S(t) | 0 \leq t < T_0\}$ of $L(X)$ is called a local α -times integrated C -semigroup on X if it is strongly continuous, $S(\cdot)C = CS(\cdot)$ and satisfies

$$(1.1) \quad S(t)S(s)x = \frac{1}{\Gamma(\alpha)} \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] (t+s-r)^{\alpha-1} S(r)Cx dr$$

for all $x \in X$ and $0 \leq t, s \leq t+s < T_0$ (see [1-2,12-14,18-21,28,30,32,35]) or called a local (0-times integrated) C -semigroup on X if it is strongly continuous, $S(\cdot)C = CS(\cdot)$ and satisfies

$$(1.2) \quad S(t)S(s)x = S(t+s)Cx$$

for all $x \in X$ and $0 \leq t, s \leq t+s < T_0$ (see [4,6,13,23,29]), where $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $S(\cdot)$ is

- (1.3) locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that $\|S(t+h) - S(t)\| \leq K_{t_0}h$ for all $0 \leq t, h, t+h \leq t_0$;
- (1.4) exponentially bounded, if $T_0 = \infty$ and there exist $K, \omega \geq 0$ such that $\|S(t)\| \leq Ke^{\omega t}$ for all $t \geq 0$;
- (1.5) nondegenerate, if $x = 0$ whenever $S(t)x = 0$ for all $0 \leq t < T_0$.

A nondegenerate local α -times integrated C -semigroup $S(\cdot)$ on X implies that $S(0) = C$ if $\alpha = 0$, and $S(0) = 0$ (zero operator on X) otherwise, and the (integral) generator $A : D(A) \subset X \rightarrow X$ of $S(\cdot)$ is a closed linear operator in X defined by $D(A) = \{x \in X | S(\cdot)x - j_\alpha(\cdot)Cx = \tilde{S}(\cdot)y_x \text{ on } [0, T_0] \text{ for some } y_x \in X\}$ and $Ax = y_x$ for all $x \in D(A)$ (see [6,13-14,23]), which is also equal to the linear operator A in X defined by $D(A) = \{x \in X | \lim_{h \rightarrow 0^+} (S(h)x - Cx)/h \in R(C)\}$ and $Ax = C^{-1} \lim_{h \rightarrow 0^+} (S(h)x - Cx)/h$ for $x \in D(A)$ when $\alpha = 0$ (see [4,23,27]). Here

$$j_\beta(t) = \frac{t^\beta}{\Gamma(\beta+1)} \text{ and } \tilde{S}(t)z = \int_0^t S(s)z ds.$$

In general, a local C -semigroup is called a C -semigroup if $T_0 = \infty$ (see [2,4,14,26,32]) or a C_0 -semigroup if $C = I$ (identity operator on X) (see [1,5]). It is known that the theory of local C -semigroup is related to another family in $L(X)$ which is called a local C -cosine function (see [2,4,8-9,24,28-29,32]). Perturbation of local (integrated) C -semigroups has been extensively studied by many authors appearing in [1,6-7,10-12,15-16,22,30-32]. Some interesting applications of this topic are also illustrated there. The well-posedness of $ACP(A, B, f, x, y)$ had been studied by many authors when $f = 0$ (see [3,6,9,17-18,20,25,32-34]). Some relationship between the well-posedness of $ACP(A, B, 0, x, y)$, a Miyadera-Feller-Phillips-Hille-Yosida type condition (see (1.6) below), and the generation of a C_0 -semigroup on $X \times X$ with generator $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ have been established in [25] when A and B are commutable on $D(B) \cap D(A)$, in [20] and [32]

for $A \in L(X)$, in [32] for $B \in L(X)$, and in [17] for the general case. In particular, Xiao and Liang [32, Theorems 2.6.1, 2.5.2 and 2.5.1] show that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ generates a C_0 -semigroup on $X \times X$ (if and) only if $B \in L(X)$ (and A generates a C_0 -semigroup on X), but the conclusion may not be true when C_0 -semigroups are replaced by local C -semigroups; and the well-posedness of $\text{ACP}(A, B, 0, x, y)$ is equivalent to A generates a C_0 -semigroup on X if $B \in L(X)$, and equivalent to B generates a cosine function on X if $A \in L(X)$. Unfortunately, a local C -semigroup may not be exponentially bounded and is not necessarily extendable to the half line $[0, \infty)$, and $\begin{pmatrix} 0 & 0 \\ B & A \end{pmatrix}$ may not be the generator of a local $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ -semigroup on $X \times X$ whenever $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is. Moreover, $\lambda \in \rho_C(A, B)$ may not imply that

$\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}$ and $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}$ are bounded even though $D(B) \cap D(A)$ is dense in X and $C = I$, and $\lambda \in \rho_C(\mathcal{T})$ implies that $\lambda \in \rho_C(A, B)$, $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}$ and $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}$ are bounded, but may not be bounded on X even though $C = I$. In particular, they are bounded on X when the assumption of $D(B) \cap D(A)$ is dense in X is added (see [17] for the case $C = I$). In this paper, we will extend the aforementioned results to the case of local C -semigroup by different methods (see Theorems 2.2 and 2.3 below). We show that for each $(x, y) \in \mathcal{D}$ $\text{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z which depends continuously differentiable on (x, y) and satisfies $Bz + Az' \in C([0, T_0], X)$ if and only if \mathcal{T} is a subgenerator of a local \mathcal{C} -semigroup on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ $\text{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y) and satisfies $Bw + Aw' \in C([0, T_0], X)$ (see Theorem 2.5 below). Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$, and \mathcal{D} is a fixed subspace of $D(B) \times D(A)$ that is dense in $X \times X$. We then prove two important lemmas (see Lemmas 2.7 and 2.8 below) which can be used to show that there exist $M, \omega > 0$ so that for each pair $x, y \in D(B) \cap D(A)$ $\text{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and $Bw + Aw' \in C([0, \infty), X)$ if and only if \mathcal{T} is a subgenerator of an exponentially bounded \mathcal{C} -semigroup on $X \times X$ if and only if there exist $M, \omega > 0$ so that $\lambda \in \rho_C(A, B)$ and

$$(1.6) \quad \|\lambda(\lambda^2 - \lambda A - B)^{-1}C\|^{(k)}, \|\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}\|^{(k)} \leq \frac{Mk!}{(\lambda - \omega)^{k+1}}$$

