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On a system of nonlinear partial functional differential equations of different types

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Abstract. We consider a system of a semilinear hyperbolic functional differential equation (where the lower order terms contain functional dependence on the unknown function) and a quasilinear parabolic functional differential equation with initial and boundary conditions. Existence of weak solutions for $t \in (0, T)$ and for $t \in (0, \infty)$ will be shown and some qualitative properties of the solutions in $(0, \infty)$ will be formulated.

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1. Introduction

In the present paper we consider weak solutions of the following system of equations:

$$u''(t) + Q(u(t)) + \varphi(x)h'(u(t)) + H(t, x; u, z) + \psi(x)u'(t) = F_1(t, x; z), \tag{1.1}$$

$$z'(t) - \sum_{j=1}^{n} D_j[a_j(t, x, Dz(t), z(t); u, z)] + a_0(t, x, Dz(t), z(t); u, z) = F_2(t, x; u) \tag{1.2}$$

$$(t, x) \in Q_T = (0, T) \times \Omega$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and we use the notations $u(t) = u(t, x)$, $z(t) = z(t, x)$ $u' = D_t u$, $z' = D_t z$ $u'' = D_t^2 u$, $Dz = (D_1 z, \dots, D_n z)$, Q may be e.g. a linear second order symmetric elliptic differential operator in the variable x; h is a C^2 function having certain polynomial growth, H contains nonlinear functional (nonlocal) dependence on u and z, with some polynomial growth and F_1 contains some functional dependence on z. Further, the functions a_j define a quasilinear elliptic differential operator in x (for fixed t) with functional dependence on u and z. Finally,

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F_2 may non-locally depending on u. (The system (1.1), (1.2) consists of a semilinear hyperbolic functional equation and a parabolic functional equation.)

This paper was motivated by some problems which were modelled by systems consisting of (functional) differential equations of different types. In [4] S. Cinca investigated a model, consiting of an elliptic, a parabolic and an ordinary nonlinear differential equation, which arise when modelling diffusion and transport in porous media with variable porosity. In [6] J.D. Logan, M.R. Petersen and T.S. Shores considered and numerically studied a similar system which describes reaction-mineralogy-porosity changes in porous media with one-dimensional space variable. J. H. Merkin, D.J. Needham and B.D. Sleeman considered in [7] a system, consisting of a nonlinear parabolic and an ordinary differential equation, as a mathematical model for the spread of morphogens with density dependent chemosensitivity. In [3], [8], [9] the existence of solutions of such systems were studied.

In Section 2 the existence of weak solutions will be proved for $t \in (0,T)$, in Section 3 some examples will be shown and in Section 4 we shall prove existence and certain properties of solutions for $t \in (0,\infty)$.

2. Solutions in $(0,T)$

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0,T) \times \Omega$. Denote by $W^{1,p}(\Omega)$ the Sobolev space of real valued functions with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^n |D_j u|^p + |u|^p\right) dx\right]^{1/p} \qquad (2 \le p < \infty).$$

The number q is defined by $1/p + 1/q = 1$. Further, let $V_1 \subset W^{1,2}(\Omega)$ and $V_2 \subset W^{1,p}(\Omega)$ be closed linear subspaces containing $C_0^\infty(\Omega))$, $V_j^\star$ the dual spaces of V_j, the duality between $V_j^\star$ and V_j will be denoted by $\langle\cdot,\cdot\rangle$, the scalar product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$. Finally, denote by $L^p(0,T;V_j)$ the Banach space of the set of measurable functions $u : (0,T) \to V_j$ with the norm

$$\|u\|_{L^p(0,T;V_j)} = \left[\int_0^T \|u(t)\|_{V_j}^p dt\right]^{1/p}$$

and $L^\infty(0,T;V_j)$, $L^\infty(0,T;L^2(\Omega))$ the set of measurable functions $u : (0,T) \to V_j$, $u : (0,T) \to L^2(\Omega)$, respectively, with the $L^\infty(0,T)$ norm of the functions $t \mapsto \|u(t)\|_{V_j}$, $t \mapsto \|u(t)\|_{L^2(\Omega)}$, respectively.

Now we formulate the assumptions on the functions in (1.1), (1.2).

(A_1). $Q : V_1 \to V_1^\star$ is a linear continuous operator such that

$$\langle Qu, v\rangle = \langle Qv, u\rangle, \quad \langle Qu, u\rangle \ge c_0 \|u\|_{V_1}^2$$

for all $u, v \in V_1$ with some constant $c_0 > 0$.

(A_2). $\varphi, \psi : \Omega \to \mathbb{R}$ are measurable functions satisfying

$$c_1 \le \varphi(x) \le c_2, \quad c_1 \le \psi(x) \le c_2 \text{ for a.a. } x \in \Omega$$

with some positive constants c_1, c_2.

(A_3). $h : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function satisfying

$$h(\eta) \geq 0, \quad |h''(\eta)| \leq \text{const}|\eta|^{\lambda-1} \text{ for } |\eta| > 1 \text{ where}$$
$$1 < \lambda \leq \lambda_0 = \frac{n}{n-2} \text{ if } n \geq 3, \quad 1 < \lambda < \infty \text{ if } n = 2.$$

(A_4). $H : Q_T \times L^2(Q_T) \times L^p(Q_T) \to \mathbb{R}$ is a function for which $(t,x) \mapsto H(t,x;u,z)$ is measurable for all fixed $u \in L^2(\Omega)$, $z \in L^p(Q_T)$, H has the Volterra property, i.e. for all $t \in [0,T]$, $H(t,x;u,z)$ depends only on the restriction of u and z to $(0,t)$. Further, the following inequality holds for all $t \in [0,T]$ and $u \in L^2(\Omega)$, $z \in L^p(Q_T)$:

$$\int_\Omega |H(t,x;u,z)|^2 dx \leq \text{const}\left[\|z\|^2_{L^p(Q_T)} + 1\right]\left[\int_0^t \int_\Omega h(u(\tau))dxd\tau + \int_\Omega h(u)dx\right];$$
$$\int_0^t \left[\int_\Omega |H(\tau,x;u_1,z) - H(\tau,x;u_2,z)|^2 dx\right] d\tau \leq M(K,z)\int_0^t \left[\int_\Omega |u_1 - u_2|^2 dx\right] d\tau$$
$$\text{if } \|u_j\|_{L^\infty(0,T;V_1)} \leq K$$

where for all fixed number $K > 0$, $z \mapsto M(K,z) \in \mathbb{R}^+$ is a bounded (nonlinear) operator.

Finally, $(z_k) \to z$ in $L^p(Q_T)$ implies

$$H(t,x;u_k,z_k) - H(t,x;u_k,z) \to 0 \text{ in } L^2(Q_T) \text{ uniformly if } \|u_k\|_{L^2(Q_T)} \leq \text{const}.$$

(A_5). $F_1 : Q_T \times L^p(Q_T) \to \mathbb{R}$ is a function satisfying $(t,x) \mapsto F_1(t,x;z) \in L^2(Q_T)$ for all fixed $z \in L^p(Q_T)$ and $(z_k) \to z$ in $L^p(Q_T$ implies that $F_1(t,x;z_k) \to F_1(t,x;z)$ in $L^2(Q_T)$.

Further,

$$\int_0^T \|F_1(\tau,x;z)\|^2_{L^2(\Omega)} d\tau \leq \text{const}\left[1 + \|z\|^{\beta_1}_{L^p(Q_T)}\right]$$

with some constant $\beta_1 > 0$.

(B_1) The functions

$$a_j : Q_T \times \mathbb{R}^{n+1} \times L^2(Q_T) \times L^p(Q_T) \to \mathbb{R} \quad (j = 0,1,\dots n),$$

are measurable in $(t,x) \in Q_T$ for all fixed $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, $u \in L^2(Q_T)$, $z \in L^p(Q_T)$ and continuous in $\xi \in \mathbb{R}^{n+1}$ for all fixed $u \in L^2(Q_T)$, $z \in L^p(Q_T)$ and a.a. fixed $(t,x) \in Q_T$.

