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Inner amenable hypergroups, invariant projections and Hahn-Banach extension theorem related to hypergroups

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Abstract. Let K be a hypergroup with a Haar measure. In the present paper we initiate the study of inner amenable hypergroups extending amenable hypergroups and inner amenable locally compact groups. We also provide characterizations of amenable hypergroups by hypergroups having the Hahn-Banach extension or monotone projection property. Finally we focus on weak*-invariant complemented subspaces of $L_\infty(K)$.

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1. Introduction

The classified theory of topological hypergroups have been well established in the 1970's by the works of Dunkl [6], Jewett [12] and Spector [29] independently. The history then observed a good interest in the study of this object in diverse areas of mathematics such as compact quantum hypergroups [2] weighted hypergroups [8, 9], amenable [13, 15, 31, 32] and commutative hypergroups [14, 24, 25]. A complete history of hypergroups can be found in [26].

Inner amenable locally compact groups G are ones possessing a mean m on $L_\infty(G)$ such that $m(R_g L_{g^{-1}} f) = m(f)$, for all $f \in L_\infty(G)$ and $g \in G$. This concept was introduced by Effros in 1975 for discrete groups and was studied by several authors [3, 4, 7, 17, 19, 21, 22]. It has been shown by Losert and Rindler that the existence of an inner invariant mean on $L_\infty(G)$ is equivalent to the existence of an asymptotically central net in $L_1(G)$ which is in the case of groups equivalent to the existence of a quasi central net in $L_1(G)$.

In section 3 we define the notion of inner amenable hypergroups extending amenable hypergroups and inner amenable locally compact groups. We say that a hypergroup K is inner amenable and m is an inner invariant mean if m is a mean on $L_\infty(K)$ and $m(L_g f) = m(R_g f)$ for all $f \in L_\infty(K)$ and all $g \in K$. An inner invariant mean m on a discrete hypergroup K is nontrivial if $m(f) \neq f(e)$ for $f \in l_\infty(K)$. In the process of constructing a discrete hypergroup with no nontrivial inner invariant mean we also define the concept of strong ergodicity of an action of a locally compact group on a hypergroup. Then we prove a relation between nontrivial inner invariant means on bounded functions of the semidirect product $K \rtimes_\tau G$ of a discrete hypergroup K and a discrete group G and strong ergodicity of the action τ. If K is commutative and τ is not strongly ergodic, then $l_\infty(K \rtimes_{\tau|_S} S)$ possesses a nontrivial inner invariant mean for each subgroup S of G, however, if τ is strongly ergodic and $l_\infty(G)$ has no nontrivial inner invariant mean, then $l_\infty(K \rtimes_\tau G)$ has no nontrivial inner invariant mean (Theorem 3.5).

Then we prove that inner amenability is an asymptotic property; there is a positive norm one net $\{\phi_\alpha\}$ in $L_1(K)$ such that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$, for all $g \in K$ if and only if K is inner amenable (Lemma 3.2), while the existence of a positive norm one net $\{\phi_\alpha\}$ in $L_2(K)$ such that $||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2 \to 0$, for all $g \in K$ only implies the inner amenability of K (Lemma 3.6) and implies the existence of a state m on $B(L_2(K))$ such that $m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g)$, for all $g \in K$ (Theorem 3.8). Furthermore, in Corollary 3.14 we characterize inner amenability of a hypergroup K in terms of compact operators; K is inner amenable if and only if there is a non-zero positive compact operator T in $B(L_\infty(K))$ such that $TL_g = TR_g$, for all $g \in K$.

Classical Hahn-Banach extension theorem and monotone extension property are well known and are widely used in several areas of mathematics. As one deals with (positive normalized) anti-actions of a semigroup on a real (partially ordered) topological vector space (with a topological vector unit), it is also interesting to know the condition under which the extension of an invariant (monotonic) linear functional is also invariant (and monotonic). In 1974 Lau characterized left amenable semigroups with these properties ([16], Theorems 1 and 2).

In section 4 we shall be concerned about hypergroup version of Hahn-Banach extension and monotone extension properties and we prove in Theorem 4.1 that $RUC(K)$ has a right invariant mean if and only if whenever $\{T_g \in B(E) \mid g \in K\}$ is a separately continuous representation of K on a Banach space E and F is a closed T_K-invariant subspace of E. If p is a continuous seminorm on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and Φ is a continuous T_K-invariant linear functional on F such that $|\Phi(x)| \leq p(x)$, then there is a continuous T_K-invariant linear functional $\tilde{\Phi}$ on E extending Φ such that $|\tilde{\Phi}(x)| \leq p(x)$, for all $x \in E$, if and only if for any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, if F is a closed $\mathscr{T}$-invariant subspace of E containing 1, and Φ is a $\mathscr{T}$-invariant monotonic linear functional on F, then there exists a $\mathscr{T}$-invariant monotonic linear functional $\tilde{\Phi}$ on E extending Φ.

The three statements above are also equivalent to an algebraic property: for any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, E contains a maximal proper $\mathscr{T}$-invariant ideal. As an application of these important geometric properties we provide a new proof of the known result; if K is a commutative hypergroup, then $UC(K)$ has an invariant mean (Corollary 4.4).

Let X be a weak*-closed left translation invariant subspace of $L_\infty(K)$. The concentration of section 5 is mainly on weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations. It turns out that similar to the locally compact groups ([18], Lemma 5.2), if X is an invariant complemented subspace of $L_\infty(K)$, then there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations if and only if $X \cap C_0(K)$ is weak*-dense in X (Theorem 5.1). This theorem has two major consequences; if K is compact, then X is invariantly complemented in $L_\infty(K)$ if and only if there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations (Corollary 5.2) and if K is commutative with connected dual, then there is no non-trivial weak*-weak*-continuous projections on $L_\infty(K)$ commuting with left translations (Corollary 5.6). Furthermore, we also characterize compact hypergroups; K is compact if and only if K is amenable and for every weak*-closed left translation invariant, invariant complemented subspace X of $L_\infty(K)$, there exists a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations (Corollary 5.4).

Finally, in section 6 we provide some remarks and related open problems.

2. Preliminaries and some notations

Throughout this manuscript, K denotes a hypergroup with a left Haar measure λ. For basic notations we refer to [12, 1]. The involution on K is denoted by $x \mapsto \check{x}$. Let L_x and R_y denote the left and right translation operators for $x, y \in K$ given by $R_y f(x) = L_x f(y) = \int f(u) d\delta_x * \delta_y(u)$, for any Borel function f on K, if this integral exists. Let $\phi * \mu(g) = \int R_{\check{k}}\phi(g) d\mu(k)$ and $\phi \circledast \mu(g) = \int \Delta(\check{k}) R_{\check{k}}\phi(g) d\mu(k)$, for $\mu \in M(K)$ and $\phi \in L_1(K)$. Then $(\phi \circledast \mu)\lambda = \phi\lambda * \mu$. We note that $\phi \circledast \mu$ is denoted by $\phi * \mu$ in the group setting. A closed subhypergroup N of K is a Weil subhypergroup if the mapping $f \mapsto T_N f$, where $(T_N f)(g * N) = \int R_n f(g) d\lambda_N(n)$ and λ_N is a left Haar measure on N is a well defined map from $C_c(K)$ onto $C_c(K/N)$ [11]. It is well known that any subgroup and any compact subhypergroup is a Weil subhypergroup ([11], p 250). If N is a closed normal subhypergroup, then K/N is a hypergroup if the convolution $\delta_{g*N} * \delta_{k*N}(f) = \int f(u * N) d\delta_g * \delta_k(u)$ $(f \in C_c(K/N))$ is independent of the choice of the representatives $g * N$ and $k * N$ [33]. The locally compact space K/N is a hypergroup if and only if N is a closed normal Weil subhypergroup of K ([33], Theorems 2.3 and 2.6). Let $(K, *)$ and $(J, .)$ be hypergroups. Then a continuous mapping $p : K \to J$ is said to be a hypergroup homomorphism if $\delta_{p(g)}.\delta_{p(k)^\check{}} = p(\delta_g * \delta_{\check{k}})$, for all $g, k \in K$. The modular function Δ is defined by $\lambda * \delta_{\check{g}} = \Delta(g)\lambda$, where λ is a left Haar measure on K and $g \in K$.

Let $CB(K)$ denote the space of all bounded continuous complex-valued functions on K and $C_c(K)$ the space of all continuous bounded functions on K with compact support. Let $LUC(K)$ $(RUC(K))$ be the space of all bounded left (right) uniformly continuous functions on K, i.e. all $f \in CB(K)$ such that the map $g \mapsto L_g f$ ($g \mapsto R_g f$) from K into $CB(K)$ is continuous when $CB(K)$ has the norm topology. Then $LUC(K)$ ($RUC(K)$) is a norm closed, conjugate closed, translation invariant subspace of $CB(K)$ containing constant functions.

Let X be a closed translation invariant subspace of $L_\infty(K)$ containing constants. Then a left invariant mean on X is a positive norm one linear functional, which is invariant under left translations and a hypergroup K is said to be amenable if there is a left invariant mean on $L_\infty(K)$. It is known that all compact and commutative hypergroups are amenable [28]. Furthermore, a closed left translation invariant complemented subspace Y of $L_\infty(K)$ is called invariant subspace, if there is a continuous projection P from $L_\infty(K)$ onto Y commuting with left translations. If Y is weak*-closed and P is weak*-weak*-continuous, then we say that Y is weak*-invariant complemented subspace of $L_\infty(K)$.

The representation $\mathscr{T} = \{T_g \mid g \in K\}$ is said to be a separately continuous representation of K on a Banach space X if $T_g : X \to X$, $T_e = I$, $||T_g|| \leq 1$, for each $g \in K$, the mapping $(g, x) \mapsto T_g x$ from $K \times X$ to X is separately continuous, and $T_{g_1} T_{g_2} x = \int T_u x d\delta_{g_1} * \delta_{g_2}(u)$, for $x \in X$ and $g_1, g_2 \in K$. If $\mathscr{T}$ is a continuous representation of K on X, then for $g \in K$, $\mu \in M(K)$, $f \in X^*$ and $\phi \in X$ define $f \,.\, g = M_g f$ by $< f \,.\, g, \phi > = < f, T_g \phi >$ and $f \,.\, \mu = M_\mu f$ by $< f \,.\, \mu, \phi > = \int < f, T_g \phi > d\mu(g)$. Then $f \,.\, \mu \in X^*$, $f \,.\, \delta_g = f \,.\, g$ and $(f \,.\, \mu) \,.\, \nu = f \,.\, (\mu * \nu)$, for $\mu, \nu \in M(K)$. Moreover, let $< N_g m, f > = < m, M_g f >$, $< N_\mu m, f > = < m, f \,.\, \mu >$ and $N_\phi = N_{\phi\lambda}$, for $\mu \in M(K)$, $\phi \in L_1(K)$, $m \in X^{**}$, $f \in X^*$ and $g \in K$. Then $N_\mu N_\nu = N_{\mu * \nu}$ and $N_\phi N_\mu = N_{\phi \circledast \mu}$, for each $\mu, \nu \in M(K)$. In addition, $||M_g|| \leq 1$, $||N_g|| \leq 1$, $||M_\mu|| \leq ||\mu||$ and $||N_\mu|| \leq ||\mu||$, for all $\mu \in M(K)$ and $g \in K$.