for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$ if and only if there exist $M, \omega > 0$ so that for each pair $x, y \in D(B) \cap D(A)$ $\text{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z with $\|z(t)\|, \|z'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and satisfies $Bz + Az' \in C([0, \infty), X)$ (see Corollary 2.6 and Theorem 2.9 below). Here $\rho_C(A, B) = \{\lambda \in \mathbb{F} \mid \lambda^2 - \lambda A - B \text{ is injective, } R(C) \subset R(\lambda^2 - \lambda A - B), \text{ and } (\lambda^2 - \lambda A - B)^{-1}C \in L(X)\}$. When $\rho(\mathcal{T})$ (resolvent set of \mathcal{T}) is nonempty, we can combine Lemma 2.4 with [23, Corollary 3.6] to show that for each $(x, y) \in D(B) \times D(A)$ $\text{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0], X)$ if and only if \mathcal{T} is the generator of a local \mathcal{C} -semigroup $\mathcal{S}(\cdot)$ on $X \times X$ if and only if for each $(x, y) \in D(B) \times D(A)$ $\text{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0], X)$ (see

Theorem 2.11 below). We then apply the modifications of [12, Theorem 2.12 and Theorem 3.2] concerning the bounded and unbounded perturbations of a locally Lipschitz continuous local once integrated C -semigroup on X (see Theorem 2.12 below) and a basic property of local C -cosine function (see [6, Theorem 2.1.11]) to obtain two new equivalence relations concerning the generations of a local C -semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and either a locally Lipschitz continuous local C -cosine function on X with subgenerator (resp., the generator) B for which A may not be bounded (see Theorem 2.13 below) or a local C -semigroup on X with subgenerator (resp., the generator) A for which B may not be bounded (see Theorem 2.16 below). Under some suitable assumptions, which can be used to show those preceding equivalence conditions which are equivalent to B is the generator of a locally Lipschitz continuous local C -cosine function on X for which A may not be bounded (see Corollaries 2.14 and 2.15 below), and are also equivalent to A is the generator of a local C -semigroup on X for which B may not be bounded (see Corollaries 2.17 and 2.18 below).

2. Abstract Cauchy problems

In this section, we consider the existence of solutions of the abstract Cauchy problem $ACP(A, B, f, x, y)$. A function u is called a (strong) solution of $ACP(A, B, f, x, y)$ if $u \in C^1([0, T_0], X)$ satisfies $ACP(A, B, f, x, y)$ (that is $u(0) = x$, $u'(0) = y$, and for a.e. $t \in (0, T_0)$, $u'(t)$ is differentiable and $u'(t) \in D(A)$, and $u''(t) = Au'(t) + Bu(t) + f(t)$ for a.e. $t \in (0, T_0)$). A linear operator A in X is called a subgenerator of a local α -times integrated C -semigroup $S(\cdot)$ if $S(t)x - j_\alpha(t)Cx = \int_0^t S(r)Axdr$ for all $x \in D(A)$ and $0 \leq t < T_0$, and $\int_0^t S(r)xdr \in D(A)$ and $A \int_0^t S(r)xdr = S(t)x - j_\alpha(t)Cx$ for all $x \in X$ and $0 \leq t < T_0$. Moreover, a subgenerator A of $S(\cdot)$ is called the maximal subgenerator of $S(\cdot)$ if it is an extension of each subgenerator of $S(\cdot)$ to $D(A)$. We next note some basic properties of a local C -semigroup, and then deduce some results about connections between the unique existence of solutions of $ACP(A, B, CBx, 0, Cy)$, $ACP\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$, and $ACP(A, B, 0, Cx, Cy)$.

Proposition 2.1. (see [4,13,23]) *Let A be the generator of a local C -semigroup $S(\cdot)$ on X . Then*

- (2.1) $S(t)S(s) = S(s)S(t)$ for $0 \leq t, s, t + s < T_0$;
- (2.2) A is closed and $C^{-1}AC = A$;
- (2.3) $S(t)x \in D(A)$ and $S(t)Ax = AS(t)x$ for $x \in D(A)$ and $0 \leq t < T_0$;
- (2.4) $\int_0^t S(r)xdr \in D(A)$ and $A \int_0^t S(r)xdr = S(t)x - Cx$ for $x \in X$ and $0 \leq t < T_0$;
- (2.5) $R(S(t)) \subset \overline{D(A)}$ for $0 \leq t < T_0$;
- (2.6) A is the maximal subgenerator of $S(\cdot)$;
- (2.7) $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator A_0 of $S(\cdot)$.

Theorem 2.2. (see [13,23]) *Let A be a closed linear operator in X such that $CA \subset AC$. Then A is a subgenerator of a local C -semigroup $S(\cdot)$ on X if and only if for each*

$x \in X$ $ACP(A, Cx, 0)$ has a unique (strong) solution $u(\cdot, x)$ in $C^1([0, T_0], X)$. In this case, we have $u(t, x) = j_0 * S(t)x (= \int_0^t S(s)x ds)$ for all $x \in X$. By slightly modifying the proof of [23, Corollary 3.5], the next theorem concerning the well-posedness of $ACP(A, f, x)$ is attained, and so its proof is omitted.

Theorem 2.3. *Let A be a closed linear operator in X such that $CA \subset AC$ and D dense in X for some subspace D of $D(A)$. Then the following are equivalent:*

- (i) A is a subgenerator of a nondegenerate local C -semigroup $S(\cdot)$ on X ;
- (ii) For each $x \in D$ $ACP(A, 0, Cx)$ has a unique solution $u(\cdot; Cx)$ in $C([0, T_0], [D(A)])$ which depends continuously on x . That is, if $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $(D, \|\cdot\|)$, then $\{u(\cdot; Cx_n)\}_{n=1}^\infty$ converges uniformly on compact subsets of $[0, T_0)$.

In this case, $u(\cdot, Cx) = S(\cdot)x$.

In the following, we always assume that A and B are biclosed linear operators in X such that $CA \subset AC$ and $CB \subset BC$.

Lemma 2.4. *Assume that \mathcal{D} is a subspace of $D(B) \times D(A)$. Then the following are equivalent:*

- (i) For each $(x, y) \in \mathcal{D}$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0], X)$;
- (ii) For each $(x, y) \in \mathcal{D}$ $ACP\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$ has a unique solution $\begin{pmatrix} u \\ v \end{pmatrix}$ in $C([0, T_0], [\mathcal{T}])$;
- (iii) For each $(x, y) \in \mathcal{D}$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0], X)$.