Further, if $(u_k) \to u$ in $L^2(Q_T)$ and $(z_k) \to z$ in $L^p(Q_T)$ then for all $\xi \in \mathbb{R}^{n+1}$, a.a. $(t,x) \in Q_T$, for a subsequence

$$a_j(t,x,\xi;u_k,z_k) \to a_j(t,x,\xi;u,z) \quad (j = 0,1,\dots,n).,$$

(B_2) For $j = 0,1,\dots,n$

$$|a_j(t,x,\xi;u,z)| \leq g_1(u,z)|\xi|^{p-1} + [k_1(u,z)](t,x),$$

where $g_1 : L^2(Q_T) \times L^p(Q_T) \to \mathbb{R}^+$ is a bounded, continuous (nonlinear) operator,

$$k_1 : L^2(Q_T) \times L^p(Q_T) \to L^q(Q_T) \text{ is continuous and}$$
$$\|k_1(u,z)\|_{L^q(Q_T)} \leq \text{const}(1 + \|u\|^\gamma_{L^2(Q_T)} + \|z\|^{p_1}_{L^p(Q_T)})$$

with some constants $\gamma > 0$, $0 < p_1 < p - 1$.

(B_3) The following inequality holds for all $t \in [0, T]$ with some constants $c_2 > 0$, $c_3 \geq 0$, $\beta \geq 0$, $\gamma_1 \geq 0$ (not depending on t, u, z):

$$\sum_{j=0}^{n} [a_j(t, x, \xi; u, z) - a_j(t, x, \xi^\star; u, z)](\xi_j - \xi_j^\star) \geq$$

$$\frac{c_2}{1 + \|u\|^\beta_{L^2(Q_T)} + \|z\|^{\gamma_1}_{L^p(Q_T)}} |\xi - \xi^\star|^p - c_3 |\xi_0 - \xi_0^\star|^2.$$

(B_4) For all fixed $u \in L^2(Q_T)$ the function

$$F_2 : Q_T \times L^2(Q_T) \to \mathbb{R} \text{ satisfies } (t, x) \mapsto F_2(t, x; u) \in L^q(Q_T),$$

$$\|F_2(t, x; u)\|_{L^q(Q_T)} \leq \text{const} \left[1 + \|u\|^\gamma_{L^2(Q_T)}\right]$$

(see (B_2)) and

$$(u_k) \to u \text{ in } L^2(Q_T) \text{ implies } F_2(t, x; u_k) \to F_2(t, x; u) \text{ in } L^q(Q_T).$$

Finally,

$$\max\{(\beta_1\beta)/2, \gamma_1\} + \max\{(\beta_1\gamma)/2, p_1\} < p - 1.$$

Theorem 2.1. *Assume (A_1) – (A_5) and (B_1) – (B_4). Then for all $u_0 \in V_1$, $u_1 \in L^2(\Omega)$, $z_0 \in L^2(\Omega)$ there exists $u \in L^\infty(0, T; V_1)$ such that*

$$u' \in L^\infty(0, T; L^2(\Omega)), \quad u'' \in L^2(0, T; V_1^\star) \text{ and } z \in L^p(0, T; V_2), \quad z' \in L^q(0, T; V_2^\star)$$

such that u satisfies (1.1) in the sense: for a.a. $t \in [0, T]$, all $v \in V_1$

$$\langle u''(t), v\rangle + \langle Q(u(t)), v\rangle + \int_\Omega \varphi(x) h'(u(t)) v dx + \int_\Omega H(t, x; u, z) v dx + \tag{2.1}$$

$$\int_\Omega \psi(x) u'(t) v dx = \int_\Omega F_1(t, x; z) v) dx$$

and the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.2}$$

Further, u, z satisfy (1.2) in the sense: for a.a. $t \in (0, T)$, all $w \in V_2$

$$\langle z'(t), w\rangle + \int_\Omega \left[\sum_{j=1}^{n} a_j(t, x, Dz(t), z(t); u, z)\right] D_j w dx + \tag{2.3}$$

$$\int_\Omega a_0(t, x, Dz(t), z(t); u, z) w dx = \int_\Omega F_2(t, x; u) w dx \text{ and}$$

$$z(0) = z_0. \tag{2.4}$$

Proof. The proof is based on the results of [11], the theory of monotone operators (see, e.g. [13]) and Schauder's fixed point theorem as follows.

Consider the problem (2.1), (2.2) for u with an arbitrary fixed $z = \tilde{z} \in L^p(Q_T)$. According to [11] assumptions (A_1) – (A_5) imply that there exists a unique solution $u = \tilde{u} \in L^\infty(0, T; V_1)$ with the properties $\tilde{u}' \in L^\infty(0, T; L^2(\Omega))$, $\tilde{u}'' \in L^2(0, T; V_1^\star)$ satisfying (2.1) and the initial condition (2.2). Then consider problem (2.3) (2.4) for

z with the above $u = \tilde{u}$ and with $z = \tilde{z}$ functional terms (see (2.6)). According to the theory of monotone operators (see, e.g., [13]) there exists a unique solution $z \in L^p(0,T;V_2)$ of (2.3), (2.4) such that $z' \in L^q(0,T;V_2^\star)$. By using the notation $S(\tilde{z}) = z$, we shall show that the operator $S : L^p(Q_T) \to L^p(Q_T)$ satisfies the assumptions of Schauder's fixed point theorem: it is continuous, compact and there exists a closed ball $B_0(R) \subset L^p(Q_T)$ such that

$$S(B_0(R)) \subset B_0(R). \tag{2.5}$$

Then Schauder's fixed point theorem will imply that S has a fixed point $z^\star \in L^p(0,T;V_2)$. Defining $u^\star$ by the solution of (2.1), (2.2) with $z = z^\star$, functions $u^\star$, $z^\star$ satisfy (2.1) – (2.4).

Lemma 2.2. *Consider problem (2.1), (2.2) for u with an arbitrary fixed $z = \tilde{z} \in L^p(Q_T)$. Assumptions (A_1) – (A_5) imply that there exists a unique $u = \tilde{u} \in L^\infty(0,T;V_1)$ such that $\tilde{u}' \in L^\infty(0,T;L^2(\Omega))$, $\tilde{u}'' \in L^2(0,T;V_1^\star)$ and (2.1), (2.2) are satisfied.*

Lemma 2.2 directly follows from Theorem 4.1 of [11].

Lemma 2.3. *Consider the following modification of problem (2.3), (2.4) with arbitrary fixed $\tilde{u} \in L^2(Q_T)$, $\tilde{z} \in L^p(Q_T)$: find $z \in L^p(0,T;V_2)$ such that $z' \in L^q(0,T;V_2^\star)$ and for a.a. $t \in [0,T]$, all $w \in V_2$*

$$\langle z'(t), w\rangle + \int_\Omega \left[\sum_{j=1}^n a_j(t,x,Dz(t),z(t);\tilde{u},\tilde{z})\right] D_j w dx + \tag{2.6}$$

$$\int_\Omega a_0(t,x,Dz(t),z(t);\tilde{u},\tilde{z}) w dx = \int_\Omega F_2(t,x;\tilde{u}) w dx,$$

$$z(0) = z_0. \tag{2.7}$$

Assumptions (B_1) – (B_4) imply that there exists a unique solution of (2.6), (2.7).

Proof. Let $a > 0$ be a fixed constant. A function z is a solution of (1.2), (2.4) if and only if $\hat{z}(t) = \mathrm{e}^{-at} z(t)$ satisfies

$$\hat{z}'(t) - \mathrm{e}^{-at} \sum_{j=1}^n D_j[a_j(t,x,\mathrm{e}^{at} D\hat{z}(t), \mathrm{e}^{at}\hat{z}(t);\tilde{u},\tilde{z})] + \tag{2.8}$$

$$\mathrm{e}^{-at} a_0(t,x,\mathrm{e}^{at} D\hat{z}(t), \mathrm{e}^{at}\hat{z}(t);\tilde{u},\tilde{z}) + a\hat{z}(t) = \mathrm{e}^{-at} F_2(t,x;\tilde{u}),$$

$$\hat{z}(0) = z_0. \tag{2.9}$$

We shall apply the theory of monotone operators to (2.8), (2.9) with sufficiently large $a > 0$.