3. Inner amenable hypergroups

Let G be a locally compact group. A mean m on $L_\infty(G)$ is called inner invariant and G is called inner amenable if $m(L_g R_{g^{-1}} f) = m(f)$, for all $g \in G$ and $f \in L_\infty(G)$ (see [7] for discrete case) which is equivalent to saying that $L_g^* m = R_g^* m$, for all $g \in G$. However, this equivalence relation breaks down when one deals with hypergroups.

We say that a *hypergroup K is inner amenable* if there exists a mean m on $L_\infty(K)$ such that $m(R_g f) = m(L_g f)$ for all $g \in K$ and $f \in L_\infty(K)$. Of course amenable hypergroups are inner amenable since each invariant mean is also an inner invariant mean. An inner invariant mean m on a non-trivial discrete hypergroup is called non-trivial if $m \neq \delta_e$, the point evaluation function on $l_\infty(K)$. If this is the case, then $m_1 = \frac{m - m(\{e\})\delta_e}{1 - m(\{e\})}$ is an inner invariant mean on $l_\infty(K)$ and $m_1(\{e\}) = 0$. Any invariant mean on $l_\infty(K)$ is a non-trivial inner invariant mean and hence any non-trivial discrete amenable hypergroup possesses a non-trivial inner invariant mean.

Example 3.1. *Let H be a nontrivial discrete amenable hypergroup and J be a discrete non-amenable hypergroup. Then $K = H \times J$ is a non-amenable hypergroup and $l_\infty(K)$ has a non-trivial inner invariant mean.*

Proof. Let H be a discrete nontrivial amenable hypergroup and J be a discrete non-amenable hypergroup. Let $K = H \times J$ with the identity (e_1, e_2). If m is an invariant mean on $l_\infty(H)$ and $f \in l_\infty(K)$, then for each $k \in J$ define a function $f_k \in l_\infty(H)$ via $f_k(g) = f(g,k)$. Furthermore, define a mean m_1 on $l_\infty(K)$ by $m_1(f) = m(f_{e_2})$. Then $m_1(f) = m(f_{e_2}) \neq f_{e_2}(e_1) = f(e_1, e_2)$. In addition, for $(g_1, g_2) \in K$ and $k \in H$ we have

$$\begin{aligned}(L_{(g_1,g_2)}f)_{e_2}(k) &= L_{(g_1,g_2)}f(k,e_2)\\ &= \sum_{(u,v)\in K} f(u,v)\delta_{(g_1,g_2)} * \delta_{(k,e_2)}(u,v)\\ &= \sum_{u\in H}\sum_{v\in J} f(u,v)\delta_{g_1} * \delta_k(u)\delta_{g_2} * \delta_{e_2}(v)\\ &= \sum_{u\in H} f_{g_2}(u)\delta_{g_1} * \delta_k(u)\\ &= L_{g_1}f_{g_2}(k).\end{aligned}$$

Hence, $(L_{(g_1,g_2)}f)_{e_2} = L_{g_1}f_{g_2}$. Similarly, $(R_{(g_1,g_2)}f)_{e_2} = R_{g_1}f_{g_2}$. Thus,

$$\begin{aligned}m_1(L_{(g_1,g_2)}f) &= m((L_{(g_1,g_2)}f)_{e_2})\\ &= m(L_{g_1}f_{g_2})\\ &= m(R_{g_1}f_{g_2})\\ &= m((R_{(g_1,g_2)}f)_{e_2})\\ &= m_1(R_{(g_1,g_2)}f).\end{aligned}$$

□

The following result shows that similar to the locally compact groups ([22], Proposition 1), inner amenability of a hypergroup is also an asymptotic property.

Lemma 3.2. *The following are equivalent:*

1. *K is inner amenable.*
2. *There is a net $\{\phi_\alpha\}$ in $L_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that*
$$||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0,$$
for all $g \in K$.
3. *There is a net $\{\psi_\beta\}$ in $L_1(K)$ with $\psi_\beta \geq 0$ such that*
$$\frac{1}{||\psi_\beta||}||L_g\psi_\beta - \Delta(g)R_g\psi_\beta||_1 \to 0,$$
for all $g \in K$.

Proof. For 3 ⇒ 2 put $\phi_\alpha = \frac{\psi_\alpha}{||\psi_\alpha||}$. We will prove the equivalence of 1 and 2. Let m be a mean on $L_\infty(K)$ such that $m(L_gf) = m(R_gf)$, for $f \in L_\infty(K)$ and $g \in K$. Then there is a net of positive norm one elements $\{q_\gamma\}$ in $L_1(K)$ such that $< L_gq_\gamma - \Delta(g)R_gq_\gamma, f >\to 0$, for each $f \in L_\infty(K)$. Let T be a map from $L_1(K)$ into $L_1(K)^K$ defined by $T\phi(g) = \Delta(g)R_g\phi - L_g\phi$, for $f \in L_\infty(K)$, $\phi \in L_1(K)$ and $g \in K$. Thus, $0 \in \overline{T(P_1(K))}$, where $P_1(K) = \{\phi \in L_1(K) \mid \phi \geq 0,\ ||\phi|| = 1\}$. Therefore, there is a net of positive norm one elements $\{\phi_\alpha\}$ in $L_1(K)$ such that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha|| \to 0$. Conversely, let m be any weak*-cluster point of $\{\phi_\alpha\}$ in $L_\infty(K)^*$. Then m is a mean on $L_\infty(K)$ such that $m(R_gf) = m(L_gf)$ for all $g \in K$ and $f \in L_\infty(K)$. □

Corollary 3.3. *Let K be a discrete hypergroup. Then the following are equivalent:*

1. *There is an inner invariant mean m on $l_\infty(K)$ such that $m(\{e\}) = 0$.*
2. *There is a net $\{\phi_\alpha\}$ in $l_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that $\phi_\alpha(e) = 0$ and that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$, for all $g \in K$.*

Let G be a locally compact group and let τ be a continuous group homomorphism from G into the topological group $Aut(K)$ of all hypergroup homomorphisms on K. The semidirect product $K \rtimes_\tau G$ of K and G is the locally compact space $K \times G$ equipped with the product topology, the convolution $\delta_{(k_1,g_1)} * \delta_{(k_2,g_2)} = \delta_{k_1} * \delta_{\tau_{g_1}(k_2)} \otimes \delta_{g_1 g_2}$ and a natural embedding of the tensor product $M(K) \otimes M(G)$ into $M(K \times G)$ [34]. In this case, there is a natural action τ of G on $L_p(K)$ $(1 \leq p \leq \infty)$ defined by $\tau_g f(k) = f(\tau_g k)$ for $f \in L_p(K)$, $g \in G$ and $k \in K$. If G and K are discrete, then we say that τ is strongly ergodic if the condition $||\tau_g \phi_\alpha - \phi_\alpha||_2 \to 0$, for some positive norm one net $\{\phi_\alpha\}$ in $l_2(K)$ and all $g \in G$ implies that $\phi_\alpha(e_1) \to 1$, where e_1 is the identity of K. In addition, a mean m on $l_\infty(K)$ is τ-invariant if $m(\tau_g f) = m(f)$, for all $g \in G$ and $f \in l_\infty(K)$. The trivial τ-invariant mean on $l_\infty(K)$ is given by $\delta_{e_1}(f) = f(e_1)$, for $f \in l_\infty(K)$ (for the corresponding definitions in the countable group setting see [4]).

The following three results are inspired by [4].

Lemma 3.4. *Let G be a discrete group and let τ be a continuous group homomorphism from G into the topological group $Aut(K)$ of all hypergroup homomorphisms on a discrete hypergroup K. Then there is a non-trivial τ-invariant mean m on $l_\infty(K)$ if and only if τ is not strongly ergodic.*

Proof. Let m be a non-trivial τ-invariant mean on $l_\infty(K)$. Without loss of generality assume $m(\delta_e) = 0$, where e is the identity of K. By a standard argument (see the proof of Lemma 3.2 for example) find a positive norm one net $\{\psi_\alpha\}$ in $l_1(K)$ such that $||\tau_g \psi_\alpha - \psi_\alpha|| \to 0$ for all $g \in G$ and $\lim_\alpha \psi_\alpha(e) = 0$. Then $\{\phi_\alpha = \psi_\alpha^{\frac{1}{2}}\}$ is a positive norm one net in $l_2(K)$, $\lim_\alpha \phi_\alpha(e) = 0$ and for $g \in G$

$$||\tau_g \phi_\alpha - \phi_\alpha||_2^2 = ||\tau_g(\psi_\alpha^{\frac{1}{2}}) - \psi_\alpha^{\frac{1}{2}}||_2^2 = ||(\tau_g \psi_\alpha)^{\frac{1}{2}} - \psi_\alpha^{\frac{1}{2}}||_2^2 \leq ||\tau_g \psi_\alpha - \psi_\alpha||_1 \to 0.$$

Therefore, τ is not strongly ergodic. Conversely, let $\{\phi_\alpha\}_{\alpha \in I}$ be a positive norm one net in $l_2(K)$ such that $||\tau_g \phi_\alpha - \phi_\alpha||_2^2 \to 0$ and that $\lim_\alpha \phi_\alpha(e) \neq 1$. Choose $\alpha_0 \in I$ such that $\phi_\alpha(e) \neq 1$ for all $\alpha \geq \alpha_0$ and put $I_1 = \{\alpha \in I \mid \alpha \geq \alpha_0\}$. Then $\{\psi_\alpha = \frac{\phi_\alpha - \phi_\alpha(e)\delta_e}{1 - \phi_\alpha(e)}\}_{\alpha \in I_1}$ is a positive norm one net in $l_2(K)$ such that $||\tau_g \psi_\alpha - \psi_\alpha||_2^2 \to 0$ and $\psi_\alpha(e) = 0$ for all $\alpha \in I_1$. Let m be a weak*-cluster point of $\{\psi_\alpha^2\}_{\alpha \in I_1}$ in $l_\infty(K)^*$ and by passing possibly to a subnet assume $m(f) = \lim < \psi_\alpha^2, f >$. Then m is a nontrivial τ-invariant mean on $l_\infty(K)$. □

Theorem 3.5. *Let $K \rtimes_\tau G$ be the semidirect product hypergroup of a discrete hypergroup K and a discrete group G.*