In this case, $w = j_0 * v$ and $z = u$.

In particular, $z, w \in C^1([0, T_0], [D(A)]) \cap C([0, T_0], [D(B)])$ if either A or B is bounded.

Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.

Proof. Since the biclosedness of A and B with $CA \subset AC$ and $CB \subset BC$ implies that \mathcal{T} is a closed linear operator in $X \times X$ so that $\mathcal{C}\mathcal{T} \subset \mathcal{T}\mathcal{C}$. Suppose that $(x, y) \in \mathcal{D}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ denotes the unique solution of $ACP\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$ in $C([0, T_0], [\mathcal{T}])$. Then v and $Bu + Av$ are continuous on $[0, T_0)$, and $u' = v$ and $v' = Bu + Av$ a.e. on $[0, T_0)$, so that $u = j_0 * v + Cx$ on $[0, T_0)$, $j_0 * v(t) \in D(B)$ for all $t \in [0, T_0)$, and $v' = Bj_0 * v + CBx$ a.e. on $[0, T_0)$. Hence, $w = j_0 * v$ is a solution of $ACP(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in C([0, T_0], X)$. The uniqueness of solutions of $ACP(A, B, CBx, 0, Cy)$ follows from the fact that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique solution of $ACP\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in $C([0, T_0], [\mathcal{T}])$. Conversely, suppose that $(x, y) \in \mathcal{D}$ and w denotes the unique solution of $ACP(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in C([0, T_0], X)$. We set $u = w + Cx$ and $v = w'$ on $[0, T_0)$. Then

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} Cx \\ Cy \end{pmatrix}, \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in D(B) \times D(A) = D(\mathcal{T})$$

for all $t \in [0, T_0)$ and $\mathcal{T} \begin{pmatrix} u \\ v \end{pmatrix}$ is continuous on $[0, T_0)$, and for a.e. $t \in (0, T_0)$ $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is differentiable and

$$\begin{aligned} \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} &= \begin{pmatrix} w'(t) \\ w''(t) \end{pmatrix} = \begin{pmatrix} w'(t) \\ Aw'(t) + Bw(t) + CBx \end{pmatrix} \\ &= \begin{pmatrix} v(t) \\ Av(t) + Bu(t) \end{pmatrix} = \mathcal{T} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \end{aligned}$$

and so $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of $\text{ACP} \left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix} \right)$ in $C([0, T_0), [\mathcal{T}])$. The uniqueness of solutions follows from the fact that 0 is the unique solution of $\text{ACP}(A, B, 0, 0, 0)$. Similarly, we can show that (ii) and (iii) are equivalent. \square

Just as an application of Theorem 2.3, the next theorem concerning the well-posedness of $\text{ACP}(A, B, f, x, y)$ is also attained.

Theorem 2.5. *Assume that \mathcal{D} is dense in $X \times X$ for some subspace \mathcal{D} of $D(B) \times D(A)$. Then the following are equivalent:*

- (i) *For each $(x, y) \in \mathcal{D}$ $\text{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y) (that is, if $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $(D(B), \|\cdot\|)$ and $\{y_n\}_{n=1}^\infty$ a Cauchy sequence in $(D(A), \|\cdot\|)$, and w_n denotes the unique solution of $\text{ACP}(A, B, CBx_n, 0, Cy_n)$, then $\{w_n(\cdot)\}_{n=1}^\infty$ and $\{w'_n(\cdot)\}_{n=1}^\infty$ both converge uniformly on compact subsets of $[0, T_0)$) and satisfies $Bw + Aw' \in C([0, T_0), X)$;*
- (ii) *\mathcal{T} is a subgenerator of a local \mathcal{C} -semigroup $\mathcal{S}(\cdot)$ on $X \times X$;*
- (iii) *For each $(x, y) \in \mathcal{D}$ $\text{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z which depends continuously differentiable on (x, y) and satisfies $Bz + Az' \in C([0, T_0), X)$.*

Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.

Proof. Since for each $(x, y) \in \mathcal{D}$ $\begin{pmatrix} u \\ v \end{pmatrix}$ is the unique solution of

$$\text{ACP} \left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix} \right)$$

in $C([0, T_0), [\mathcal{T}])$ if and only if for each $(x, y) \in \mathcal{D}$ $u = w + Cx$ and $v = w'$ on $[0, T_0)$, and w is the unique solution of $\text{ACP}(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in C([0, T_0), X)$. By Theorem 2.3, we also have $\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{S}(\cdot) \begin{pmatrix} x \\ y \end{pmatrix}$. Consequently, \mathcal{T} is a subgenerator of a local \mathcal{C} -semigroup on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ $\text{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y) . Similarly, we can show that (ii) and (iii) are equivalent. \square

Corollary 2.6. *Assume that \mathcal{D} is dense in $X \times X$ for some subspace \mathcal{D} of $D(B) \times D(A)$. Then the following are equivalent:*

- (i) There exist $M, \omega > 0$ such that for each $(x, y) \in \mathcal{D} ACP(A, B, CBx, 0, Cy)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and $Bw + Aw' \in C([0, \infty), X)$;
- (ii) \mathcal{T} is a subgenerator of an exponentially bounded C -semigroup on $X \times X$;
- (iii) There exist $M, \omega > 0$ such that for each $(x, y) \in \mathcal{D} ACP(A, B, 0, Cx, Cy)$ has a unique solution z with $\|z(t)\|, \|z'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and satisfies $Bz + Az' \in C([0, T_0), X)$.

Lemma 2.7. Assume that $\lambda \in \rho_C(\mathcal{T})$ (C -resolvent set of \mathcal{T}). Then

- (i) $\lambda \in \rho_C(A, B)$;
- (ii) $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})$ and $(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}$ are closable, and their closures are bounded and have the same domain;
- (iii) $(\lambda - \mathcal{T})^{-1}C = \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)}) & (\lambda^2 - \lambda A - B)^{-1}C \\ (\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix}$
on $D((\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})) \times X$, and on $X \times X$ if $D(B) \cap D(A)$ is dense in X .