Define (with fixed $\tilde{u} \in L^2(Q_T)$, $\tilde{z} \in L^p(Q_T)$, $t \in [0,T]$) operator $\hat{A}_{\tilde{u},\tilde{z}}$ by

$$\langle \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}), w\rangle = \int_\Omega \mathrm{e}^{-at} \sum_{j=1}^n a_j(t,x,\mathrm{e}^{at} D\hat{z}(t), \mathrm{e}^{at}\hat{z}(t);\tilde{u},\tilde{z}) D_j w dx +$$

$$\int_\Omega \mathrm{e}^{-at} a_0(t,x,\mathrm{e}^{at} D\hat{z}(t), \mathrm{e}^{at}\hat{z}(t);\tilde{u},\tilde{z}) w dx + a\int_\Omega \hat{z} w dx,$$

$$\hat{z} \in L^p(0,T;V_2), \quad w \in V_2.$$

By (B_1), (B_2) operator $\hat{A}_{\tilde{u},\tilde{z}} : L^p(0,T;V_2) \to L^q(0,T;V_2^\star)$ is bounded and demicontinuous (see, e.g. [13]). Further, it is uniformly monotone if $a > 0$ is sufficiently large.

Indeed, by (B_3), for arbitrary $\hat{z}_1, \hat{z}_2 \in L^p(0,T;V_2)$

$$\int_0^T \langle \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}_1) - \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}_2), \hat{z}_1 - \hat{z}_2 \rangle dt = \tag{2.10}$$

$$\int_{Q_T} \mathrm{e}^{-2at} \sum_{j=1}^n [a_j(t,x,\mathrm{e}^{at} D\hat{z}_1(t), \mathrm{e}^{at}\hat{z}_1(t); \tilde{u}, \tilde{z}) -$$
$$a_j(t,x,\mathrm{e}^{at} D\hat{z}_2(t), \mathrm{e}^{at}\hat{z}_2(t); \tilde{u}, \tilde{z})] \mathrm{e}^{at} D_j(\hat{z}_1 - z_2) dt dx +$$
$$\int_{Q_T} \mathrm{e}^{-2at} [a_0(t,x,\mathrm{e}^{at} D\hat{z}_1(t), \mathrm{e}^{at}\hat{z}_1(t); \tilde{u}, \tilde{z}) -$$
$$a_0(t,x,\mathrm{e}^{at} D\hat{z}_2(t), \mathrm{e}^{at}\hat{z}_2(t); \tilde{u}, \tilde{z})] \mathrm{e}^{at} (\hat{z}_1 - \hat{z}_2) dt dx \geq$$
$$\frac{c_2}{1 + \|\tilde{u}\|^\beta_{L^2(Q_T)} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}} \int_{Q_T} \mathrm{e}^{-2at} [\mathrm{e}^{at} |D\hat{z}_1 - D\hat{z}_2|^p + \mathrm{e}^{at} |\hat{z}_1 - \hat{z}_2|^p] dt dx -$$
$$c_3 \int_{Q_T} |\hat{z}_1 - \hat{z}_2|^2 dt dx + a \int_{Q_T} |\hat{z}_1 - \hat{z}_2|^2 dt dx \geq$$
$$\frac{c_2'}{1 + \|\tilde{u}\|^\beta_{L^2(Q_T)} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}} \int_{Q_T} [|D\hat{z}_1 - D\hat{z}_2|^p + |\hat{z}_1 - \hat{z}_2|^p] dt dx$$

with some constant $c_2' > 0$ (depending on T) if $a > 0$ is sufficiently large.

Consequently, according to the theory of monotone operators (see, e.g. [13]) problem (2.8), (2.9) for $\hat{z}$ has a unique weak solution, thus (2.6), (2.7) has a unique solution.

By using Lemmas 2.2, 2.3 we may define operator $S : L^p(Q_T) \to L^p(Q_T)$ as follows. Let $\tilde{z} \in L^p(Q_T)$ be an arbitrary element. By Lemma 2.2 there exists a unique solution $\tilde{u}$ of (2.1), (2.2). According to Lemma 2.3 there exists a unique solution z of (2.6), (2.7). Operator S is defined by $S(\tilde{z}) = z$.

Lemma 2.4. *The operator $S : L^p(Q_T) \to L^p(Q_T)$ is compact.*

Proof. Let $(\tilde{z}_k)$ be a bounded sequence in $L^p(Q_T)$ and consider the (unique) solution $\tilde{u}_k$ of (2.1), (2.2) with fixed $z = \tilde{z}_k$. We show that $(\tilde{u}_k)$ is bounded in $L^\infty(0,T;V_1)$ and $(\tilde{u}_k')$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Indeed, applying the arguments in the proof of Theorem 2.1 in [11], one gets the solutions $\tilde{u}_k$ of (2.1), (2.2) as the (weak) limit of Galerkin approximations

$$\tilde{u}_{mk}(t) = \sum_{l=1}^m g^k_{lm}(t) w_l \text{ where } g^k_{lm} \in W^{2,2}(0,T)$$

and $w_1, w_2, \ldots$ is a linearly independent system in V_1 such that the linear combinations are dense in V_1, further, the functions $\tilde{u}_{mk}$ satisfy (for $j = 1, \ldots, m$)

$$\langle \tilde{u}''_{mk}(t), w_j \rangle + \langle Q(\tilde{u}_{mk}(t)), w_j \rangle + \int_\Omega \varphi(x) h'(\tilde{u}_{mk}(t)) w_j dx + \tag{2.11}$$

$$\int_\Omega H(t,x;\tilde{u}_{mk},\tilde{z}_k)w_j dx + \int_\Omega \psi(x)\tilde{u}'_{mk}(t)w_j dx = \int_\Omega F_1(t,x;\tilde{z}_k)w_j dx,$$

$$\tilde{u}_{mk}(0) = u_{m0}, \quad \tilde{u}'_{mk}(0) = u_{m1} \tag{2.12}$$

where u_{m0}, u_{m1} $(m = 1, 2, \dots)$ are linear combinations of $w_1, w_2, \dots w_m$, satisfying $(u_{m0}) \to u_0$ in V_1 and $(u_{m1}) \to u_1$ in $L^2(\Omega)$ as $m \to \infty$.

Multiplying (2.11) by $(g^k_{lm})'(t)$, summing with respect to j and integrating over $(0,t)$, by Young's inequality we find

$$\frac{1}{2}\|\tilde{u}'_{mk}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\langle Q(\tilde{u}_{mk}(t)), \tilde{u}_{mk}(t)\rangle + \int_\Omega \varphi(x)h(\tilde{u}_{mk}(t))dx + \tag{2.13}$$

$$\int_0^t \left[\int_\Omega H(\tau,x;\tilde{u}_{mk},\tilde{z}_k)\tilde{u}'_{mk}(\tau)dx\right]d\tau + \int_0^t \left[\int_\Omega \psi(x)|\tilde{u}'_{mk}(\tau)|^2 dx\right]d\tau =$$

$$\int_0^t \left[\int_\Omega F_1(\tau,x;\tilde{z}_k)\tilde{u}'_{mk}(\tau)dx\right]d\tau + \frac{1}{2}\|\tilde{u}'_{mk}(0)\|^2_H + \frac{1}{2}\langle Q(\tilde{u}_{mk}(0)), \tilde{u}_{mk}(0)\rangle +$$

$$\int_\Omega \varphi(x)h(\tilde{u}_{mk}(0))dx \le \frac{1}{2}\int_0^T \|F_1(\tau,x;\tilde{z}_k)\|^2_{L^2(\Omega)}d\tau + \frac{1}{2}\int_0^T \|\tilde{u}'_{mk}(\tau)\|^2_{L^2(\Omega)} + \text{const}$$

where the constant is not depending on m, k, t. (See [11].)