1. *If K is commutative and τ is not strongly ergodic, then for each subgroup S of G, $l_\infty(K \rtimes_{\tau|_S} S)$ possesses a non-trivial inner invariant mean.*
2. *If τ is strongly ergodic and $l_\infty(G)$ has no non-trivial inner invariant mean, then $l_\infty(K \rtimes_\tau G)$ has no non-trivial inner invariant mean.*

Proof. 1. Assume that there exists a subgroup S of G such that $l_\infty(K \rtimes_{\tau|_S} S)$ has no non-trivial inner invariant mean. Let m be a mean on $l_\infty(K)$ such that $m(\tau_g f) = m(f)$, for all $g \in S$ and $f \in l_\infty(K)$. We will show that m is trivial. For $f \in l_\infty(K \rtimes_{\tau|_S} S)$ and $g \in S$ define a function $f_g \in l_\infty(K)$ by $f_g(k) = f(k, g)$,

$(k \in K)$. Let $M(f) = m(f_{e_2})$, for $f \in l_\infty(K \rtimes_{\tau|_S} S)$. Then M is a mean on $l_\infty(K \rtimes_{\tau|_S} S)$. For $f \in l_\infty(K \rtimes_{\tau|_S} S)$, $(k_1, g_1) \in K \rtimes_{\tau|_S} S$ and $k \in K$

$$\begin{aligned}(L_{(k_1,g_1)}f)_{e_2}(k) &= L_{(k_1,g_1)}f(k, e_2)\\ &= \sum_{(u,v)} f(u, v)\delta_{(k_1,g_1)} * \delta_{(k,e_2)}(u, v)\\ &= \sum_u \sum_v f(u, v)\delta_{k_1} * \delta_{\tau_{g_1}k}(u)\delta_{g_1e_2}(v)\\ &= \sum_u f(u, g_1)\delta_{k_1} * \delta_{\tau_{g_1}k}(u)\\ &= \sum_u f_{g_1}(u)\delta_{k_1} * \delta_{\tau_{g_1}k}(u)\\ &= L_{k_1}f_{g_1}(\tau_{g_1}k)\\ &= \tau_{g_1}(L_{k_1}f_{g_1})(k).\end{aligned}$$

Moreover,

$$\begin{aligned}(R_{(k_1,g_1)}f)_{e_2}(k) &= R_{(k_1,g_1)}f(k, e_2)\\ &= \sum_{(u,v)} f(u, v)\delta_{(k,e_2)} * \delta_{(k_1,g_1)}(u, v)\\ &= \sum_u \sum_v f(u, v)\delta_k * \delta_{\tau_{e_2}k_1}(u)\delta_{e_2g_1}(v)\\ &= \sum f_{g_1}(u)\delta_k * \delta_{k_1}(u)\\ &= L_{k_1}f_{g_1}(k),\end{aligned}$$

since K is commutative. Hence,

$$\begin{aligned}M(L_{(k_1,g_1)}f) &= m((L_{(k_1,g_1)}f)_{e_2})\\ &= m(\tau_{g_1}(L_{k_1}f_{g_1}))\\ &= m(L_{k_1}f_{g_1})\\ &= m((R_{(k_1,g_1)}f)_{e_2})\\ &= M(R_{(k_1,g_1)}f).\end{aligned}$$

Therefore, M is inner invariant. Then M is trivial, i.e, $M(f) = f(e_1, e_2)$. For $f \in l_\infty(K)$ let $f_1(k, g) = f(k)$ if $g = e_2$ and zero otherwise, $((k, g) \in K \rtimes_{\tau|_S} S)$. Then $(f_1)_{e_2}(k) = f_1(k, e_2) = f(k)$. Thus, $f(e_1) = f_1(e_1, e_2) = M(f_1) = m((f_1)_{e_2}) = m(f)$ which means that m is trivial. Consequently, τ is strongly ergodic by Lemma 3.4.

2. Suppose m is a non-trivial inner invariant mean on $l_\infty(K \rtimes_\tau G)$ and assume without loss of generality that $m(\delta_{(e_1,e_2)}) = 0$, where (e_1, e_2) is the identity of $K \rtimes_\tau G$. Then $m(R_{(e_1,g^{-1})}L_{(e_1,g)}h) = m(h)$, for all $h \in l_\infty(K \rtimes_\tau G)$ and $(e_1, g) \in K \rtimes_\tau G$. For $f \in l_\infty(K)$ let $f_1(k, g) = f(k)$ if $g = e_2$ and zero otherwise, $((k, g) \in K \rtimes_\tau G)$. Then $f_1 \in l_\infty(K \rtimes_\tau G)$. We will show that $m(\chi_{K\rtimes_\tau e_2}) = 0$. If not, then m_1 with

$$m_1(f) = \frac{m(f_1)}{m(\chi_{K\rtimes_\tau e_2})},\ (f \in l_\infty(K))$$

is a mean on $l_\infty(K)$ and $m_1(\delta_{e_1}) = 0$. For $(k_1, g_1), (e_1, g) \in K \rtimes_\tau G$ and $f \in l_\infty(K)$

$$\begin{aligned}R_{(e_1,g)}(\tau_g f)_1(k_1, g_1) &= \sum_{(u,v)}(\tau_g f)_1(u, v)\delta_{(k_1,g_1)} * \delta_{(e_1,g)}(u, v)\\ &= \sum_u \sum_v (\tau_g f)_1(u, v)\delta_{k_1} * \delta_{e_1}(u)\delta_{g_1g}(v)\\ &= (\tau_g f)_1(k_1, g_1g)\end{aligned}$$

Hence,

$$R_{(e_1,g)}(\tau_g f)_1(k_1,g_1)=\begin{cases}\tau_g f(k_1)=f(\tau_g(k_1)) & \text{if } g_1g=e_2,\\ 0 & \text{if } g_1g\neq e_2.\end{cases} \tag{1}$$

In addition,

$$\begin{aligned}L_{(e_1,g)}f_1(k_1,g_1) &=\sum_{(u,v)} f_1(u,v)\delta_{(e_1,g)}*\delta_{(k_1,g_1)}(u,v)\\ &=\sum_u\sum_v f_1(u,v)\delta_{e_1}*\delta_{\tau_g(k_1)}(u)d\delta_{gg_1}(v)\\ &=f_1(\tau_g(k_1),gg_1)\end{aligned}$$

Thus,

$$L_{(e_1,g)}(f)_1(k_1,g_1)=\begin{cases}f(\tau_g(k_1)) & \text{if } gg_1=e_2,\\ 0 & \text{if } gg_1\neq e_2.\end{cases} \tag{2}$$

Therefore, $R_{(e_1,g)}(\tau_g f)_1=L_{(e_1,g)}f_1$. In other words

$$(\tau_g f)_1=R_{(e_1,g^{-1})}L_{(e_1,g)}f_1.$$

Now observe that

$$\begin{aligned}m_1(\tau_g f) &=\frac{m((\tau_g f)_1)}{m(\chi_{K\rtimes_\tau e_2})}\\ &=\frac{m(R_{(e_1,g^{-1})}L_{(e_1,g)}f_1)}{m(\chi_{K\rtimes_\tau e_2})}\\ &=\frac{m(f_1)}{m(\chi_{K\rtimes_\tau e_2})}\\ &=m(f).\end{aligned}$$

A contradiction with the strong ergodicity of τ (Lemma 3.4). Consequently, $m(\chi_{K\rtimes_\tau e_2})=0$. For a subset C of G let $m_2(\chi_C)=m(\chi_{K\rtimes_\tau C})$ and let m_3 be an extension of m_2 to a mean on $l_\infty(G)$. Then m_3 is a mean on $l_\infty(G)$ and $m_3(\delta_{e_2})=m(\chi_{K\rtimes_\tau e_2})=0$. Furthermore, m_3 is also inner invariant since m_3 is an extension of m_2 and

$$(K\times gCg^{-1})=(e_1,g)(K\times C)(e_1,g^{-1})$$

for each $g\in G$ and each subset C of G.

□

Lemma 3.6. *The following conditions hold:*

1. *If there is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha\geq 0$ and $||\phi_\alpha||_2=1$ such that $||L_g\phi_\alpha-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2\to 0$, for all $g\in K$, then K is inner amenable.*
2. *If K is unimodular and there is a net $\{V_\alpha\}$ of Borel subsets of K with $0<\lambda(V_\alpha)<\infty$ such that $||\frac{L_g\chi_{V_\alpha}}{\lambda(V_\alpha)}-\frac{R_g\chi_{V_\alpha}}{\lambda(V_\alpha)}||_1\to 0$ for all $g\in K$, then there is a net $\{\psi_\alpha\}$ in $L_2(K)$ with $\psi_\alpha\geq 0$ and $||\psi_\alpha||_2=1$ such that $||L_g\psi_\alpha-R_g\psi_\alpha||_2\to 0$, for all $g\in K$.*

Proof. (1): For each α put $\psi_\alpha=\phi_\alpha^2$. Then for $g,k\in K$

$$\begin{aligned}&\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)\\ &=L_g\phi_\alpha^2(k)+\Delta(g)R_g\phi_\alpha^2(k)-2\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)\\ &=(L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k))^2+L_g\phi_\alpha^2(k)\\ &+\Delta(g)R_g\phi_\alpha^2(k)-(L_g\phi_\alpha)^2(k)-\Delta(g)(R_g\phi_\alpha)^2(k)\end{aligned}$$

Hence,

$$\begin{aligned}
&-[\int\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)d\lambda(k)]\\
&=-[\int(L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k))^2 d\lambda(k)\\
&+\int L_g\phi_\alpha^2(k)d\lambda(k)+\int\Delta(g)R_g\phi_\alpha^2(k)d\lambda(k)\\
&-\int(L_g\phi_\alpha)^2(k)d\lambda(k)-\int\Delta(g)(R_g\phi_\alpha)^2(k)d\lambda(k)]\\
&\leq-||L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)||_2^2-||\phi_\alpha||_2^2\\
&-||\phi_\alpha||_2^2+||\phi_\alpha||_2^2+||\phi_\alpha||_2^2\to 0,
\end{aligned}$$

because

$$\begin{aligned}
\int\Delta(g)(R_g\phi_\alpha)^2(k)d\lambda(k)\quad &=<\Delta(g)R_g\phi_\alpha,R_g\phi_\alpha>\\
&=<\phi_\alpha,R_{\check{g}}R_g\phi_\alpha>\\
&\leq||\phi_\alpha||_2^2
\end{aligned}$$

and each ϕ_α is positive. In addition,

$$\begin{aligned}
&\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)R_g\phi_\alpha^2(k)\\
&\leq\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)(R_g\phi_\alpha)^2(k)\\
&=[L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)]\ \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k),
\end{aligned}$$

by Holder's inequality. Thus,

$$\begin{aligned}
&\int|\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\\
&\leq\Delta^{\frac{1}{2}}(g)||R_g\phi_\alpha||_2\ ||L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)||_2\to 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&||L_g\psi_\alpha-\Delta(g)R_g\psi_\alpha||_1\\
&=\int|L_g\phi_\alpha^2(k)-\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\\
&\leq\int|\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)|d\lambda(k)\\
&+\int|2\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-2\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\to 0,
\end{aligned}$$

since,

$$\begin{aligned}
&\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)\\
&=\int\int[\phi_\alpha^2(u)+\Delta(g)\phi_\alpha^2(v)-2\Delta^{\frac{1}{2}}(g)\phi_\alpha(u)\phi_\alpha(v)]d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)\\
&=L_g\phi_\alpha^2(k)-\Delta(g)R_g\phi_\alpha^2(k)+2\Delta(g)R_g\phi_\alpha^2(k)-2\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha^2(k)L_g\phi_\alpha^2(k).
\end{aligned}$$