Proof. To show that $\lambda^2 - \lambda A - B$ is closed. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(B) \cap D(A)$ which converges to x in X and $\{(\lambda^2 - \lambda A - B)x_n\}_{n=1}^{\infty}$ converges to y in X . Then $\begin{pmatrix} x_n \\ \lambda x_n \end{pmatrix} \in D(\mathcal{T})$, $\begin{pmatrix} x_n \\ \lambda x_n \end{pmatrix} \rightarrow \begin{pmatrix} x \\ \lambda x \end{pmatrix}$, and

$$(\lambda - \mathcal{T}) \begin{pmatrix} x_n \\ \lambda x_n \end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda^2 - \lambda A - B)x_n \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

By the closedness of $\lambda - \mathcal{T}$, we have $\begin{pmatrix} x \\ \lambda x \end{pmatrix} \in D(\mathcal{T})$ and

$$\begin{pmatrix} 0 \\ (\lambda^2 - \lambda A - B)x \end{pmatrix} = (\lambda - \mathcal{T}) \begin{pmatrix} x \\ \lambda x \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix},$$

and so $(\lambda^2 - \lambda A - B)x = y$. Hence, $\lambda^2 - \lambda A - B$ is closed. To show that $\lambda^2 - \lambda A - B$ is injective. Suppose that $(\lambda^2 - \lambda A - B)x = 0$. Then

$$(\lambda - \mathcal{T}) \begin{pmatrix} x \\ \lambda x \end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda^2 - \lambda A - B)x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and so $\begin{pmatrix} x \\ \lambda x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence, $x = 0$, which implies that $\lambda^2 - \lambda A - B$ is injective.

To show that $R(C) \subset R(\lambda^2 - \lambda A - B)$. Suppose that $z \in X$ is given. Then

$$(\lambda - \mathcal{T}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ Cz \end{pmatrix}$$

for some $(x, y) \in D(\mathcal{T}) = D(B) \times D(A)$, so that $\lambda x - y = 0$ and $-Bx + (\lambda - A)y = Cz$. Hence, $x \in D(B) \cap D(A) (= D(\lambda^2 - \lambda A - B))$ and $(\lambda^2 - \lambda A - B)x = Cz$, which implies that $R(C) \subset R(\lambda^2 - \lambda A - B)$. Consequently, $\lambda \in \rho_C(A, B)$.

To show that $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})$ and $(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}$ are closable, we need only to show that $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})$ or $(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}$ is closable. We will show that

$$(2.8) \quad (\lambda - \mathcal{T})^{-1}\mathcal{C} = \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A) & (\lambda^2 - \lambda A - B)^{-1}C \\ (\lambda^2 - \lambda A - B)^{-1}CB & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix}$$

on $D(B) \cap D(A)$ first or equivalently,

$$\begin{aligned} (\lambda - \mathcal{T}) \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A) & (\lambda^2 - \lambda A - B)^{-1}C \\ (\lambda^2 - \lambda A - B)^{-1}CB & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} Cx \\ Cy \end{pmatrix} \\ &= \mathcal{C} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

for all $x, y \in D(B) \cap D(A)$. Suppose that $x, y \in D(B) \cap D(A)$ are given. Then by the fact $B(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x = (\lambda - A)(\lambda^2 - \lambda A - B)^{-1}CBx$ that we have

$$\begin{aligned} &(\lambda - \mathcal{T}) \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A) & (\lambda^2 - \lambda A - B)^{-1}C \\ (\lambda^2 - \lambda A - B)^{-1}CB & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \lambda & -I \\ -B & \lambda - A \end{pmatrix} \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A) & (\lambda^2 - \lambda A - B)^{-1}C \\ (\lambda^2 - \lambda A - B)^{-1}CB & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \lambda & -I \\ -B & \lambda - A \end{pmatrix} \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x + (\lambda^2 - \lambda A - B)^{-1}Cy \\ (\lambda^2 - \lambda A - B)^{-1}CBx + \lambda(\lambda^2 - \lambda A - B)^{-1}Cy \end{pmatrix} \\ &= \begin{pmatrix} Cx \\ Cy \end{pmatrix}. \end{aligned}$$

Suppose that $x_n \in D(B) \cap D(A)$, $x_n \rightarrow 0$ in X , and $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \rightarrow y$ in X . Then

$$\begin{aligned} (\lambda^2 - \lambda A - B)^{-1}CBx_n &= (\lambda^2 - \lambda A - B)^{-1}C(B + \lambda A - \lambda^2)x_n \\ &\quad + (\lambda^2 - \lambda A - B)^{-1}C(\lambda^2 - \lambda A)x_n \\ &= Cx_n + (\lambda^2 - \lambda A - B)^{-1}C(\lambda^2 - \lambda A)x_n \\ &\rightarrow \lambda y, \end{aligned}$$

and so

$$\begin{aligned} (\lambda - \mathcal{T})^{-1}\mathcal{C} \begin{pmatrix} x_n \\ 0 \end{pmatrix} &= \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A) & (\lambda^2 - \lambda A - B)^{-1}C \\ (\lambda^2 - \lambda A - B)^{-1}CB & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} \begin{pmatrix} x_n \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \\ (\lambda^2 - \lambda A - B)^{-1}CBx_n \end{pmatrix} \\ &\rightarrow \begin{pmatrix} y \\ \lambda y \end{pmatrix} = (\lambda - \mathcal{T})^{-1}\mathcal{C} \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence, $y = 0$, which implies that $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)_{D(B) \cap D(A)}$ is closable.

To show that $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}$ is bounded.

Let $x \in D(\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}})$ be given.