By using (A_2), (A_4), (A_5) and the Cauchy-Schwarz inequality, we obtain from (2.13)

$$\frac{1}{2}\|\tilde{u}'_{mk}(t)\|^2_{L^2(\Omega)} + \frac{c_0}{2}\|\tilde{u}_{mk}(t))\|^2_{V_1} + c_1\int_\Omega h(\tilde{u}_{mk}(t))dx \le \tag{2.14}$$

$$\int_0^T \|F_1(\tau,x;\tilde{z}_k)\|^2_{L^2(\Omega)}d\tau +$$

$$\text{const}\left\{1 + \int_0^t \|\tilde{u}'_{mk}(\tau)\|^2_{L^2(\Omega)}d\tau + \int_0^t \left[\int_\Omega h(\tilde{u}_{mk}(\tau))dx\right]d\tau\right\}.$$

Consequently,

$$\|\tilde{u}'_{mk}(t)\|^2_{L^2(\Omega)} + \int_\Omega h(\tilde{u}_{mk}(t))dx \le$$

$$\text{const}\left\{1 + \int_0^t [\|\tilde{u}'_{mk}(\tau)\|^2_{L^2(\Omega)} + \int_\Omega h(\tilde{u}_{mk}(\tau))dx]\right\}$$

where the constant is not depending on k, m, t. Thus by Gronwall's lemma

$$\|\tilde{u}'_{mk}(t)\|^2_{L^2(\Omega)} + \int_\Omega h(\tilde{u}_{mk}(t))dx \le \text{const} \tag{2.15}$$

and so by (A_1) and (2.14)

$$\|\tilde{u}_{mk}(t)\|_{V_1} \le \text{const} \tag{2.16}$$

where the constants are not depending on k, m, t. The inequalities (2.15), (2.16) imply that the weak limits $\tilde{u}_k$, $\tilde{u}'_k$ of $(\tilde{u}_{mk})$ and $(\tilde{u}'_{mk})$, respectively, are bounded in $L^\infty(0,T;V_1)$, $L^\infty(0,T;L^2(\Omega))$, respectively.

Consequently, by the well known compact imbedding theorem (see [5]) there is a subsequence of $(\tilde{u}_k)$, again denoted by $(\tilde{u}_k)$, for simplicity, which is convergent in $L^2(Q_T)$ to some $\tilde{u}$ and $(\tilde{u}_k) \to \tilde{u}$ a.e. in Q_T.

Consider the sequence of solutions z_k of (2.6) (2.7) with $\tilde{u} = \tilde{u}_k$, $\tilde{z} = \tilde{z}_k$. We show that the sequence z_k is bounded in $L^p(0,T;V_2)$. Indeed, for the functions $\hat{z}_k = \mathrm{e}^{-at} z_k$ we have

$$\langle \hat{z}_k', w\rangle + \langle \hat{A}_{\tilde{u}_k,\tilde{z}_k}(\hat{z}_k), w\rangle = \langle \mathrm{e}^{-at} F_2(t,x;\tilde{u}_k), w\rangle, \tag{2.17}$$

thus, integrating (2.17) over $(0,T)$ with $w = \hat{z}_k$ one obtains

$$\frac{1}{2}\|\hat{z}_k(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\hat{z}_k(0)\|^2_{L^2(\Omega)} + \int_0^T \langle \hat{A}_{\tilde{u}_k,\tilde{z}_k}(\hat{z}_k), \hat{z}_k\rangle dt = \tag{2.18}$$

$$\int_0^T \langle \mathrm{e}^{-at} F_2(t,x;\tilde{u}_k), w\rangle dt.$$

Applying the inequality (2.10) to $\hat{z}_1 = \hat{z}_k$ and $\hat{z}_2 = 0$, we obtain

$$\frac{\text{const}}{1 + \|\tilde{u}_k\|^{\beta}_{L^2(Q_T)} + \|\tilde{z}_k\|^{\gamma_1}_{L^p(Q_T)}} \int_{Q_T} [|D\hat{z}_k|^p + |\hat{z}_k|^p] dt \le \tag{2.19}$$

$$\int_0^T \langle \hat{A}_{\tilde{u}_k,\tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}_k,\tilde{z}_k}(0), \hat{z}_k - 0\rangle dt =$$

$$\int_0^T \langle \hat{A}_{\tilde{u}_k,\tilde{z}_k}(\hat{z}_k), \hat{z}_k\rangle dt - \int_0^T \hat{A}_{\tilde{u}_k,\tilde{z}_k}(0), \hat{z}_k\rangle dt.$$

By (2.18)

$$\left|\int_0^T \langle \hat{A}_{\tilde{u}_k,\tilde{z}_k}(\hat{z}_k), \hat{z}_k\rangle dt\right| \le \left|\int_0^T \langle \mathrm{e}^{-at} F_2(t,x;\tilde{u}_k), w\rangle dt\right| + \text{const} \le \tag{2.20}$$

$$\text{const}\|F_2(t,x;\tilde{u}_k)\|_{L^q(Q_T)} \|\hat{z}_k\|_{L^p(Q_T)}$$

and by (B_2)

$$\left|\int_0^T \hat{A}_{\tilde{u}_k,\tilde{z}_k}(0), \hat{z}_k\rangle dt\right| \le \text{const}\|\hat{z}_k\|_{L^p(Q_T)} \tag{2.21}$$

Hence by (2.19), (2.20), (B_4), $(\hat{z}_k)$ is bounded in $L^p(0,T;V_2)$ (as $p > 1$ and $\|\tilde{u}_k\|_{L^2(Q_T)}$, $\|\tilde{z}_k\|_{L^p(Q_T)}$ are bounded).

Further, the equality (2.17) implies that $(\hat{z}_k')$ is bounded in $L^q(0,T;V_2^\star)$. So by the well known compact imbedding theorem (see [5]) there is a subsequence of $(\hat{z}_k)$ which is convergent in $L^p(Q_T)$. Therefore, the corresponding subsequence of (z_k) is convergent, too in $L^p(Q_T)$.

Lemma 2.5. *The operator $S : L^p(Q_T) \to L^p(Q_T)$ is continuous.*

Proof. Assume that

$$(\tilde{z}_k) \to \tilde{z} \text{ in } L^p(Q_T). \tag{2.22}$$

Now we show that for the solutions $\tilde{u}_k$ of (2.1), (2.2) with $z = \tilde{z}_k$

$$(\tilde{u}_k) \to \tilde{u} \text{ in } L^2(Q_T) \tag{2.23}$$

and a.e. in Q_T for a subsequence where $\tilde{u}$ is the solution of (2.1), (2.2) with $z = \tilde{z}$.

In the proof of (2.23) we use the (uniqueness) Theorem 4.1 of [11]. Since $(\tilde{z}_k)$ is bounded in $L^p(0,T;V_2)$, $(\tilde{u}_k)$ is bounded in $L^2(Q_T)$ (see the proof of Lemma 2.4).

Further, $\tilde{u}$ and $\tilde{u}_k$ are weak solutions of (1.1) (i.e. of (2.1) with $z = \tilde{z}$ and $z = \tilde{z}_k$, respectively and satisfy the initial conditions (2.2), thus

$$\tilde{u}''(t) + Q(\tilde{u}(t)) + \varphi(x)h'(\tilde{u}(t)) + H(t, x; \tilde{u}, \tilde{z}) + \psi(x)\tilde{u}'(t) = F_1(t, x; \tilde{z}), \tag{2.24}$$

$$\tilde{u}_k''(t) + Q(\tilde{u}_k(t)) + \varphi(x)h'(\tilde{u}_k(t)) + H(t, x; \tilde{u}_k, \tilde{z}) + \psi(x)\tilde{u}_k'(t) = F_1(t, x; \tilde{z}_k) + H(t, x; \tilde{u}_k, \tilde{z}) - H(t, x; \tilde{u}_k, \tilde{z}_k). \tag{2.25}$$

Theorem 4.1 of [11] implies that for the solutions $\tilde{u}$ of (2.24) and $\tilde{u}_k$ of (2.25) we have for any $s \in [0, T]$ an estimation of the form

$$\|\tilde{u}_k(s) - \tilde{u}(s)\|^2_{L^2(\Omega)} \le \text{const} \int_{Q_T} \left| \int_0^t [F_1(\tau, x; \tilde{z}_k) - F_1(\tau, x; \tilde{z})] d\tau \right|^2 dtdx + \text{const} \int_{Q_T} \left| \int_0^t [H(\tau, x; \tilde{u}_k, \tilde{z}_k) - H(\tau, x; \tilde{u}_k, \tilde{z})] d\tau \right|^2 dtdx$$

where the right hand side is converging to 0 as $k \to \infty$ by (A_4), (A_5).

So we have proved (2.23).