By Lemma 3.2 then K is inner amenable. The rest follows by a similar argument as in ([28], Theorem 4.3) if K is unimodular. □

Remark 3.7. *Let K be a discrete hypergroup. If there is a positive norm one net $\{\phi_\alpha\}$ in $l_2(K)$ with $\phi_\alpha(e)\to 0$ such that $||L_g\phi_\alpha-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2\to 0$, for all $g\in K$, $l_\infty(K)$ has a non-trivial inner invariant mean.*

Theorem 3.8. *The following are equivalent:*

1. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha\geq 0$ and $||\phi_\alpha||_2=1$ such that $||L_g\phi_\alpha-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2\to 0$, for all $g\in K$.*

2. *There is a net* $\{\phi_\alpha\}$ *in* $L_2(K)$ *with* $\phi_\alpha \geq 0$ *and* $||\phi_\alpha||_2 = 1$ *such that for each* $g \in K$

$$|\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0$$

and

$$|\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0.$$

In this case K *is inner amenable and there is a state* m *on* $B(L_2(K))$ *such that* $m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g)$, *for all* $g \in K$, *where* L_g (R_g) *is the left (right) translation operator on* $L_2(K)$.

Proof. If (1) holds, then for $g \in K$

$$\begin{aligned}
&|\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ |\\
&= |\ < L_g\phi_\alpha, L_g\phi_\alpha > - < L_g\phi_\alpha, \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha >\ |\\
&= |\ < L_g\phi_\alpha, L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha >\ |\\
&\leq ||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha|| \to 0.
\end{aligned}$$

Similarly, $|\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \to 0$, for $g \in K$. Conversely, for each $g \in K$ we have

$$\begin{aligned}
&||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2\\
&=< L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha >\\
&= ||L_g\phi_\alpha||_2^2 + ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - 2 < L_g\phi_\alpha, \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha >\\
&= ||L_g\phi_\alpha||_2^2 + ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - 2\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\\
&\leq |\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ |\\
&+|\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \to 0.
\end{aligned}$$

For each $T \in B(L_2(K))$ let $m_\alpha T =< T\phi_\alpha, \phi_\alpha >$ and let m be a weak*-cluster point of the net $\{m_\alpha\}$ in $B(L_2(K))^*$. Without loss of generality assume that $mT = \lim_\alpha m_\alpha(T)$. Then m is a state on $B(L_2(K))$ and for $g \in K$

$$\begin{aligned}
&|m(L_g) - m(\Delta^{\frac{1}{2}}(g)R_g)|\\
&= |\lim_\alpha < L_g\phi_\alpha, \phi_\alpha > - \lim_\alpha < \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, \phi_\alpha > |\\
&= |\lim_\alpha < L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, \phi_\alpha > |\\
&\leq \lim_\alpha ||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha|| = 0.
\end{aligned}$$

In addition, K is inner amenable by Lemma 3.6. □

It is known that the amenability of a locally compact group G can be characterized by the existence of a state m on $B(L_2(K))$ with $m(L_g) = 1$, for all $g \in G$ ([3], Theorem 2). By a similar method as in the proof of Theorem 3.8 we have the following:

Remark 3.9. *If* K *satisfies Reiter's condition* P_2, *then there is a state* m *on* $B(L_2(K))$ *such that* $m(L_g) = 1$, *for all* $g \in K$.

Let G be a locally compact group. Then G is an $[IN]$-group if and only if G possesses a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in G$. However, one may not expect this equivalence relation to hold in the hypergroup setting. A hypergroup K is called $[IN]$-hypergroup if there is a compact neighborhood V of e such that $g * V = V * g$, for all $g \in K$. It is easy to see that each of compact or commutative hypergroups are $[IN]$-hypergroups and possess a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$. For a discrete hypergroup K the situation is quite different: although K is an $[IN]$-hypergroup, we have that $L_g\delta_e = R_g\delta_e$, for all $g \in K$ if and only if $\delta_g * \delta_{\check{g}}(e) = \delta_{\check{g}} * \delta_g(e)$, for all $g \in K$.

Corollary 3.10. *Let K be a hypergroup possessing a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$. Let Q_V be the operator on $L_2(K)$ given by $Q_V f =< f, \chi_V > .\chi_V$ for $f \in L_2(K)$. Then the following are equivalent:*

1. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$, $< \phi_\alpha, \chi_V >= 0$ and $||\phi_\alpha||_2 = 1$ such that*
$$||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2 \to 0,$$
for all $g \in K$.
2. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$, $< \phi_\alpha, \chi_V >= 0$ and $||\phi_\alpha||_2 = 1$ such that for $g \in K$*
$$|\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0,$$
and
$$||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0.$$

In this case

a. *There is an inner invariant mean m on $L_\infty(K)$ with*
$$m(\chi_V) = 0.$$
b. *There is a state m on $B(L_2(K))$ such that $m(Q_V) = 0$ and*
$$m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g),$$
for all $g \in K$.
c. *The operators $id - Q_V$ and $id + Q_V$ are not in the C^*-algebra generated by $\{L_g - \Delta^{\frac{1}{2}}(g)R_g \mid g \in K\}$.*

Proof. We will show b $\Rightarrow$ c, for all other parts we refer to the proof of Theorem 3.8. Let
$$T = \sum_{i=1}^{n} \lambda_i(L_{g_i} - \Delta^{\frac{1}{2}}(g_i)R_{g_i}).$$
Then $m(T) = 0$ and hence
$$||T - (id - Q_V)|| \geq |m(T) - m(id - Q_V)| = 1.$$
Similarly, $||T - (id + Q_V)|| \geq 1$. Thus, $id - Q_V$ and $id + Q_V$ are not in the C^*-algebra generated by $\{L_g - \Delta^{\frac{1}{2}}(g)R_g \mid g \in K\}$. ☐

Remark 3.11. *Let K be a unimodular hypergroup possessing a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$ and let $1 \leq p < \infty$. Then there is a compact operator T in $B(L_p(K))$ such that $L_gT = R_gT$, $L_{\check{k}}TL_g = R_{\check{k}}TR_g$ and $TL_g = TR_g$, for all $g, k \in K$.*

Proof. Let $Tf := < \chi_V, f > \chi_V$. Then for $f \in L_p(K)$ and $g, k \in K$,

$$\begin{aligned} L_{\check{k}}TL_gf &= < \chi_V, L_gf > L_{\check{k}}\chi_V \\ &= < L_{\check{g}}\chi_V, f > L_{\check{k}}\chi_V \\ &= < R_{\check{g}}\chi_V, f > R_{\check{k}}\chi_V \\ &= < \chi_V, R_gf > R_{\check{k}}\chi_V \\ &= R_{\check{k}}TR_gf. \end{aligned}$$

Hence, $L_{\check{k}}TL_g = R_{\check{k}}TR_g$, for all $g, k \in K$. Similarly we can prove other parts. □

Example 3.12.

1. *Let $K = H \vee J$ be the hypergroup join of a compact group H and a discrete commutative hypergroup J. Then there is a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$.*
2. *Let $K = H \vee J$ be the hypergroup join of a finite commutative hypergroup H and a discrete group J. Then $\delta_g * \delta_{\check{g}}(e) = \delta_{\check{g}} * \delta_g(e)$, for all $g \in K$ and hence $L_g\delta_e = R_g\delta_e$, for all $g \in K$. since*

$$\delta_{\check{j}} * \delta_j(e) = \sum_{g \in H} \frac{1}{\delta_{\check{g}} * \delta_g(e)}\delta_g = \delta_j * \delta_{\check{j}}(e),$$

for $j \in J$.

Lau and Paterson in ([19], Theorem 2) proved that a locally compact group G is inner amenable if and only if there exists a non-zero compact operator in $\mathscr{A}_\infty^{'}$, where

$$\mathscr{A}_\infty^{'} = \{T \in B(L_\infty(G)) \mid L_{g^{-1}}R_gT = TL_{g^{-1}}R_g, \ \forall g \in G\}.$$

We note that

$$\mathscr{A}_\infty^{'} = \{T \in B(L_\infty(G)) \mid R_gTR_{g^{-1}} = L_gTL_{g^{-1}}, \ \forall g \in G\}$$

which is not the case as we step beyond the groundwork of locally compact groups. The following is an extension of ([19], Theorem 2):

Remark 3.13. *The following conditions hold:*

1. *If K is inner amenable, then there is a compact operator T in $B(L_\infty(K))$ such that $T(h) = 1$, for some $h \in L_\infty(K)$,*

$$L_{\check{n}}TL_g = R_{\check{m}}TR_g, \ TL_g = TR_g,$$

for all $g, n, m \in K$ and $T(f) \geq 0$, for $f \geq 0$.

2. *If there is a non-zero operator T in $B(L_\infty(K))$ such that*

$$TL_g = TR_g,$$

for all $g \in K$ and $T(f) \geq 0$, for $f \geq 0$, then K is inner amenable and $T(f) \geq 0$, for $f \geq 0$.

Proof. 1. If m is an inner invariant mean on $L_\infty(K)$, then the operator T in $B(L_\infty(K))$ defined by $T(f) = m(f)1$, for $f \in L_\infty(K)$ is the desired operator.

2. Let m be a mean on $L_\infty(K)$. Then $m \circ T$ is an inner invariant positive linear functional on $L_\infty(K)$. Let $f_0 \in L_\infty(K)$ such that $T(f_0) > 0$. Then f_0 can be decomposed into positive elements and if $f \geq 0$, then $T(f) \leq ||f||T(1)$. Hence, $m \circ T(1) \neq 0$ and $\frac{m\circ T}{m\circ T(1)}$ is an inner invariant mean on $L_\infty(K)$.