Then $(x_n, (\lambda^2 - \lambda A - B)^{-1}CBx_n) \rightarrow (x, \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}x)$ for some $x_n \in D(B) \cap D(A)$, and so

$$(\lambda - \mathcal{T})^{-1}\mathcal{C} \begin{pmatrix} x_n \\ 0 \end{pmatrix} = \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \\ (\lambda^2 - \lambda A - B)^{-1}CBx_n \end{pmatrix} \rightarrow (\lambda - \mathcal{T})^{-1}\mathcal{C} \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Hence, $\{(\lambda^2 - \lambda A - B)^{-1}(\lambda - A)x_n\}_{n=1}^\infty$ and $\{(\lambda^2 - \lambda A - B)^{-1}Bx_n\}_{n=1}^\infty$ both converge. By the closedness of $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}$ and $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}$, we have $x \in D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})})$ and

$$(\lambda - \mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})x} \\ (\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}x \end{pmatrix},$$

which implies that $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}$ is bounded and

$$D(\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}) \subset D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}).$$

Similarly, we can show that $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}$ is bounded and

$$D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}) \subset D(\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}),$$

which implies that

$$\begin{aligned} (\lambda - \mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ y \end{pmatrix} &= (\lambda - \mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ 0 \end{pmatrix} + (\lambda - \mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} 0 \\ y \end{pmatrix} \\ &= \begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

for all $(x, y) \in D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}) \times X$. Combining this with the closedness of $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}$ and the denseness of $D(B) \cap D(A)$ in X , we have

$$\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}} \in L(X). \quad \square$$

Lemma 2.8. *Assume that $\lambda \in \rho_C(A, B)$. Then*

- (i) $\lambda - \mathcal{T}$ is injective;
- (ii) $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})$ and $(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}$ are closable and their closures have the same domain, and

$$(\lambda - \mathcal{T})\begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} = \mathcal{C}$$

on $D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}) \times X$;

- (iii) $\lambda \in \rho_C(\mathcal{T})$ and

$$(\lambda - \mathcal{T})^{-1}\mathcal{C} = \begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix},$$

if $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})} \in L(X)$.

In particular, the conclusion of (iii) holds when A or B in $L(X)$, or $D(B) \cap D(A)$ is dense in X with $AB = BA$ on $D(B) \cap D(A)$.

Proof. To show that $\lambda - \mathcal{T}$ is injective. Suppose that $(\lambda - \mathcal{T})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $\lambda x - y = 0$ and $-Bx + (\lambda - A)y = 0$, so that $\lambda x = y$ and $-Bx + (\lambda^2 - \lambda A)x = 0$. Hence, $x = 0 = y$, which implies that $\lambda - \mathcal{T}$ is injective. Just as in the proof of Lemma 2.7, we will apply (2.8) to show that $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})$ and

$(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}$ are closable. Suppose that $\{x_n\}_{n=1}^\infty$ is a sequence in $D(B) \cap D(A)$ which converges to 0 in X and $\{(\lambda^2 - \lambda A - B)(\lambda - A)x_n\}_{n=1}^\infty$ converges to y in X . Then

$$(\lambda^2 - \lambda A - B)^{-1}CBx_n = -Cx_n + (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \rightarrow \lambda y,$$

and so $\begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \\ (\lambda^2 - \lambda A - B)^{-1}CBx_n \end{pmatrix} \rightarrow \begin{pmatrix} y \\ \lambda y \end{pmatrix}$. Hence,

$$(\lambda - \mathcal{T}) \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \\ (\lambda^2 - \lambda A - B)^{-1}CBx_n \end{pmatrix} = \begin{pmatrix} Cx_n \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By the closedness of \mathcal{T} , we have $\begin{pmatrix} y \\ \lambda y \end{pmatrix} \in D(\mathcal{T})$ and $(\lambda - \mathcal{T}) \begin{pmatrix} y \\ \lambda y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which together with the injectivity of $\lambda - \mathcal{T}$ implies that $y = 0$.

Consequently, $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})$ is closable. Similarly, we can show that $(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}$ is closable. Just as in the proof of Lemma 2.7, we will show that

$$D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}) = D(\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}),$$

and for each $x \in D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})})$

$$\begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}x \\ (\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}x \end{pmatrix} \in D(\mathcal{T}).$$

Suppose that $x \in D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})})$ is given. Then $x_n \rightarrow x$ and $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \rightarrow \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}x$ for some sequence $\{x_n\}_{n=1}^\infty$ in $D(B) \cap D(A)$, and so

$$(\lambda^2 - \lambda A - B)^{-1}CBx_n \rightarrow -Cx + \overline{\lambda(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}x.$$

Hence, $x \in D(\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}})$, which implies that

$$D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}) \subset D(\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}).$$

Similarly, we can show that

$$D(\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}}) \subset D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}).$$

Since

$$\begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \\ (\lambda^2 - \lambda A - B)^{-1}CBx_n \end{pmatrix} \rightarrow \begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}x \\ (\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}x \end{pmatrix}$$

and

$$(\lambda - \mathcal{T}) \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}C(\lambda - A)x_n \\ (\lambda^2 - \lambda A - B)^{-1}CBx_n \end{pmatrix} = \begin{pmatrix} Cx_n \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} Cx \\ 0 \end{pmatrix}.$$

By the closedness of $\lambda - \mathcal{T}$, we have

$$(\lambda - \mathcal{T}) \begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B) \cap D(A)})}x \\ (\lambda^2 - \lambda A - B)^{-1}CB_{D(B) \cap D(A)}x \end{pmatrix} = \begin{pmatrix} (\lambda^2 - \lambda A - B)^{-1}Cx \\ \lambda(\lambda^2 - \lambda A - B)^{-1}Cx \end{pmatrix} = \mathcal{C} \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Consequently,

$$(\lambda - \mathcal{T}) \left(\begin{array}{c} \overline{(\lambda^2 - \lambda A - B)^{-1} C (\lambda - A_{D(B) \cap D(A)})} \\ \overline{(\lambda^2 - \lambda A - B)^{-1} C B_{D(B) \cap D(A)}} \end{array} \quad \begin{array}{c} (\lambda^2 - \lambda A - B)^{-1} C \\ \lambda (\lambda^2 - \lambda A - B)^{-1} C \end{array} \right) = \mathcal{C}$$

on $D(\overline{(\lambda^2 - \lambda A - B)^{-1} C (\lambda - A_{D(B) \cap D(A)})}) \times X$. □

Since $\overline{(\lambda^2 - \lambda A - B)^{-1} C (\lambda - A_{D(B) \cap D(A)})} = \overline{[(\lambda^2 - \lambda A - B)^{-1} C B_{D(B) \cap D(A)}] \frac{1}{\lambda} + \frac{1}{\lambda} C}$ and $(\lambda^2 - \lambda A - B)^{-1} C = [\lambda (\lambda^2 - \lambda A - B)^{-1} C] \frac{1}{\lambda}$, we can combine Lemma 2.7 with Lemma 2.8 and [1, Theorem 2.4.1] or [32, Theorem 1.2.1] to obtain the next new Miyadera-Feller-Phillips-Hille-Yosida type theorem concerning the generation of an exponentially bounded \mathcal{C} -semigroup on $X \times X$.