Now we show that (2.22), (2.23) imply:

$$(z_k) \to z \text{ in } L^p(Q_T), \text{ i.e. } (\hat{z}_k) \to \hat{z} \text{ in } L^p(Q_T) \tag{2.26}$$

for the solutions of (2.6), (2.7) and (2.8), (2.9), respectively (in the case of $z_k, \hat{z}_k$, instead of $\tilde{u}, \tilde{z}$ we have $\tilde{u}_k, \tilde{z}_k$). Since

$$\langle (\hat{z}_k - \hat{z})', \hat{z}_k - \hat{z} \rangle + \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle = \langle \mathrm{e}^{-at} F_2(t, x; \tilde{u}_k) - \mathrm{e}^{-at} F_2(t, x; \tilde{u}), \hat{z}_k - \hat{z} \rangle,$$

integrating over $(0, T)$ with respect to t, we find

$$\frac{1}{2}\|\hat{z}_k(T) - \hat{z}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\hat{z}_k(0) - \hat{z}(0)\|^2_{L^2(\Omega)} + \int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt = \int_0^T \langle \mathrm{e}^{-at} F_2(t, x; \tilde{u}_k) - \mathrm{e}^{-at} F_2(t, x; \tilde{u}), \hat{z}_k - \hat{z} \rangle dt \tag{2.27}$$

where by (2.10)

$$\int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt = \int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt + \int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt \ge \frac{c_2'}{1 + \|\tilde{u}_k\|^{\beta}_{L^2(Q_T)} + \|\tilde{z}_k\|^{\gamma_1}_{L^p(Q_T)}} \int_{Q_T} [|D\hat{z}_k - D\hat{z}|^p + |\hat{z}_k - \hat{z}|^p] dtdx + \int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt. \tag{2.28}$$

By (2.22), (B_1), (B_2), Vitali's theorem and Hölder's inequality

$$\lim_{k\to\infty}\int_0^T \langle \hat{A}_{\tilde{u}_k,\tilde{z}_k}(\hat{z}) - \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}), \hat{z}_k - \hat{z}\rangle dt = 0 \tag{2.29}$$

as $\|\hat{z}_k - \hat{z}\|_{L^p(Q_T)}$ is bounded. Similarly, the right hand side of (2.27) is coverging to 0 by (B_4). Therefore, (2.27) – (2.29) imply (2.26).

Lemma 2.6. *There is a closed ball*

$$\overline{B_R(0)} = \{z \in L^p(Q_T) : \|z\|_{L^p(Q_T)} \le R\}$$

such that $S(\overline{B_R(0)}) \subset \overline{B_R(0)}$.

Proof. According to (2.14) we have for the sequence $(\tilde{u}_m)$ of Galerkin approximation of the solution of (2.1), (2.2) (with $z = \tilde{z}$)

$$\frac{1}{2}\|\tilde{u}_m'(t)\|^2_{L^2(\Omega)} + \frac{c_0}{2}\|\tilde{u}_m(t)\|^2_{V_1} + c_1\int_\Omega h(\tilde{u}_m(t))dx \le \tag{2.30}$$

$$\frac{1}{2}\int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau + \text{const}\int_0^t \|\tilde{u}_m'(\tau)\|^2_{L^2(\Omega)}d\tau +$$

$$\int_0^t \left[\int_\Omega h(\tilde{u}_m(\tau))dx\right] d\tau + \text{const}$$

where the constants are not depending on $m, t, \tilde{z}$. Hence, by Gronwall's lemma one obtains

$$\|\tilde{u}_m'(t)\|^2_H + \int_\Omega h(\tilde{u}_m(t))dx \le \text{const}\left[1 + \int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau\right] + \tag{2.31}$$

$$\text{const}\int_0^t \left[1 + \int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau\cdot \mathrm{e}^{t-s}\right] ds =$$

$$\text{const}\left[1 + \int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau\right]$$

where the constants are independent of $m, t, \tilde{z}$. Thus by (2.30) and (A_5) we find

$$\|\tilde{u}_m(t)\|^2_{V_1} \le \text{const}\left[1 + \int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau\right] \le \text{const}\left[1 + \|\tilde{z}\|^{\beta_1}_{L^p(0,T;V_2)}\right]$$

which implies (for the solution $\tilde{u}$ of (2.1), (2.2), the limit of $(\tilde{u}_m)$)

$$\|\tilde{u}\|^2_{L^2(Q_T)} \le \text{const}\left[1 + \|\tilde{z}\|^{\beta_1}_{L^p(Q_T)}\right]. \tag{2.32}$$

On the other hand, similarly to (2.19) – (2.21), by (B_2), (B_4) we have for $\hat{z}(t) = \mathrm{e}^{-at}z(t)$ (where z is the solution of (2.3), (2.4))

$$\frac{\text{const}}{1 + \|\tilde{u}\|^{\beta}_{L^2(Q_T)} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}}\int_{Q_T}[|D\hat{z}|^p + |\hat{z}|^p]dt \le$$

$$\int_0^T \langle \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}), \hat{z}\rangle dt - \int_0^T \langle \hat{A}_{\tilde{u},\tilde{z}}(0), \hat{z}\rangle dt \le$$

$$\text{const} + \text{const}\|F_2(t,x;\tilde{u})\|_{L^q(Q_T)}\|\hat{z}\|_{L^p(Q_T)} + \text{const}\|k_1(\tilde{u},\tilde{z})\|_{L^q(Q_T)}\|\hat{z}\|_{L^p(Q_T)} \leq$$
$$\text{const} + \text{const}\left(1 + \|\tilde{u}\|^{\gamma}_{L^2(Q_T)} + \|\tilde{z}\|^{p_1}_{L^p(Q_T)}\right)\|\hat{z}\|_{L^p(Q_T)} \leq$$
$$\text{const} + \text{const}\left(1 + \|\tilde{z}\|^{\beta_1\gamma/2}_{L^p(Q_T)} + \|\tilde{z}\|^{p_1}_{L^p(Q_T)}\right)\|\hat{z}\|_{L^p(Q_T)} \leq$$
$$\tilde{c}_1 + \tilde{c}_2\left(1 + \|\tilde{z}\|^{\max\{(\beta_1\gamma)/2,p_1\}}_{L^p(Q_T)}\right)\|\hat{z}\|_{L^p(Q_T)}.$$

Thus for $\|\hat{z}\|_{L^p(Q_T)} \geq \tilde{c}_1/\tilde{c}_2$

$$\|\hat{z}\|^{p-1}_{L^p(Q_T)} \leq \text{const}\left[1 + \|\tilde{u}\|^{\beta}_{L^2(Q_T)} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}\right]\left[1 + \|\tilde{z}\|^{\max\{(\beta_1\gamma)/2,p_1\}}_{L^p(Q_T)}\right] \leq \tag{2.33}$$
$$\text{const}\left[\left(1 + \|\tilde{z}\|^{\beta_1}_{L^p(Q_T)}\right)^{\beta/2} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}\right]\cdot\left[1 + \|\tilde{z}\|^{\max\{(\beta_1\gamma)/2,p_1\}}_{L^p(Q_T)}\right] \leq$$
$$\text{const}[1 + \|\tilde{z}\|^{\delta}_{L^p(Q_T)}]$$

where

$$\delta = \max\{(\beta_1\beta)/2, \gamma_1\} + \max\{(\beta_1\gamma)/2, p_1\}. \tag{2.34}$$

By (B_4) $\delta < p - 1$, thus for sufficiently large R

$$\tilde{z} \in \overline{B_R(0)} = \left\{\tilde{z} \in L^p(Q_T), \quad \|\tilde{z}\|_{L^p(Q_T)} \leq R\right\}$$

implies

$$\|z\|_{L^p(Q_T)} \leq R, \text{ i.e. } z \in \overline{B_R(0)}.$$

(The norm of $\|z\|_{L^p(Q_T)}$ can be estimated by $\|\hat{z}\|_{L^p(Q_T)}$, multiplied by a constant.) So the proof of Lemma 2.6 is completed.

Finally, Lemmas 2.4 - 2.6 and Schauder's fixed point theorem imply that S has a fixed point and, consequently, there exists a solution of (2.1) – (2.4).

3. Examples

Let the operator Q be defined by

$$\langle Qu, v\rangle = \int_\Omega \left[\sum_{j,l=1}^n a_{jl}(x)(D_l u)(D_j v) + d(x)uv\right] dx +$$

where $a_{jl}, d \in L^\infty(\Omega)$, $a_{jl} = a_{lj}$, $\sum_{j,l=1}^n a_{jl}(x)\xi_j\xi_l \geq c_0|\xi|^2$, $d \geq c_0$ with some positive constant c_0. Then, clearly, assumption (A_1) is satisfied.

If h is a C^2 function such that $h(\eta) = |\eta|^{\lambda+1}$ if $|\eta| > 1$ then $(A_3)$is satisfied.