□

Corollary 3.14. *K is inner amenable if and only if there is a non-zero compact operator T in $B(L_\infty(K))$ such that $TL_g = TR_g$, for all $g \in K$ and $T(f) \geq 0$, for $f \geq 0$.*

Corollary 3.15. *Let G be a locally compact group. Then G is inner amenable if and only if there is a non-zero operator T in $\mathscr{A}_\infty^{'}$ such that $TL_g = TR_g$, for all $g \in G$ and $T(f) \geq 0$, for $f \geq 0$.*

We say that K satisfies *central Reiter's condition* P_1, if there is a net $\{\phi_\alpha\}$ in $L_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that

$$||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$$

uniformly on compact subsets of K. By Lemma 3.2 if K satisfies central Reiter's condition P_1, then K is inner amenable. Sinclair ([27], page 47) in particular called a net $\{\phi_\alpha\}$ in $L_1(G)$ quasi central if $||\mu * \phi_\alpha - \phi_\alpha * \mu|| \to 0$, for all $\mu \in M(G)$, where G is a locally compact group. We say that the net $\{\phi_\alpha\}$ in $L_1(K)$ is *quasi central* if

$$||\mu * \phi_\alpha - \phi_\alpha \circledast \mu|| \to 0,$$

for all $\mu \in M(K)$.

One note the distinction between the condition $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$ uniformly on compacta and the (equivalent for groups, but not for hypergroups) condition $||\phi_\alpha - \Delta(g)L_{\check{g}}R_g\phi_\alpha||_1 \to 0$ uniformly on compacta. For the group case please see ([30], Theorem 4.2).

Remark 3.16. *If the net $\{\phi_\alpha\}$ in $L_1(K)$ satisfies central Reiter's condition P_1, then*

1. *For given $\{\psi_i\}_{i=1}^n \subseteq L_1(K)$ and $\epsilon > 0$, there is an element $\phi \in L_1(K)$ such that $||\psi_i * \phi - \phi * \psi_i|| < \epsilon$, for $i = 1, 2, ..., n$.*
2. *The net $\{\phi_\alpha\}$ is a quasi central net in $L_1(K)$.*

Proof. (1): Let $\epsilon > 0$ be given and let C_i be compact subsets of K such that $\int_{K\setminus C_i} |\psi_i|(g)d\lambda(g) < \epsilon$. Let $C = \bigcup_{i=1}^n C_i$ and let $\alpha \in I$ be such that $||L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|| < \epsilon$, for all $g \in C$. Then

$$\begin{aligned}
&||\psi_i * \phi_\alpha - \phi_\alpha * \psi_i||_1 \\
&= \int |\int \psi_i(g)L_{\check{g}}\phi_\alpha(k)d\lambda(g) - \int \psi_i(g)\Delta(\check{g})R_{\check{g}}\phi_\alpha(k)d\lambda(g)|d\lambda(k) \\
&\leq \int |\psi_i(g)| \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) \\
&= \int_{K\setminus C} |\psi_i(g)| \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) \\
&+ \int_C |\psi_i(g)| \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) \\
&< \epsilon^2 + \epsilon\ Max_{i=1,\dots,n}||\psi_i||_1
\end{aligned}$$

(2): Without loss of generality assume that $\mu \in M(K)$ has a compact support C. Let $\epsilon > 0$ be given and let $\alpha \in I$ be such that $||L_{\check{g}}\phi_\alpha - \Delta(\check{g})R_{\check{g}}\phi_\alpha|| < \epsilon$, for all

$g \in C$. Then

$$\begin{aligned}&||\mu * \phi_\alpha - \phi_\alpha \circledast \mu||\\ &= \int | \int (L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha) d\mu(g)| d\lambda(k)\\ &\leq \int \int_C |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha| d\mu(g) d\lambda(k)\\ &+ \int \int_{K\setminus C} |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha| d\mu(g) d\lambda(k)\\ &\leq \epsilon ||\mu||.\end{aligned}$$

□

Losert and Rindler called a net $\{\phi_\alpha\}$ in $L_1(G)$, G is a locally compact group, asymptotically central if $\frac{1}{||\phi_\alpha||}(\Delta(g)R_g L_{g^{-1}}\phi_\alpha - \phi_\alpha) \to 0$ weakly for all $g \in G$ [21].

We say that the net $\{\phi_\alpha\}$ in $L_1(K)$ is *asymptotically central* if

$$\frac{1}{||\phi_\alpha||}(\Delta(g)R_g L_{\check{g}}\phi_\alpha - \phi_\alpha) \to 0$$

weakly for all $g \in K$. In addition, we say that the net $\{\phi_\alpha\}$ in $L_1(K)$ is *hypergroup asymptotically central* if

$$\frac{1}{||\phi_\alpha||}(\Delta(g)R_g\phi_\alpha - L_g\phi_\alpha) \to 0$$

weakly for all $g \in K$. The reason for our definition is that

$$Z(L_1(K)) = \{\phi \in L_1(K) \mid \Delta(g)R_g\phi = L_g\phi,\ \forall g \in K\},$$

where $Z(L_1(K))$ is the algebraic center of the hypergroup algebra $L_1(K)$. Then it is easy to see that if K is discrete and unimodular or commutative, then any approximate identity in $L_1(K)$ is hypergroup asymptotically central and hence $L_1(K)$ is Arens semi-regular (see [10], page 45 for the definition).

Remark 3.17. *If $L_1(K)$ has an asymptotically central bounded approximate identity, then K is an inner amenable locally compact group.*

Proof. Let $\{\phi_\alpha\}$ be an asymptotically central bounded approximate identity for $L_1(K)$ and m be a weak*-cluster point of $\{\phi_\alpha\}$ in $L_\infty(K)^*$. Without loss of generality assume that ϕ_α's are real-valued and $\lim_\alpha < \phi_\alpha, f > = < m, f >$ for each $f \in L_\infty(K)$. Then $m(L_g R_{\check{g}} f) = m(f)$, for each $f \in L_\infty(K)$ and $g \in K$. In addition,

$$m(\phi * f) = \lim < \phi_\alpha, \phi * f > = \lim < \check{\Delta}\check{\phi} * \phi_\alpha, f > = < \check{\Delta}\check{\phi}, f > = \phi * f(e),$$

for $\phi \in L_1(K)$ and $f \in L_\infty(K)$. Thus, $m(f) = f(e)$, for each $f \in C_0(K)$ ([28], Lemma 2.2). Therefore,

$$\delta_g * \delta_{\check{g}}(f) = R_{\check{g}} f(g) = L_g R_{\check{g}} f(e) = m(L_g R_{\check{g}} f) = m(f) = \delta_e(f),$$

for $f \in C_0(K)$. i.e. $\delta_g * \delta_{\check{g}} = \delta_e$, for all $g \in K$ and hence $G(K) = K$. It follows then by the proof of ([21], Theorem 2) that the locally compact group K is also inner amenable. □

In 1991, Lau and Paterson characterized inner amenable locally compact groups G in terms of a fixed point property of an action of G on a Banach space ([17], Theorem 5.1). This characterization can be extended naturally to hypergroups and we have:

Remark 3.18. *The following are equivalent:*

1. *K is inner amenable.*
2. *Whenever $\{T_g \in B(E) \mid g \in K\}$ is a separately continuous representation of K on a Banach space E as contractions, there is some*
$$T \in \overline{\{N_\phi \mid \phi \in L_1(K),\ ||\phi|| = 1,\ \phi \geq 0\}}^{w^*.o.t}$$
such that
$$N_g T = T N_g,$$
for all $g \in K$.

Remark 3.19. *Let N be a closed normal Weil subhypergroup of K. If K is inner amenable, then K/N is also inner amenable.*

Proof. Define a linear isometry ϕ from $L_\infty(K/N)$ to the subspace
$$\{f \in L_\infty(K) \mid R_g f = R_k f,\ g \in k * N,\ k \in K\}$$
of $L_\infty(K)$ by $\phi(f) = f \circ \pi$, where π is the quotient map from K onto K/N. Then
$$\begin{aligned}
&\textstyle\int |L_g(\phi f)(k) - \phi(L_{g*N} f)(k)| d\lambda(k) \\
&= \textstyle\int | \int f(u * N) d\delta_g * \delta_k(u) - (L_{g*N} f) \circ \pi(k) | d\lambda(k) \\
&= \textstyle\int | \int f(u * N) d\delta_{g*N} * \delta_{k*N}(u * N) - L_{g*N} f(k * N) | d\lambda(k) \\
&= 0,
\end{aligned}$$
since N is a Weil subhypergroup. Thus, $\phi(L_{g*N} f) = L_g(\phi f)$ for $f \in L_\infty(K/N)$ and $g \in K$. Similarly, $\phi(R_{g*N} f) = R_g(\phi f)$ for $f \in L_\infty(K/N)$ and $g \in K$. Let m be an inner invariant mean on $L_\infty(K)$ and define $m_1(f) = m(\phi f)$, $f \in L_\infty(K/N)$. Then m_1 is a mean on $L_\infty(K/N)$. In addition, for $f \in L_\infty(K/N)$ and $g \in K$
$$\begin{aligned}
m_1(L_{g*N} f) &= m(\phi(L_{g*N} f)) \\
&= m(L_g \phi f) \\
&= m(R_g \phi f) \\
&= m(\phi(R_{g*N} f)) \\
&= m_1(R_{g*N} f).
\end{aligned}$$
□

4. Hahn-Banach extension and monotone extension properties

It is the purpose of this section to provide a hypergroup version of Hahn-Banach extension property and monotone extension property by which amenable hypergroups can be characterized.

Let E be a partially ordered Banach space over $\mathbb{R}$. An element $1 \in E$ is called a topological order unit if for each $f \in E$ there exists $\lambda > 0$ such that $-\lambda 1 \leq f \leq \lambda 1$ and the set $\{f \in E \mid 1 \leq f \leq 1\}$ is a neighbourhood of E and a proper subspace I of E is said to be a proper ideal if $[0, f] \subseteq I$, for each $f \in E$. Moreover, a separately continuous linear representation $\mathscr{T} = \{T_g \mid g \in K\}$ of K on E is positive if $T_g f \geq 0$ for all $g \in K$ and $f \geq 0$. $\mathscr{T}$ is normalized if $T_g 1 = 1$ for all $g \in K$.