Theorem 2.9. *Assume that $D(B) \cap D(A)$ is dense in X . Then \mathcal{T} is a subgenerator of an exponentially bounded \mathcal{C} -semigroup on $X \times X$ if and only if there exist $M, \omega > 0$ such that $\lambda \in \rho_C(A, B)$ and (1.6) holds for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$.*

Just as a result in [17, Theorem 2] for the case of C_0 -semigroup, we can combine Corollary 2.6 with Theorem 2.9 to obtain the next corollary.

Corollary 2.10. *Assume that $D(B) \cap D(A)$ is dense in X . Then the following statements are equivalent:*

- (i) *There exist $M, \omega > 0$ such that for each pair $x, y \in D(B) \cap D(A)$, $ACP(A, B, CBx, 0, Cy)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \leq M e^{\omega t} (\|x\| + \|y\|)$ for all $t \geq 0$ and $Bw + Aw' \in C([0, \infty), X)$;*
- (ii) *\mathcal{T} is a subgenerator of an exponentially bounded \mathcal{C} -semigroup on $X \times X$;*
- (iii) *There exist $M, \omega > 0$ such that $\lambda \in \rho_C(A, B)$ and (1.6) holds for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$;*
- (iv) *There exist $M, \omega > 0$ such that for each pair $x, y \in D(B) \cap D(A)$, $ACP(A, B, 0, Cx, Cy)$ has a unique solution z with $\|z(t)\|, \|z'(t)\| \leq M e^{\omega t} (\|x\| + \|y\|)$ for all $t \geq 0$ and satisfies $Bz + Az' \in C([0, T_0), X)$.*

Combining Lemma 2.4 with [23, Corollary 3.6], the next theorem is also attained.

Theorem 2.11. *Assume that $\rho(\mathcal{T})$ (resolvent set of \mathcal{T}) is nonempty. Then the following are equivalent:*

- (i) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0), X)$;*
- (ii) *\mathcal{T} is the generator of a local \mathcal{C} -semigroup $\mathcal{S}(\cdot)$ on $X \times X$;*
- (iii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0), X)$.*

By modifying slightly the proofs of [12, Theorem 2.12 and Theorem 3.2], the next theorem is also attained, and so its proof is omitted.

Theorem 2.12. *Let B be a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated \mathcal{C} -semigroup on X . Assume that A is a bounded linear operator from $\overline{D(B)}$ into $R(C)$ or a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$ and $A + B$ is closed. Then $A + B$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated \mathcal{C} -semigroup on X .*

Since B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C -cosine function on X if and only if $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated \mathcal{C} -semigroup on $X \times X$ (see [6, Theorem 2.1.11]); and A is a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$ implies that

$$R\left(C^{-1}\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\right) = R\left(\begin{pmatrix} 0 & 0 \\ 0 & C^{-1}A \end{pmatrix}\right) \subset D\left(\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}\right) = D(B) \times D(A),$$

we can apply Theorem 2.12 to obtain the next new result concerning the generations of a locally Lipschitz continuous local C -cosine function on X with subgenerator (resp., the generator) B and a local \mathcal{C} -semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ for which A may not be bounded.

Theorem 2.13. *Assume that A is a bounded linear operator from $\overline{D(B)}$ into $R(C)$ or a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$. Then \mathcal{T} is a subgenerator (resp., the generator) of a local \mathcal{C} -semigroup on $X \times X$ only if B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C -cosine function on X . The "if part" is also true when the assumption of $D(B)$ is dense in X is added.*

Proof. Suppose that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local \mathcal{C} -semigroup on $X \times X$. Then it is also a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated \mathcal{C} -semigroup on $X \times X$, and so $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated \mathcal{C} -semigroup on $X \times X$. Hence, B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C -cosine function on X . Conversely, suppose that $D(B)$ is dense in X and B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C -cosine function on X . Then $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated \mathcal{C} -semigroup on $X \times X$, and so $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated \mathcal{C} -semigroup on $X \times X$. Hence, it is also a subgenerator (resp., the generator) of a local \mathcal{C} -semigroup on $X \times X$. \square

Combining Theorem 2.11 with Theorem 2.13, we can obtain the next two corollaries.

Corollary 2.14. *Assume that $\rho(A, B)$ is nonempty and $A \in L(X)$. Then the following are equivalent:*

- (i) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w in $C([0, T_0], [D(B)])$;*
- (ii) *\mathcal{T} is the generator of a local \mathcal{C} -semigroup on $X \times X$;*
- (iii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z in $C([0, T_0], [D(B)])$.*

Moreover, (i)-(iii) imply

(iv) B is the generator of a locally Lipschitz continuous local C -cosine function on X if $R(A) \subset R(C)$, and (i)-(iv) are equivalent if the assumption of $D(B)$ is dense in X is also added. Here $[D(B)]$ denotes the Banach space $D(B)$ with norm $|\cdot|$ defined by $|x| = \|x\| + \|Bx\|$ for $x \in D(B)$.

Corollary 2.15. Assume that $D(B) \cap D(A)$ is dense in X , $\rho(A, B)$ nonempty, and $AB = BA$ on $D(B) \cap D(A)$. Then the following are equivalent:

- (i) For each $(x, y) \in D(B) \times D(A)$ ACP($A, B, CBx, 0, Cy$) has a unique solution w such that $Bw + Aw' \in C([0, T_0], X)$;
- (ii) \mathcal{T} is the generator of a local C -semigroup on $X \times X$;
- (iii) For each $(x, y) \in D(B) \times D(A)$ ACP($A, B, 0, Cx, Cy$) has a unique solution z such that $Bz + Az' \in C([0, T_0], X)$.

Moreover, (i)-(iii) are equivalent to

(iv) B is the generator of a locally Lipschitz continuous local C -cosine function on X if A is a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$.

Since B is a bounded linear operator from $[D(A)]$ into $R(C)$ so that $R(C^{-1}B) \subset D(A)$ implies that

$$R\left(C^{-1}\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\right) = R\left(\begin{pmatrix} 0 & 0 \\ C^{-1}B & 0 \end{pmatrix}\right) \subset D\left(\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}\right) = D(B) \times D(A),$$

we can combine Theorem 2.11 with Theorem 2.13 to obtain the next new result concerning the generations of a local C -semigroup on X with subgenerator (resp., the generator) A and a local C -semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ for which B may not be bounded.