The condition (A_4) is satisfied e.g. if

$$H(t,x;u,z) = \chi(t,x)g_1(L_1 z)g_2(L_2 u) \text{ where } \chi \in L^\infty(Q_T),$$
$$L_1 : L^p(0,T;V_2) \to L^2(Q_T), \quad L_2 : L^2(Q_T) \to L^2(Q_T)$$

are continuous linear operators (with the Volterra property); g_1 is a globally Lipschitz bounded function, g_2 is a globally Lipschitz function. In the particular case when

$$L_2 \text{ is an } L^2(Q_T) \to L^\infty(Q_T) \text{ bounded linear operator} \tag{3.1}$$

then g_2 may be a locally Lipschitz function satisfying

$$|g_2(\eta)| \le \text{const}|\eta|^{(\lambda+1)/2} \text{ for } |\eta| > 1.$$

The operator L_2 has the property (3.1) e.g. if

$$(L_2 u)(t,x) = \int_{Q_t} \tilde{K}(t,x;\tau,\xi)u(\tau,\xi)d\tau d\xi \text{ where}$$

$$\int_{Q_T} |\tilde{K}(t,x;\tau,\xi)|^2 d\tau d\xi \le \text{const for all } (t,x) \in Q_T.$$

The operator $F_1 : Q_T \times L^p(0,T;V_2) \to \mathbb{R}$ may have the form

$$F_1(t,x;z) = f_1(t,x,L_3 z)$$

where $f_1(t,x,\mu)$ is measurable in (t,x), continuous in μ and

$$|f_1(t,x,\mu)| \le \text{const}|\mu|^{\beta_1/2} + \tilde{f}_1(t,x) \text{ where}$$

$$0 \le \beta_1 \le 2, \quad \tilde{f}_1 \in L^2(Q_T), \quad L_3 : L^p(0,T;V_2) \to L^2(Q_T)$$

is a linear continuous operator. Then (A_5) is fulfilled. In the particular case when

$$L_3 \text{ is } L^p(0,T;V_2) \to L^\infty(Q_T)$$

linear and continuous then $\beta_1 \le 2$ is not assumed.

Now we formulate examples for a_j satisfying (B_1) – (B_3):

$$a_j(t,x,\xi;u,z) = \alpha(t,x,L_4 u, L_5 z)\xi_j|\zeta|^{p-2}, \quad j = 1,\dots,n \text{ where } \zeta = (\xi_1,\dots \xi_n),$$

$\alpha(t,x,\nu_1,\nu_2)$ is measurable in (t,x), continuous in ν_1,ν_2 and satisfies

$$\frac{\text{const}}{1+|\nu_1|^\beta+|\nu_2|^{\gamma_1}} \le \alpha(t,x,\nu_1,\nu_2) \le \text{const}(1+|\nu_1|^\gamma+|\nu_2|^{p_1})$$

with some positive constants, $L_4, L_5 : L^2(Q_T) \to L^\infty(Q_T)$ are continuous linear operators,

$$a_0(t,x,\xi;u,z) = \alpha_0(t,x,L_6 u, L_7 z)\xi_0|\xi_0|^{p-2} + \alpha_1(z),$$

where $\alpha_0(t,x,\nu_1,\nu_2)$ is measurable in (t,x), continuous in ν_1,ν_2,

$$\frac{\text{const}}{1+|\nu_1|^\beta+|\nu_2|^{\gamma_1}} \le \alpha_0(t,x,\nu_1,\nu_2) \le \text{const}(1+|\nu_1|^\gamma+|\nu_2|^{p_1})$$

with some positive constants, $L_6, L_7 : L^2(Q_T) \to L^\infty(Q_T)$ are continuous linear operators and α_1 is a globally Lipschitz function. If the values of α, α_0 are between two positive constants then $L_4 - L_7$ may be $L^2(Q_T) \to L^2(Q_T)$ continuous linear operators.

Finally, the function $F_2 : Q_T \times L^2(Q_T) \to \mathbb{R}$ may have the form

$$F_2(t,x;u) = f_2(t,x,L_8 u)$$

where $f_2(t,x,\mu)$ is measurable in (t,x), continuous in μ and

$$|f_2(t,x,\mu)| \le \text{const}|\mu|^\gamma + \tilde{f}_2(t,x),$$

$$0 \le \gamma \le 1, \quad \tilde{f}_2 \in L^2(Q_T) \text{ and } L_8 : L^2(Q_T) \to L^2(Q_T)$$

is a continuous linear operator. Then (B_4) is satisfied. In the particular case when

$$L_8 \text{ is an } L^2(Q_T) \to L^\infty(Q_T) \text{ bounded linear operator}$$

then $\gamma \le 1$ is not assumed.

4. Solutions in $(0, \infty)$

Now we formulate an existence theorem with respect to solutions for $t \in (0, \infty)$. Denote by $L^p_{loc}(0, \infty; V_1)$ the set of functions $u : (0, \infty) \to V_1$ such that for each fixed finite $T > 0$, their restrictions to $(0, T)$ satisfy $u|_{(0,T)} \in L^p(0, T; V_1)$ and let $Q_\infty = (0, \infty) \times \Omega$, $L^\alpha_{loc}(Q_\infty)$ the set of functions $u : Q_\infty \to \mathbb{R}$ such that $u|_{Q_T} \in L^\alpha(Q_T)$ for any finite T.

Now we formulate assumptions on H, F_1, a_j, F_2.

$(\tilde{A}_4)$ The function $H : Q_\infty \times L^2_{loc}(Q_\infty) \times L^p_{loc}(Q_\infty) \to \mathbb{R}$ is such that for all fixed $u \in L^2_{loc}(Q_\infty)$, $z \in L^p_{loc}(Q_\infty)$ the function $(t, x) \mapsto H(t, x; u, z)$ is measurable, H has the Volterra property (see (A_4)) and for each fixed finite $T > 0$, the restriction H_T of H to $Q_T \times L^2(Q_T) \times L^p(Q_T)$ satisfies (A_4).

Remark. Since H has the Volterra property, this restriction H_T is well defined by the formula

$$H_T(t, x; \tilde{u}, \tilde{z}) = H(t, x; u, z), \quad (t, x) \in Q_T \quad \tilde{u} \in L^2(Q_T), \tilde{z} \in L^p(Q_T)$$

where $u \in L^2_{loc}(Q_\infty)$, $z \in L^p_{loc}(Q_\infty)$ may be any function satisfying $u(t, x) = \tilde{u}(t, x)$, $z(t, x) = \tilde{z}(t, x)$ for $(t, x) \in Q_T$.

$(\tilde{A}_5)$ $F_1 : Q_\infty \times L^p_{loc}(Q_\infty) \to \mathbb{R}$ has the Volterra property and for each fixed finite $T > 0$, the restriction of F_1 to $(0, T)$ satisfies (A_5).

$(\tilde{B})$ $a_j : Q_\infty \times \mathbb{R}^{n+1} \times L^2_{loc}(Q_\infty) \times L^p_{loc}(Q_\infty) \to \mathbb{R}$ $(j = 0, 1, \dots, n)$ have the Volterra property and for each finite $T > 0$, their restrictions to $(0, T)$ satisfy (B_1) – (B_3).

$(\tilde{B}_4)$ $F_2 : Q_\infty \times L^2_{loc}(Q_\infty) \to \mathbb{R}$ has the Volterra property and for each fixed finite $T > 0$, the restriction of F_2 to $(0, T)$ satisfies (B_4).

Theorem 4.1. *Assume (A_1) – (A_3), $(\tilde{A}_4)$, $(\tilde{A}_5)$, $(\tilde{B})$, $(\tilde{B}_4)$. Then for all $u_0 \in V_1$, $u_1 \in L^2(\Omega)$ there exist*

$$u \in L^\infty_{loc}(0, \infty; V_1), \quad z \in L^p_{loc}(0, \infty; V_2) \textit{ such that}$$

$$u' \in L^\infty_{loc}(0, \infty; L^2(\Omega)), \quad u'' \in L^2_{loc}(0, \infty; V_1^\star), \quad z' \in L^q_{loc}(0, \infty; V_2^\star),$$

(2.1) – (2.4) hold for a.a. $t \in (0, \infty)$ and the initial condition (2.2) is fulfilled.