Theorem 4.1. *The following are equivalent:*

1. *$RUC(K)$ has a right invariant mean.*
2. *Let $\{T_g \in B(E) \mid g \in K\}$ be a separately continuous representation of K on a Banach space E and let F be a closed T_K-invariant subspace of E. Let p be a continuous seminorm on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and Φ be a continuous linear functional on F such that $|\Phi(x)| \leq p(x)$ and $\Phi(T_g x) = \Phi(x)$ for $g \in K$ and $x \in F$. Then there is a continuous linear functional $\tilde{\Phi}$ on E such that*
   (a) $\tilde{\Phi}|_F \equiv \Phi$.
   (b) $|\tilde{\Phi}(x)| \leq p(x)$ for each $x \in E$.
   (c) $\tilde{\Phi}(T_g x) = \tilde{\Phi}(x)$ for $g \in K$ and $x \in E$.
3. *For any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, if F is a closed $\mathscr{T}$-invariant subspace of E containing 1, and Φ is a $\mathscr{T}$-invariant monotonic linear functional on F, then there exists a $\mathscr{T}$-invariant monotonic linear functional $\tilde{\Phi}$ on E extending Φ.*
4. *For any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, E contains a maximal proper $\mathscr{T}$-invariant ideal.*

Proof. $1 \Rightarrow 2$: By Hahn-Banach extension theorem there is a continuous linear functional Φ_1 on E such that $|\Phi_1(x)| \leq p(x)$ for each $x \in E$ and $\Phi_1|_F \equiv \Phi$. For each $f \in E$ define a continuous bounded function $h_{\Phi_1,f}$ on K via $h_{\Phi_1,f}(g) = \Phi_1(T_g f)$. Let $\{g_\alpha\}$ be a net in K converging to e. Then

$$\begin{aligned} ||R_{g_\alpha} h_{\Phi_1,f} - h_{\Phi_1,f}|| &= \sup_{g\in K} |R_{g_\alpha} h_{\Phi_1,f}(g) - h_{\Phi_1,f}(g)| \\ &= \sup_{g\in K} |\textstyle\int \Phi_1(T_u f) d\delta_g * \delta_{g_\alpha}(u) - \Phi_1(T_g f)| \\ &= \sup_{g\in K} |\Phi_1(T_g T_{g_\alpha} f) + \Phi_1(-T_g f)| \\ &\leq \sup_{g\in K} p(T_g T_{g_\alpha} f - T_g f) \\ &\leq p(T_{g_\alpha} f - f) \to 0, \end{aligned}$$

since $\Phi_1 \in E^*$. Hence, $h_{\Phi_1,f} \in RUC(K)$ ([28], Remark 2.3). Let m be a right invariant mean on $RUC(K)$ and let $\tilde{\Phi}(f) = m(h_{\Phi_1,f})$, for $f \in E$. Then $\tilde{\Phi}|_F \equiv \Phi$ since $h_{\Phi_1,f}(g) = \Phi_1(T_g f) = \Phi(f)$, for $f \in F$. Furthermore, $|\tilde{\Phi}(f)| \leq \sup_{g\in K} |\Phi_1(T_g f)| \leq p(f)$, for $f \in E$ and

$$\begin{aligned} h_{\Phi_1,T_g f}(k) &= \Phi_1(T_k T_g f) \\ &= \textstyle\int \Phi_1(T_u f) d\delta_k * \delta_g(u) \\ &= \textstyle\int h_{\Phi_1,f}(u) d\delta_k * \delta_g(u) \\ &= R_g h_{\Phi_1,f}(k). \end{aligned}$$

Thus,

$$\tilde{\Phi}(T_g f) = m(h_{\Phi_1,T_g f}) = m(R_g h_{\Phi_1,f}) = m(h_{\Phi_1,f}) = \tilde{\Phi}(f).$$

$2 \Rightarrow 1$: Let $E = RUC(K)$, $F = \mathbb{C}.1$ and consider the continuous representation $\{R_g \mid g \in K\}$ of K on $RUC(K)$. Define a seminorm p on E by $p(f) = ||f||$. Then $p(R_g f) \leq p(f)$, for $f \in E$ and $g \in K$. In addition, δ_a is a left invariant mean on F for a given $a \in K$ with $|\delta_a(f)| \leq p(f)$. Therefore, there is some $m \in RUC(K)^*$ such that $m|_F \equiv \delta_a$, $m(f) \leq ||f||$ and $m(R_g f) = m(f)$, for $f \in E$ and $g \in K$. Then m is a right invariant mean on $RUC(K)$ because $m(1) = \delta_a(1) = 1 = ||m||$.

For all other parts we refer to ([16], Theorem 2) and a similar argument as above. □

Let $CB_{\mathbb{R}}(K)$ denote all bounded continuous real-valued functions on K and $UC_{\mathbb{R}}(K)$ ($RUC_{\mathbb{R}}(K)$) denote all functions in $CB_{\mathbb{R}}(K)$ which are (right) uniformly continuous. It is easy to see that $UC_{\mathbb{R}}(K)$ and $RUC_{\mathbb{R}}(K)$ are norm-closed translation invariant subspace of $CB_{\mathbb{R}}(K)$ containing constants. However, in contrast to the group case, $RUC_{\mathbb{R}}(K)$ need not be a Banach lattice in general. The following result is a consequence of Theorem 4.1 and the proof of ([16], Theorem 1).

Remark 4.2. *Let K be a hypergroup such that $RUC_{\mathbb{R}}(K)$ is a Banach lattice. Then the following are equivalent:*

1. *$RUC(K)$ has a right invariant mean.*
2. *For any linear action $\mathscr{T}$ of K on a Banach space E, if U is a $\mathscr{T}$-invariant open convex subset of E containing a $\mathscr{T}$-invariant element, and M is a $\mathscr{T}$-invariant subspace of E which does not meet U, then there exists a closed $\mathscr{T}$-invariant hyperplane H of E such that H contains M and H does not meet U.*
3. *For any contractive action $\mathscr{T} = \{T_g \in B(E) \mid g \in K\}$ of K on a Hausdorff Banach space E, any two points in $\{f \in E \mid T_g f = f, \ \forall g \in K\}$ can be separated by a continuous $\mathscr{T}$-invariant linear functional on E.*

Example 4.3. *1. Let K be a hypergroup such that the maximal subgroup $G(K)$ is open. Then $RUC_{\mathbb{R}}(K)$ is a Banach lattice.*

2. Let $K = H \vee J$ be the hypergroup join of a compact hypergroup H and a discrete hypergroup J. Then $RUC_{\mathbb{R}}(K) = CB_{\mathbb{R}}(K)$ is a Banach lattice.

Proof. To see 1, let $f, h \in RUC_{\mathbb{R}}(K)$ and $\{g_\alpha\}$ be a net in K converging to e. Then $g_\alpha \in G(K)$, for some α_0 and all $\alpha \geq \alpha_0$ since $G(K)$ is open. Thus, $R_{g_\alpha}(f \vee h) = R_{g_\alpha} f \vee R_{g_\alpha} h$ for $\alpha \geq \alpha_0$. Therefore, the mapping

$$g \mapsto (R_g f, R_g h) \mapsto R_g f \vee R_g h$$

from K to $CB_{\mathbb{R}}(K)$ is continuous at e and hence $f \vee h \in RUC_{\mathbb{R}}(K)$. □

Next we use Theorem 4.1 to prove that $UC(K)$ has an invariant mean, for any commutative hypergroup K.

Corollary 4.4. *Let K be a commutative hypergroup. Then $UC(K)$ has an invariant mean.*

Proof. Let $\mathscr{T} = \{T_g \in B(E) \mid g \in K\}$ be a separately continuous representation of K on a real Banach space E and let F be a closed $\mathscr{T}$-invariant subspace of E. Let p be a continuous sublinear map on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and ϕ be a continuous $\mathscr{T}$-invariant linear functional on F such that $\phi(x) \leq p(x)$ for $x \in F$. Define a representation $\{T_\mu \in B(E) \mid \mu \in M_1^c(K)\}$ of $M_1^c(K)$, the probability measures with compact support on K, on E via

$$T_\mu x = \int T_g x d\mu(g).$$

Then $T_{\mu*\nu} = T_\mu T_\nu$, for $\mu, \nu \in M_1^c(K)$. In addition,

$$p(T_\mu x) = p(\int T_g x d\mu(g)) \leq \int p(T_g x) d\mu(g) \leq p(x).$$

Define a real valued function q on E via

$$q(x) = \inf\{\frac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x)\},$$

where the inf is taken over all finite collection of probability measures with compact support $\{\mu_1, ..., \mu_m\}$ on K. Then $q(x) \leq p(x)$ for $x \in E$ since for each $m \in \mathbb{N}$,

$$\tfrac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x) \quad \leq \tfrac{1}{m} \big[p(T_{\mu_1} x) + ... + p(T_{\mu_m} x)\big] \leq p(x).$$

Moreover, q is sublinear. In fact for $m \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$ and $x \in E$,

$$\tfrac{1}{m} p(T_{\mu_1}(\alpha x) + ... + T_{\mu_m}(\alpha x)) = \tfrac{1}{m} \alpha p(T_{\mu_1} x + ... + T_{\mu_m} x).$$

Thus, $q(\alpha x) = \alpha q(x)$ for $\alpha \in \mathbb{R}^+$ and $x \in E$. To see that $q(x+y) \leq q(x) + q(y)$, let $x, y \in E$ and $\epsilon > 0$ be given. Choose probability measures $\mu_1, ..., \mu_m, \nu_1, ..., \nu_n$ on K with compact support such that

$$\frac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x) \leq q(x) + \epsilon,$$

and

$$\frac{1}{n} p(T_{\nu_1} x + ... + T_{\nu_n} x) \leq q(y) + \epsilon.$$

Consider the set $\mathcal{K} = \{\nu_j * \mu_i \mid j = 1, ..., n,\ i = 1, ..., m\}$. Then

$$\begin{aligned} \tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} x\Big] &= \tfrac{1}{nm} p\big[\textstyle\sum_{j=1}^n T_{\nu_j}(\sum_{i=1}^m T_{\mu_i} x)\big] \\ &\leq \tfrac{1}{nm} \textstyle\sum_{j=1}^n p\big[T_{\nu_j}(\sum_{i=1}^m T_{\mu_i} x)\big] \\ &\leq \tfrac{1}{nm} \textstyle\sum_{j=1}^n p\big[\sum_{i=1}^m T_{\mu_i} x\big] \\ &= \tfrac{1}{m} p\big[\textstyle\sum_{i=1}^m T_{\mu_i} x\big] \\ &\leq q(x) + \epsilon, \end{aligned}$$

and similarly, $\frac{1}{nm} p\Big[\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} y\Big] \leq q(y) + \epsilon$. Hence,

$$\begin{aligned} &\tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i}(x+y)\Big] \\ &= \tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} x + \sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} y\Big] \\ &\leq \tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} x\Big] + \tfrac{1}{nm} p\Big[\textstyle\sum_{j=1}^n \sum_{i=1}^m T_{\nu_j * \mu_i} y\Big] \\ &\leq q(x) + q(y) + 2\epsilon. \end{aligned}$$