Theorem 2.16. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ or a bounded linear operator from $[D(A)]$ into $R(C)$ so that $R(C^{-1}B) \subset D(A)$. Then \mathcal{T} is a subgenerator (resp., the generator) of a local C -semigroup on $X \times X$ if and only if A is a subgenerator (resp., the generator) of a local C -semigroup on X .

Proof. Clearly, $C\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}C$ on $X \times D(A)$

(resp., $C^{-1}\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}C = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$) is equivalent to $CA = AC$ on $D(A)$ (resp.,

$C^{-1}AC = A$). Suppose that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a

local C -semigroup on $X \times X$. Then $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator)

of a local C -semigroup $\mathcal{S}(\cdot)$ on $X \times X$. For each pair $x, y \in X$, we set

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = j_0 * \mathcal{S}(t) \begin{pmatrix} x \\ y \end{pmatrix}$$

for all $0 \leq t < T_0$. Then

$$\begin{pmatrix} u \\ v \end{pmatrix} \in C^1([0, T_0], X \times X) \cap C([0, T_0], [\mathcal{T}]), \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} Cx \\ Cy \end{pmatrix} = \begin{pmatrix} v(t) \\ Av(t) \end{pmatrix} + \begin{pmatrix} Cx \\ Cy \end{pmatrix}$$

for all $0 \leq t < T_0$, so that $u(0) = 0 = v(0)$, $u'(t) = v(t) + Cx$ and $v'(t) = Av(t) + Cy$ for all $0 \leq t < T_0$. Hence, v is a solution of $ACP(A, Cy, 0)$ in $C^1([0, T_0], X) \cap C([0, T_0], [D(A)])$, $u(0) = 0$, and $u' = v$ on $[0, T_0]$. To show that A is a subgenerator (resp., the generator) of a local C -semigroup on X , we remain only to show that 0 is the unique solution of $ACP(A, 0, 0)$ in $C^1([0, T_0], X) \cap C([0, T_0], [D(A)])$ (see Theorem 2.2). To this end. Suppose that v is a solution of $ACP(A, 0, 0)$ in $C^1([0, T_0], X) \cap C([0, T_0], [D(A)])$. We set $u = j_0 * v$, then $u(0) = 0 = v(0)$ and

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ Av(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

for all $0 \leq t < T_0$. The uniqueness of solutions of $ACP(A, 0, 0)$ follows from the uniqueness of solutions of $ACP\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$. Conversely, suppose that A is a subgenerator (resp., the generator) of a local C -semigroup $S(\cdot)$ on X . To show that $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local C -semigroup on $X \times X$, we need only to show that for each pair $x, y \in X$, $ACP\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ has a unique solution in $C^1([0, T_0], X \times X) \cap C\left([0, T_0], \left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$. To do this. For each pair $x, y \in X$, we set $v(t) = j_0 * S(t)y$ and $u(t) = j_0 * v(t) + tCx$ for all $0 \leq t < T_0$. Then $u(0) = 0 = v(0)$, and $v'(t) = S(t)y = Av(t) + Cy$ and $u'(t) = v(t) + Cx$ for all $0 \leq t < T_0$, so that

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} v(t) + Cx \\ Av(t) + Cy \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} Cx \\ Cy \end{pmatrix}$$

for all $0 \leq t < T_0$. Hence, $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of $ACP\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in $C^1([0, T_0], X \times X) \cap C\left([0, T_0], \left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$. The uniqueness of solutions of $ACP\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in $C^1([0, T_0], X \times X) \cap C\left([0, T_0], \left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$ follows from the uniqueness of solutions of $ACP(A, 0, 0)$. Consequently, $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local C -semigroup on $X \times X$, which implies that \mathcal{T} is a subgenerator (resp., the generator) of a local C -semigroup on $X \times X$. \square

Corollary 2.17. *Assume that $\rho(A, B)$ is nonempty and $B \in L(X)$. Then the following are equivalent:*

- (i) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w in $C^1([0, T_0], [D(A)])$;*
- (ii) *\mathcal{T} is the generator of a local C -semigroup on $X \times X$;*
- (iii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z in $C^1([0, T_0], [D(A)])$.*

Moreover, (i)-(iii) are equivalent to

- (vi) *A is the generator of a local C -semigroup on X ,*
if $R(B) \subset R(C)$.

Corollary 2.18. *Assume that $D(B) \cap D(A)$ is dense in X , $\rho(A, B)$ nonempty, and $AB = BA$ on $D(B) \cap D(A)$. Then the following are equivalent:*

- (i) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0], X)$;*
- (ii) *\mathcal{T} is the generator of a local C -semigroup on $X \times X$;*
- (iii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0], X)$.*

Moreover, (i)-(iii) are equivalent to

- (iv) *A is the generator of a local C -semigroup on X ,*

if B is a bounded linear operator from $[D(A)]$ into $R(C)$ so that $R(C^{-1}B) \subset D(A)$.

We end this paper with a simple illustrative example. Let $S(\cdot) (= \{S(t) | 0 \leq t < 1\})$ be a family of bounded linear operators on c_0 (family of all convergent sequences in \mathbb{F} with limit 0,) defined by $S(t)x = \{e^{-n}e^{nt}x_n\}_{n=1}^{\infty}$, then $S(\cdot)$ is a local C -semigroup on c_0 with generator A defined by $Ax = \{nx_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$ with $\{nx_n\}_{n=1}^{\infty} \in c_0$. Here $C = S(0)$. Let $\{p_n\}_{n=1}^{\infty} \in l^{\infty}$ with $\{e^n p_n\}_{n=1}^{\infty} \in l^{\infty}$, and B be a bounded linear operator from $[D(A)]$ into $R(C)$ defined by $Bx = \{nx_n p_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in D(A)$, then $R(C^{-1}B) \subseteq D(A)$, $CB = BC$ on $\overline{D(A)}$, and $B : D(A) \subset c_0 \rightarrow c_0$ can be extended to a bounded linear operator on $\overline{D(A)} = c_0$. Applying Corollary 2.17, we get that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is the generator of a local C -semigroup on $c_0 \times c_0$.