Assume that the following additional conditions are satisfied: there exist H^∞, $F_1^\infty \in L^2(\Omega)$, $u_\infty \in V_1$, a bounded function $\tilde{\beta}$, belonging to $L^2(0, \infty; L^2(\Omega))$ such that

$$Q(u_\infty) = F_1^\infty - H^\infty, \tag{4.1}$$

$$|H(t, x; u, z) - H^\infty| \le \tilde{\beta}(t, x), \quad |F_1(t, x; z) - F_1^\infty(x)| \le \tilde{\beta}(t, x) \tag{4.2}$$

for all fixed $u \in L^2_{loc}(Q_\infty)$, $z \in L^p_{loc}(Q_\infty)$). Further, there exist functions

$$a_j^\infty : \Omega \times \mathbb{R}^{n+1} \times L^2(\Omega) \to \mathbb{R}, \quad j = 1, \dots, n \qquad F_2^\infty : \Omega \times L^2(\Omega) \to \mathbb{R}$$

such that for each fixed $z_0 \in V_2$, $z \in L^p_{loc}(Q_\infty)$ *and* $u \in L^2_{loc}(Q_\infty)$, $w_0 \in V_1$ *with the property*

$$\lim_{t\to\infty} \|u(t) - w_0\|_{L^2(\Omega)} = 0$$

for the functions

$$\varphi_j(t) = \|a_j(t, x, Dz_0, z_0; u, z) - a_j^\infty(x, Dz_0, z_0; w_0)\|_{L^q(\Omega)}, \quad j = 0, 1, \dots, n, \tag{4.3}$$

$$\psi(t) = \|F_2(t, x; u) - F_2^\infty(x; w_0)\|_{L^q(\Omega)} \tag{4.4}$$

we have

$$\lim_{t\to\infty} \varphi_j(t) = 0, \quad \lim_{t\to\infty} \psi(t) = 0. \tag{4.5}$$

Finally, (B_3) *is satisfied such that the following inequalities hold for all* $t > 0$ *with constants* $c_2 > 0$, $\beta > 0$, *not depending on* t*:*

$$\sum_{j=0}^n [a_j(t, x, \xi; u, z) - a_j(t, x, \xi^\star; u, z)][\xi_j - \xi_j^\star] \tag{4.6}$$

$$\frac{c_2}{1 + \|u\|^\beta_{L^2(Q_t \setminus Q_{t-a})}} |\xi - \xi^\star|^p$$

with some fixed $a > 0$ *(finite delay).*

Then for the above solutions u, z *we have*

$$u \in L^\infty(0, \infty; V_1), \tag{4.7}$$

$$\|u'(t)\|_{L^2(\Omega)} \leq const e^{-c_1 t} \tag{4.8}$$

where c_1 *is given in* (A_2) *and there exists* $w_0 \in V_1$ *such that*

$$u(T) \to w_0 \text{ in } L^2(\Omega) \text{ as } T \to \infty, \quad \|u(T) - w_0\|_{L^2(\Omega)} \leq const e^{-c_1 T} \tag{4.9}$$

and w_0 *satisfies*

$$Q(w_0) + \varphi h'(w_0) = F_1^\infty - H^\infty. \tag{4.10}$$

Finally, there exists a unique solution $z_0 \in V_2$ *of*

$$\sum_{j=1}^n \int_\Omega a_j^\infty(x, Dz_0, z_0; w_0) D_j v dx + \int_\Omega a_0^\infty(x, Dz_0, z_0; w_0) v dx = \tag{4.11}$$

$$\int_\Omega F_2^\infty(x; w_0) v dx \text{ for all } v \in V_2$$

(where w_0 *is the solution of (4.10)) and*

$$\lim_{t\to\infty} \|z(t) - z_0\|_{L^2(\Omega)} = 0, \quad \lim_{T\to\infty} \int_{T-b}^{T+b} \|z(t) - z_0\|^p_{V_2} dt = 0 \tag{4.12}$$

for arbitrary fixed $b > 0$*. If*

$$\varphi_j, \psi \in L^q(0, \infty) \text{ then } z \in L^p(0, \infty; V_2). \tag{4.13}$$

Proof. Let $(T_k)_{k\in\mathbb{N}}$ be a monotone increasing sequence, converging to $+\infty$. According to Theorem 2.1, there exist solutions u_k, z_k of (2.1) – (2.4) for $t \in (0, T_k)$. The Volterra property of H, F_1, a_j, F_2 implies that the restrictions of u_k, z_k to $t \in (0, T_l)$ with $T_l < T_k$ satisfy (2.1) – (2.4) for $t \in (0, T_l)$.

Now consider the restrictions $u_k|_{(0,T_1)}$, $z_k|_{(0,T_1)}$, $k = 2, 3, \ldots$. Applying (2.33), (2.34) and $\delta < p - 1$ to $T = T_1$ and $\tilde{z} = z_k|_{(0,T_1)}$ we obtain that the sequence

$$\left(z_k|_{(0,T_1)}\right)_{k\in\mathbb{N}} \text{ is bounded in } L^p(Q_{T_1}) \tag{4.14}$$

thus by Lemma 2.4 there is a subsequence $(z_{1k})_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ such that the sequence of restrictions $(z_{1k}|_{(0,T_1)})_{k\in\mathbb{N}}$ is convergent in $L^p(Q_{T_1})$.

Now consider the restrictions $z_{1k}|_{(0,T_2)}$ By using the above arguments, we find that there exists a subsequence $(z_{2k})_{k\in\mathbb{N}}$ of $(z_{21})_{k\in\mathbb{N}}$ such that $(z_{2k}|_{(0,T_2)})_{k\in\mathbb{N}}$ is convergent in $L^p(Q_{T_2})$.

Thus for all $l \in \mathbb{N}$ we obtain a subsequence $(z_{lk})_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ such that $(z_{lk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is convergent in $L^p(Q_{T_l})$. Then the diagonal sequence $(z_{kk})_{k\in\mathbb{N}}$ is a subsequence of $(z_k)_{k\in\mathbb{N}}$ such that for all fixed $l \in \mathbb{N}$, $(z_{kk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is convergent in $L^p(Q_{T_l})$ to some $z^\star \in L^p_{loc}(Q_\infty)$. Since z_{ll} is a fixed point of $S = S_l : L^p(Q_{T_l}) \to L^p(Q_{T_l})$ and S_l is continuous thus the limit $z^\star|_{(0,T_l)}$ in $L^p(Q_{T_l})$ of $(z_{kk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is a fixed point of $S = S_l$.

Consequently, the solutions $u_l^\star$ of (2.1), (2.2) when z is the restriction of $z^\star$ to $(0, T_l)$ and the restriction of $z^\star$ to $(0, T_l)$ satisfy (2.1) – (2.4) for $t \in (0, T_l)$. Since for $m < l$, $u_l^\star|_{(0,T_m)} = u_m^\star$ (by the Volterra property of H, F_1, a_j, F_2), we obtain $u^\star \in L^2_{loc}(Q_\infty)$ such that for all fixed l, $u^\star|_{(0,T_l)}$, $z^\star|_{(0,T_l)}$ satisfy (2.1) – (2.4) for $t \in (0, T_l)$, so the first part of Theorem 4.1 is proved.

Now assume that the additional conditions (4.1) - (4.6) are satisfied. Then we obtain (4.7) – (4.10) for $u = u^\star$, $z = z^\star$ by using the arguments of the proof of Theorem 3.2 in [11]. For convenience we formulate the main steps of the proof.