Therefore,

$$q(x+y) \leq q(x) + q(y).$$

For $\mu \in M_1^c(K)$, $x \in E$ and $m \in \mathbb{N}$,

$$\begin{aligned} &\tfrac{1}{m} p(T_{\mu_1} T_\mu x + ... + T_{\mu_m} T_\mu x) \\ &= \tfrac{1}{m} p(T_\mu T_{\mu_1} x + ... + T_\mu T_{\mu_m} x) \\ &\leq \tfrac{1}{m} p(T_{\mu_1} x + ... + T_{\mu_m} x). \end{aligned}$$

Hence, $q(T_\mu x) \leq q(x)$. Furthermore, for each $m \in \mathbb{N}$

$$\tfrac{1}{m}p(T_{\mu_1}x + ... + T_{\mu_m}x) \quad \leq \tfrac{1}{m}[p(T_{\mu_1}x) + ... + p(T_{\mu_m}x)] \leq p(x).$$

Thus, $q(x) \leq p(x)$. By Hahn-Banach extension theorem there is a continuous linear functional $\tilde{\phi}$ on E such that $\tilde{\phi}(x) \leq q(x)$ for each $x \in E$ and $\tilde{\phi}|_F \equiv \phi$. For $x \in E$, $n \in \mathbb{N}$ and $\mu \in M_1^c(K)$

$$\begin{aligned}
&q(x - T_\mu x)\\
&\leq \tfrac{1}{n+1}p\Big[\Big(T_e(x - T_\mu x) + T_\mu(x - T_\mu x)\\
&+T_\mu T_\mu(x - T_\mu x) + ... + \underbrace{T_\mu T_\mu ... T_\mu}_{\text{n times}}(x - T_\mu x)\Big)\Big]\\
&= \tfrac{1}{n+1}p(x + \underbrace{T_\mu T_\mu ... T_\mu}_{\text{n+1 times}}(-x))\\
&\leq \tfrac{1}{n+1}[p(x) + p(-x)] \to 0.
\end{aligned}$$

Therefore, $\tilde{\phi}(x - T_\mu x) \leq q(x - T_\mu x) \leq 0$. Since $\tilde{\phi}$ is linear By replacing x by $-x$, one has $\tilde{\phi}(T_\mu x) = \tilde{\phi}(x)$. In particular, $\tilde{\phi}(T_g x) = \tilde{\phi}(x)$ for $g \in K$ and $x \in E$. Therefore, $UC(K)$ has an invariant mean (Theorem 4.1). □

5. Weak*-invariant complemented subspaces of $L_\infty(K)$

Let X be a weak*-closed left translation invariant, invariant complemented subspace of $L_\infty(K)$. Then this section provides a connection between X being invariantly complemented in $L_\infty(K)$ by a weak*-weak*-continuous projection and the behavior of $X \cap C_0(K)$.

Theorem 5.1. *Let X be a weak*-closed, left translation invariant, invariant complemented subspace of $L_\infty(K)$. Then the following are equivalent:*

1. *There exists a weak*-weak*-continuous projection Q from $L_\infty(K)$ onto X commuting with left translations.*
2. *$X \cap C_0(K)$ is weak* dense in X.*

Proof. Let P be a continuous projection from $L_\infty(K)$ onto X commuting with left translations. We first observe that $P(LUC(K)) \subseteq LUC(K)$. In fact if $f \in LUC(K)$ and $\{g_\alpha\}$ is a net in K such that $g_\alpha \to g \in K$, then

$$||L_{g_\alpha}Pf - L_gPf|| = ||P(L_{g_\alpha}f - L_gf)|| \leq ||P||\ ||L_{g_\alpha}f - L_gf|| \to 0.$$

Thus, $P|_{C_0(K)}$ is a bounded operator from $C_0(K)$ into $CB(K)$. Define a bounded linear functional on $C_0(K)$ by $\psi_1(f) := (P\check{f})(e)$. Let $\mu \in M(K)$ be such that $(Pf)(e) = \int \check{f}(x)d\mu(x)$, for each $f \in C_0(K)$. Then for $x \in K$ and $f \in C_0(K)$,

$$(Pf)(x) = L_xPf(e) = PL_xf(e) = \int L_xf(\check{y})d\mu(y) = f * \mu(x).$$

Hence, $P(f) = f * \mu$, for $f \in C_0(K)$. Define an operator $T : L_1(K) \to L_1(K)$ via $T(h) := h * \check{\mu}$.

Then $Q = T^*$ is weak*-weak*-continuous and $< Qf, h >=< f, h * \check{\mu} >=< f * \mu, h >$, for $h \in L_1(K)$ and $f \in C_0(K)$. Thus, $Q(f) = f * \mu$ for $f \in C_0(K)$. In addition, Q commutes with left translations on $L_\infty(K)$, since for $h \in L_1(K)$ and $f \in L_\infty(K)$

$$\begin{aligned} < QL_x f, h > &=< L_x f, h * \check{\mu} > \\ &=< f, (L_{\check{x}} h) * \check{\mu} > \\ &=< Q(f), L_{\check{x}} h > \\ &=< L_x Q(f), h > . \end{aligned}$$

We will show that Q is a projection. For $f \in C_0(K) \cap X$, and $h \in L_1(K)$,

$$\begin{aligned} < f * \mu, h > &= [(f * \mu) * \check{h}](e) \\ &= [f * (h * \check{\mu})\check{}\,](e) \\ &= [(h * \check{\mu}) * \check{f}](e) \\ &= \textstyle\int (h * \check{\mu})(x) \check{f}(\check{x}) dx \\ &=< f, h * \check{\mu} > . \end{aligned}$$

Hence,

$$< Q(f), h > \quad =< f, h * \check{\mu} >=< f * \mu, h >=< P(f), h >=< f, h > .$$

If $X \cap C_0(K)$ is weak* dense in X, let $\{f_\alpha\}$ be a net in $X \cap C_0(K)$ such that $f_\alpha \to f$ in the weak*-topology of $L_\infty(K)$. Then, $Q(f) = f$ since Q is weak*-continuous.

Moreover, for $f \in C_0(K)$ and $h \in X^\perp$,

$$< Q(f), h > \quad =< f, h * \check{\mu} >=< f * \mu, h >=< P(f), h >= 0.$$

Thus, $< Q(f), h >= 0$, for each $f \in L_\infty(K)$ and $h \in X^\perp$, since $C_0(K)$ is weak*-dense in $L_\infty(K)$. i.e. $Q(f) \in X$.

Conversely, if Q is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations, then there exists some $\mu \in M(K)$ such that $Q^*|_{L_1(K)}(h) = h * \mu$, for $h \in L_1(K)$ ([1], Theorem 1.6.24). Hence, for $f \in C_0(K)$ we have $Q(f) = f * \check{\mu}$ which is in $C_0(K) \cap X$ ([1], Theorem 1.2.16, iv). Then $C_0(K) \cap X$ is weak*-dense in $X = \{Q(f) \mid f \in L_\infty(K)\}$ since $C_0(K)$ is weak*-dense in $L_\infty(K)$ and Q is weak*-weak*-continuous. □

As a direct consequence of Theorem 5.1 we have the following result:

Corollary 5.2. *Let K be a compact hypergroup and let X be a weak*-closed left translation invariant subspace of $L_\infty(K)$. Then X is invariantly complemented if and only if there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations.*

Corollary 5.3. *Let K be a compact hypergroup and let X be a left translation invariant w^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB. Then X is the range of a weak*-weak*-continuous projection commuting with left translations.*

Proof. This follows from ([31], Corollary 3.13, Lemma 3.9) and Theorem 5.1. □

Corollary 5.4. *The following are equivalent:*

1. *K is compact.*

2. *K is amenable and for every weak*-closed left translation invariant, invariant complemented subspace X of $L_\infty(K)$, there exists a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations.*

Proof. If K is compact, then item 2 follows from ([31], Lemma 3.9) and ([28], Example 3.3). Conversely, consider the one-dimensional subspace $X = \mathbb{C}.1$. Then X is a weak*-closed left translation invariant, invariant complemented subspace of $L_\infty(K)$, since K is amenable. If P is a weak*-weak*-continuous projection from $L_\infty(K)$ onto $\mathbb{C}.1$ commuting with left translations, then there is some $\phi \in L_1(K)$ such that $P(f) = \delta_\phi(f)$ for $f \in L_\infty(K)$. Hence, $\delta_\phi(1) = 1$ and $< \delta_\phi, L_g f >=< \delta_\phi, f >$. i.e., $L_g\phi = \phi$, for $g \in K$. In particular, $L_g\phi(e) = \phi(g) = \phi(e)$, for all $g \in K$. Therefore, $1 = \delta_\phi(1) = \int_K \phi(g)d\lambda(g) = \phi(e)\lambda(K)$ which means that K is compact. □

Commutative hypergroups with connected dual can be found in the study of hypergroups constructed on $\mathbb{R}_+$. In fact any Sturm-Liouville hypergroup on $\mathbb{R}_+$ associated with a function $A : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying certain conditions falls in this range ([36], Theorem 4.4). If K is a commutative hypergroup, then $\widehat{K}$ carries a dual hypergroup structure if $\widehat{K}$ can be equipped with a hypergroup structure such that the functions δ_g with $\delta_g(\xi) = \xi(g)$, for $\xi \in \widehat{K}$ are characters of $\widehat{K}$ for all $g \in K$. In addition, K is said to be a Pontryagin hypergroup if $\widehat{K}$ carries a dual hypergroup structure and $\widehat{\widehat{K}}$ can be identified with K. One knows that all Bessel-Kingman hypergroups are Pontryagin hypergroup. ([35], p 483). Let $M_0(K)$ denote the class of all closed subsets of K which contain a support of a non-zero measure in $M(K)$ with the Fourier-Stieltjes transform vanishing at infinity and let $\Delta(X) = \widehat{K} \cap X$.