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References

- [1] Arendt, W., Batty, C.J.K., Hieber, H., Neubrander, F., *Vector-Valued Laplace Transforms and Cauchy Problems*, 96, Birkhauser Verlag, Basel-Boston-Berlin, 2001.
- [2] DeLaubenfuls, R., *Integrated Semigroups, C -Semigroups and the Abstract Cauchy Problem*, Semigroup Forum, **41**(1990), 83-95.
- [3] Fattorini, H.O., *The Cauchy Problem*, Addison-Wesley, Reading, Mass., 1983.

- [4] Gao, M.C., *Local C -Semigroups and C -Cosine Functions*, Acta Math. Sci., **19**(1999), 201-213.
- [5] Goldstein, J.A., *Semigroups of Linear Operators and Applications*, Oxford, 1985.
- [6] Kostic, M., *Generalized Semigroups and Cosine Functions*, Mathematical Institute Belgrade, 2011.
- [7] Kostic, M., *Abstract Volterra Integro-Differential Equations*, Taylor and Francis Group, 2015.
- [8] Kostic, M., *Convolved C -Cosine Functions and C -Semigroups. Relations with Ultradistribution and Hyperfunction Sines*, J. Math. Anal. Appl., **338**(2008), 1224-1242.
- [9] Kuo, C.-C., Shaw, S.-Y., *C -Cosine Functions and the Abstract Cauchy Problem I, II*, J. Math. Anal. Appl., **210**(1997), 632-646, 647-666.
- [10] Kuo, C.-C., *Additive Perturbations of Local C -Semigroups*, Acta Math. Sci., **35B**(2015), no. 6, 1566-1576.
- [11] Kuo, C.-C., *Multiplicative Perturbations of Local C -Semigroups*, Proc. Mathematical Sci., **125**(2015), no. 1, 45-55.
- [12] Kuo, C.-C., *Perturbation Theorems for Local Integrated Semigroups*, Studia Math., **197**(2010), 13-26.
- [13] Kuo, C.-C., *Local K -Convolved C -Semigroups and Abstract Cauchy Problems*, Taiwanese J. Math., **19**(2015), no. 4, 1227-1245.
- [14] Kuo, C.-C., Shaw, S.-Y., *On α -Times Integrated C -Semigroups and the Abstract Cauchy Problem*, Studia Math., **142**(2000), 201-217.
- [15] Liang, J., Xiao, T.-J., Li, F., *Multiplicative Perturbations of Local C -Regularized Semigroups*, Semigroup Forum, **72**(2006), no. 3, 375-386.
- [16] Li, Y.-C., Shaw, S.-Y., *Perturbation of Nonexponentially-Bounded α -Times Integrated C -Semigroups*, J. Math. Soc. Japan, **55**(2003), 1115-1136.
- [17] Mel'nikova, I.-V., *The Method of Integrated Semigroups for the Cauchy Problems in Banach spaces*, Siberian Math. J., **40**(1999), 100-109.
- [18] Mel'nikova, I.-V., Filinkov, A.I., *Abstract Cauchy Problems: Three Approaches*, Chapman Hill/CRC, 2001.
- [19] Neubrander, F., *Integrated Semigroups and their Applications to the Abstract Cauchy Problem*, Pacific J. Math., **135**(1988), 111-155.
- [20] Neubrander, F., *Integrated Semigroups and their Applications to Complete Second Order Cauchy Problems*, Semigroup Forum, **38**(1989), 233-251.
- [21] Nicasie, S., *The Hille-Yosida and Trotter-Kato Theorems for Integrated Semigroups*, J. Math. Anal. Appl., **180**(1993), 303-316.
- [22] Shaw, S.-Y., Kuo, C.-C., Li, Y.-C., *Perturbation of Local C -Semigroups*, Nonlinear Analysis, **63**(2005), 2569-2574.
- [23] Shaw, S.-Y., Kuo, C.-C., *Generation of Local C -Semigroups and Solvability of the Abstract Cauchy Problems*, Taiwanese J. Math., **9**(2005), 291-311.
- [24] Shaw, S.-Y., Li, Y.-C., *Characterization and generator of local C -Cosine and C -Sine Functions*, Inter. J. Evolution Equations, **1**(2005), no. 4, 373-401.
- [25] Sova, M., *Linear Differential Equations in Banach spaces*, Rozprawy Ceskoslovenske Acad. Vd. Rada. Mat. Prirod. Ved., **85**(1975), 1-150.
- [26] Tanaka, N., Miyadera, I., *C -semigroups and the Abstract Cauchy Problem*, J. Math. Anal. Appl., **170**(1992), 196-206.

- [27] Tanaka, N., Okazawa, N., *Local C -Semigroups and Local Integrated Semigroups*, Proc. London Math. Soc., **61**(1990), no. 3, 63-90.
- [28] Takenaka, T., Piskarev, S., *Local C -Cosine Families and N -Times Integrated Local Cosine Families*, Taiwanese J. Math., **8**(2004), 515-546.
- [29] Wang, S.-W., Gao, M.-C., *Automatic Extensions of Local Regularized Semigroups and Local Regularized Cosine Functions*, Proc. London Math. Soc., **127**(1999), 1651-1663.
- [30] Wang, S.-W., Wang, M.-Y., Shen, Y., *Perturbation Theorems for Local Integrated Semigroups and Their Applications*, Studia Math., **170**(2005), 121-146.
- [31] Xiao, T.-J., Liang, J., *Multiplicative Perturbations of C -Regularized Semigroups*, Comput. Math. Appl., **41**(2001), 1215-1221.
- [32] Xiao, T.-J., Liang, J., *The Cauchy Problem for Higher-Order Abstract Differential Equations*, Lectures Notes in Math., 1701, Springer, 1998.
- [33] Xiao, T.-J., Liang, J., *On Complete Second Order Linear Differential Equations in Banach spaces*, Pacific J. Math., **142**(1990), 175-195.
- [34] Xiao, T.-J., Liang, J., *Differential Operators and C -Wellposedness of Complete Second Order Abstract Cauchy Problems*, Pacific J. Math., **186**(1998), 167-200.
- [35] Xiao, T.-J., Liang, J., *Approximations of Laplace Transforms and Integrated Semigroups*, J. Funct. Anal., **172**(2000), 202-220.

Chung-Cheng Kuo
Department of Mathematics
Fu Jen University
New Taipei City, Taiwan 24205
e-mail: cckuo@math.fju.edu.tw