Lemma 5.5. *Let K be a commutative hypergroup such that the dual space $\widehat{K}$ is connected and let X be a weak*-closed translation invariant, invariant complemented subspace of $L_\infty(K)$. Then $X = L_\infty(K)$ or $C_0(K) \cap X = \{0\}$.*

Proof. Let P be a continuous projection from $L_\infty(K)$ onto X commuting with left translations. Then it follows from the proof of Theorem 5.1 that $P|_{C_0(K)}(f) = f * \mu \in C_0(K)$, for some $\mu \in M(K)$. Hence, $\hat{\mu} = (\mu * \mu)\hat{} = \hat{\mu}.\hat{\mu}$ ([12], 7.3.E). Therefore, $\hat{\mu}(\xi) = 0$ or 1, for $\xi \in \widehat{K}$. Then $\hat{\mu} \equiv 0$ or $\hat{\mu} \equiv 1$, since $\xi \mapsto \hat{\mu}(\xi)$ is continuous on $\widehat{K}$ ([12], 7.3.E) and $\widehat{K}$ is connected. Consequently, $X \cap C_0(K) = \{0\}$ or $X = L_\infty(K)$. □

Corollary 5.6. *Let K be a commutative hypergroup such that $\widehat{K}$ is connected. Then there is no non-trivial weak*-weak*-continuous projection from $L_\infty(K)$ into $L_\infty(K)$ commuting with translations.*

Proof. This follows from Theorem 5.1 and Lemma 5.5. □

Corollary 5.7. *Let K be a commutative Pontryagin hypergroup such that $\widehat{K}$ is connected. Then there is no proper weak*-closed translation invariant, invariant complemented subspace X of $L_\infty(K)$ with $\Delta(X) \in M_0(\widehat{K})$.*

Proof. This follows from Lemma 5.5. □

Corollary 5.7 has the following immediate consequence:

Corollary 5.8. *Let K be a commutative Pontryagin hypergroup such that $\widehat{K}$ is connected. Then there is no non-trivial, invariant complemented ideal I of $L_1(K)$ with $\Delta(I^{\perp}) \in M_0(\widehat{K})$.*

6. Miscellaneous Remarks and Open Problems

Let A be a closed translation invariant subalgebra of $L_\infty(K)$ containing constant functions. In what follows we provide an equivalent condition for A to possess a multiplicative left invariant mean. This equivalence is given in terms of a fixed point property which is a generalization of Mitchell fixed point theorem ([23], Theorem 1).

Definition 6.1. *Let A be a closed translation-invariant subalgebra of $L_\infty(K)$ containing constant functions. Let E be a separated locally convex topological vector space and Y be a compact subset of E. Let X be the space of all probability measures on Y. Let $\{T_g \mid g \in K\}$ be a continuous representation of K on X. Suppose that $B := \{y \in Y \mid T_g y \in Y, \forall g \in K\} \neq \emptyset$ and for each $y \in B$, define $h_{y,\phi}(g) = \phi(T_g y)$, for $g \in K$ and $\phi \in CB(Y)$. It is easy to see that $h_{y,\phi}$ is continuous and $||h_{y,\phi}|| \leq ||\phi||$. Therefore, $h_y : \phi \mapsto h_{y,\phi}$ is a bounded linear operator from $CB(Y)$ into $CB(K)$. Let $Y_1 := \{y \in B \mid h_y(CB(Y)) \subseteq A\}$.*

The family $\mathscr{T}$ is an $E-E$-representation of (K, A) on X if $B \neq \emptyset$ and $Y_1 \neq \emptyset$,

Definition 6.2. *The pair (K, A) has the common fixed point property on compacta with respect to $E-E$-representations if, for each compact subset Y of a separated locally convex topological vector space E and for each $E-E$-representation of K, A on X, there is in Y a common fixed point of the family $\mathscr{T}$.*

Remark 6.3. *Let A be a closed translation-invariant subalgebra of $L_\infty(K)$ containing constant functions. Then the following are equivalent:*

1. *A has a multiplicative left invariant mean.*
2. *The pair (K, A) has the common fixed point property on compacta with respect to $E-E$-representations.*

Proof. Let $\mathscr{T}$ be an $E-E$-representation of (K, A) on X. Then there exists an element $y \in Y$ such that $h_y(CB(Y)) \subseteq A$ and $T_g y \in Y$ for all $g \in K$. Let h_y^* be the adjoint of h_y and let m be a multiplicative left invariant mean on A. Then $< h_y^* m, 1 >= 1$, where 1 is the constant 1 function on Y. Also $h_y(f_1 f_2) = (h_{y,f_1})(h_{y,f_2})$, for $f_1, f_2 \in CB(Y)$ and $g \in K$. In addition, since m is multiplicative, $h_y^* m$ is a nonzero multiplicative linear functional on $CB(Y)$ and $< h_y^*(m), \bar{h} >= \overline{< h_y^*(m), h >}$, Thus, there exists an element $x_y \in Y$ such that $f(x_y) =< h_y^* m, f >=< m, h_{y,f} >$, for all $f \in CB(Y)$.

For each $g \in K$, define a map $\Psi_g : E^* \to CB(Y)$ via $(\Psi_g f)(z) =< f, T_g z >$, for $f \in E^*$, $z \in Y$. Then $h_{y,\Psi_g f} = L_g[h_{y,f}]$ since $f \in E^*$. Hence, $T_g x_y = x_y$, for each $g \in K$ since m is left translation invariant and E^* separates point of E.

Conversely, let $E = A^*$ and Y be the set of all multiplicative means on A. Then $X = Mean(A)$. Define $(g, m) \mapsto L_g^* m$ from $K \times Mean(A)$ into $Mean(A)$, where $Mean(A)$ has the weak*-topology of A^*. Then $\mathscr{T} = \{L_g^* \mid g \in K\}$ is a separately continuous representation of K on X. We note that each $\phi \in CB(Y)$ corresponds to an element $f_\phi \in A$ such that $\phi(m) = m(f_\phi)$, for $m \in Y$. Let $P(K) = \{g \in K \mid \delta_k * \delta_g$ is a point mass measuse, δ_{kg}, $\forall k \in K\}$, $g \in P(K)$ and $k \in K$. Then

$$\delta_{g_{L_K}}\phi(k) = \phi(L_k^*\delta_g) = \phi(\delta_{kg}) = \delta_{kg}(f_\phi) = R_g f_\phi(k).$$

Hence, $\delta_{g_{L_K}}\phi \in A$, since A is right translation invariant. i.e, $\delta_{g_{L_K}}(CB(Y)) \subseteq A$, for $g \in P(K)$. Thus, $\mathscr{T}$ is an $E-E$-representation of K, A on X. Therefore, there is some $m_0 \in Y$ such that $L_g^* m_0 = m_0$, for all $g \in K$. □

Let T be a bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$. Then T commutes with convolution from the left if $T(\phi * f) = \phi * T(f)$, for all $\phi \in L_1(K)$ and $f \in L_\infty(K)$. The following can be proved by a similar argument as in ([20], Theorem 2).

Remark 6.4. *The following are equivalent:*

1. *K is compact.*
2. *Any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with convolution from the left is weak*-weak* continuous.*

Using bounded approximate identity of $L_1(K)$, one can show that any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with convolution from the left also commutes with left translations. However, the converse is not true in general. For instance, if K is a direct product $G \times J$ of any locally compact non-discrete group G which is amenable as a discrete group and a finite hypergroup J, then for any left invariant mean m on $L_\infty(K)$ which is not topological left invariant, the operator $T(f) := m(f).1$ commutes with left translations but not with convolutions from the left.

It is important to note that in contrast to the group case, there is a class of compact commutative hypergroups for which any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ commuting with convolution is weak*-weak* continuous:

Example 6.5. *Fix $0 < a \leq \frac{1}{2}$ and let H_a be the hypergroup on $\mathbb{Z}_+ \cup \{\infty\}$ given by $\delta_m * \delta_n = \delta_{min(n,m)}$, for $m \neq n \in \mathbb{Z}_+$, $\delta_\infty * \delta_m = \delta_m * \delta_\infty = \delta_m$ and $\delta_n * \delta_n = \frac{1-2a}{1-a}\delta_n + \sum_{k=n+1}^{\infty} a^k \delta_k$ [5]. Then any bounded linear operator from $L_\infty(H_a)$ into $L_\infty(H_a)$ commuting with translations is weak*-weak* continuous.*

Proof. Let T be a bounded linear operator from $L_\infty(H_a)$ into $L_\infty(H_a)$ commuting with translations. For each $\phi \in L_1(K)$ and $n \in \mathbb{Z}_+$ define a function ϕ_n on K which coincide with ϕ on $\{0, 1, ..., n\}$ and zero otherwise. Then $||\phi_n - \phi||_1 \to 0$. In addition, for each $f \in L_\infty(K)$ we have $||T(\phi_n * f) - T(\phi * f)|| \to 0$ and $||\phi_n * Tf - \phi * Tf|| \to 0$ ([12], 6.2 C). For each $f \in L_\infty(K)$

$$\begin{aligned} T(\phi_n * f) &= T(\textstyle\sum_{k=0}^{n} \phi(k)(1-a)a^k L_{\check{k}} f) \\ &= \textstyle\sum_{k=0}^{n} \phi(k)(1-a)a^k T(L_{\check{k}} f) \\ &= \textstyle\sum_{k=0}^{n} \phi(k)(1-a)a^k L_{\check{k}} Tf \\ &= \phi_n * Tf \end{aligned}$$

we have that $T(\phi * f) = \phi * Tf$. Now the result follows from Remark 6.4. □

The following problems are still open:

Question 6.6. *Let K be a compact hypergroup such that $L_\infty(K)$ has a unique left invariant mean. Let T be a bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with left translations. Can we conclude that T commutes with convolution from the left?*

Question 6.7. *Let G be a locally compact group. Then $L_1(G)$ is Arens semi-regular if and only if G is abelian or discrete (*[21]*, Theorem 1). Can we characterize hypergroups for which $L_1(K)$ is Arens semi-regular?*

Question 6.8. *Is there any non-inner amenable hypergroup K such that $Z(L_1(K))$ is non-trivial?*

Question 6.9. *Let K be a hypergroup such that $L_1(K)$ has a positive non-trivial center. Is there a compact neighbourhood V of the identity with $\Delta(g)R_g\chi_V = L_g\chi_V$?*

Question 6.10. *Let K be a connected, inner amenable hypergroup. Is K amenable?*

We say that a hypergroup K is topologically inner amenable if there exists a mean m on $L_\infty(K)$ such that $m((\Delta\phi)\check{} * f) = m(f * \check{\phi})$ for any positive norm one element ϕ in $L_1(K)$ and any $f \in L_\infty(K)$. It is easy to see that any inner invariant mean on $UC(K)$ is topologically inner invariant since

$$\begin{aligned} m(f * \check{\phi}) &= \int < m, R_g f\phi(g) > d\lambda(g) \\ &= \int < m, L_g f\phi(g) > d\lambda(g) \\ &=< m, \int L_g f\phi(g) d\lambda(g) > \\ &=< m, \int L_g f\phi(g)\Delta(g) d\check{\lambda}(g) > \\ &= m((\Delta\phi)\check{} * f). \end{aligned}$$

. However, on the space $L_\infty(K)$ the relation between topological inner invariant means and inner invariant means is not clear.

Question 6.11. *Let m be a topological inner invariant mean on $L_\infty(K)$. Is m also an inner invariant mean?*

Question 6.12. *Let K be an inner amenable hypergroup. Is there any topological inner invariant mean on $L_\infty(K)$?*

Question 6.13. *Let K be an inner amenable hypergroup. Does K satisfy central Reiter's condition P_1? (see (*[22]*, Remark) for the group case).*

Question 6.14. *Let K be a compact hypergroup. Can we have an exact description of weak*-closed left translation invariant complemented subspaces of $L_\infty(K)$?*

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