The sequence $(z_{kk})|_{k\in\mathbb{N}}$ is bounded in $L^p(0, T_l; V_2)$ for each fixed l by (2.19) – (2.21), (B_4), (4.14)), consequently, from (2.13) (with $\tilde{z}_k = z_{kk}$) we obtain for the solutions u_{kk} of (2.1), (2.2) with $\tilde{z} = z_{kk}$ (since u_{kk} is the limit of the Galerkin approximations $\tilde{u}_{mk}$)

$$\frac{1}{2}\|u'_{kk}(t)\|^2_H + \frac{1}{2}\langle Q(u_{kk}(t)), u_{kk}(t)\rangle + \int_\Omega \varphi(x) h(u_{kk}(t))dx + \tag{4.15}$$

$$\int_0^t \left[\int_\Omega \psi(x)|u'_{kk}(\tau)|^2 dx\right] d\tau + \int_0^t \left[\int_\Omega H(\tau, x; u_{kk}, z_{kk}) u'_{kk}(\tau) dx\right] d\tau =$$

$$\int_0^t \left[\int_\Omega F_1(\tau, x; z_{kk}) u'_{kk}(\tau) dx\right] d\tau + \frac{1}{2}\|u'_{kk}(0)\|^2_H + \frac{1}{2}\langle Q(u_{kk}(0)), u_{kk}(t)\rangle +$$

$$\int_\Omega \varphi(x) h(u_{kk}(0))dx$$

for all $t > 0$. Hence we find by (4.1), (4.2) and Young's inequality for $w_{kk} = u_{kk} - u_\infty$

$$\frac{1}{2}\|w'_{kk}(t)\|^2_{L^2(\Omega)} + \frac{c_0}{2}\|u_{kk}(t))\|^2_{V_1} + c_1 \int_\Omega h(u_{kk}(t))dx + \text{const} \int_0^t \left[\int_\Omega |w'_{kk}|^2 dx\right] d\tau \le \tag{4.16}$$

$$\text{const}\left\{\int_0^t \|F_1(\tau,x;z_{kk})-F_1^\infty\|_H^2 d\tau + \int_0^t \|H(\tau,x;u_{kk}z_{kk})-H^\infty\|_H^2 d\tau\right\}+$$

$$\varepsilon\int_0^t\left[\int_\Omega |w_{kk}'|^2dx\right]d\tau+\frac{1}{2}\|u_{kk}'(0)\|_H^2+\frac{1}{2}\langle Q(u_{kk}(0)),u_{kk}(0)\rangle+c_2\int_\Omega h(u_{kk}(0))dx\le$$

$$\varepsilon\int_0^t\left[\int_\Omega |w_{kk}'|^2dx\right]d\tau+\text{const}+C(\varepsilon)\|\tilde\beta\|^2_{L^2(0,\infty;H)}.$$

Choosing sufficiently small $\varepsilon > 0$, we obtain

$$\int_0^t\left[\int_\Omega |w_{kk}'|^2dx\right]d\tau\le \text{const} \tag{4.17}$$

and thus by (4.16)

$$\|u_{kk}'(t)\|^2_{L^2(\Omega)}+\tilde c\int_0^t \|u_{kk}'(\tau)\|^2_{L^2(\Omega)}d\tau\le c^\star$$

with some positive constants $\tilde c$ and $c^\star$ not depending on k and $t \in (0,\infty)$. Hence by Gronwall's lemma we obtain (4.8) and by (4.16) we find (4.7).

It is not difficult to show that

$$\|u(T_2)-u(T_1)\|_H\le\int_{T_1}^{T_2}\|u'(t)\|_H dt \tag{4.18}$$

(see [11]), thus (4.8) implies (4.9) and by $u \in L^\infty(0,\infty;V_1)$, the limit w_0 of $u(t)$ as $t\to\infty$ must belong to V_1.

In order to prove (4.10) we apply equation (1.1) to $v\chi_{T_k}(t)$ with arbitrary fixed $v \in V_1$ where $\lim_{k\to\infty}(T_k) = +\infty$ and

$$\chi_{T_k}(t)=\chi(t-T_k),\quad \chi\in C_0^\infty,\quad supp\chi\subset[0,1],\quad \int_0^1\chi(t)dt=1.$$

Then by (4.8) one obtains (4.10) as $k\to\infty$.

Now we show that there exists a unique solution $z_0 \in V_2$ of (4.11). This statement follows from the fact that the operator (applied to $z_0 \in V_2$) on the left hand side of (4.11) is bounded, demicontinuous and uniformly monotone (see, e.g. [13]) by (B_1), (B_2), (4.9), (4.5), (4.6).

Finally, we show (4.12). By (4.6) we have

$$\frac{1}{2}\frac{d}{dt}\|z(t)-z_0\|_H^2+\frac{c_2}{1+\|u\|_{L^2(Q_t\setminus Q_{t-a})}}\|z(t)-z_0\|^p_{V_2}\le \tag{4.19}$$

$$\int_\Omega\sum_{j=1}^n[a_j(t,x,Dz,z;u,z)-a_j(t,x,Dz_0,z_0;u,z)](D_jz-D_jz_0)dx+$$

$$\int_\Omega[a_0(t,x,Dz,z;u,z)-a_0(t,x,Dz_0,z_0;u,z)](z-z_0)dx=$$

$$\int_\Omega[F_2(t,x;u)-F_2^\infty(x,w_0)](z-z_0)dx-$$

$$\int_\Omega\sum_{j=1}^n[a_j(t,x,Dz_0,z_0;u,z)-a_j^\infty(x,Dz_0,z_0;w_0)](D_jz-D_jz_0)dx-$$

$$\int_\Omega [a_0(t,x,Dz_0,z_0;u,z) - a_0^\infty(t,x,Dz_0,z_0;w_0)](z-z_0)dx \le$$
$$C(\varepsilon)\|F_2(t,x;u) - F_2^\infty(x,w_0)\|_{L^q(\Omega)} + \varepsilon\|z(t)-z_0\|_{L^p(\Omega)} +$$
$$C(\varepsilon)\sum_{j=1}^n \|a_j(t,x,Dz_0,z_0;u,z) - a_j^\infty(x,Dz_0,z_0;w_0)\|^q_{L^q(\Omega)} + \varepsilon\|D_j z(t) - D_j z_0\|^p_{L^p(\Omega)} +$$
$$C(\varepsilon)\|a_0(t,x,Dz_0,z_0;u,z) - a_0^\infty(x,Dz_0,z_0;w_0)\|^q_{L^q(\Omega)} + \varepsilon\|z(t)-z_0\|^p_{L^p(\Omega)}.$$

Since $\|u\|^\beta_{L^2(Q_t\setminus Q_{t-a})}$ is bounded for $t\in(0,\infty)$ by (4.9) and

$$\|z(t)-z_0\|_{V_2} \ge \text{const}\|z(t)-z_0\|_{L^2(\Omega)}$$

with some positive constant, thus by (4.3) – (4.5), (4.19) with sufficiently small $\varepsilon > 0$ we obtain for

$$y(t) = \|z(t)-z_0\|^2_H$$

the inequality

$$y'(t) + c^\star[y(t)]^{p/2} \le g(t) \tag{4.20}$$

where $c^\star$ is a positive constant and $\lim_\infty g = 0$.

The inequality (4.20) implies the first part of (4.12):

$$\lim_\infty y = 0 \tag{4.21}$$

(see [10]). Integrating (4.19) with respect to t over $(T-b, T+b)$ we obtain the second part of (4.12) by (4.21). Integrating (4.19) with respect to t over $(0,T)$, by (4.21) we obtain (4.13) as $T\to\infty$.

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References

[1] Adams, R.A., *Sobolev spaces*, Academic Press, New York - San Francisco - London, 1975.

[2] Berkovits, J., Mustonen, V., *Topological degreee for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems*, Rend. Mat. Ser. VII, **12**(1992), 597-621.

[3] Besenyei, A., *On nonlinear systems containing nonlocal terms*, PhD Thesis, Eötvös Loránd University, Budapest, 2008.

[4] Cinca, S., *Diffusion und Transport in porösen Medien bei veränderlichen Porosität*, Diplomawork, Univ. Heidelberg, 2000.

[5] Lions, J.L., *Quelques méthodes de résolution des problemes aux limites non linéaires*, Dunod Gauthier-Villars, Paris, 1969.

[6] Logan, J.D., Petersen, M.R., Shores, T.S., *Numerical study of reaction-mineralogy-porosity changes in porous media*, Appl. Math. Comput., **127**(2002), 149-164.

[7] Merkin, J.H., Needham, D.J., Sleeman, B.D., *A mathematical model for the spread of morphogens with density dependent chemosensitivity*, Nonlinearity, **18**(2005), 2745-2773.

[8] Simon, L., *On some singular systems of parabolic functional equations*, Math. Bohem., **135**(2010), 123-132.

[9] Simon, L., *On singular systems of parabolic functional equations*, Operator Theory: Advances and Applications, **216**(2011), 317-330.

[10] Simon, L., *Application of monotone type operators to parabolic and functional parabolic PDE's*, Handbook of Differential Equations. Evolutionary Equations, Vol. 4, Elsevier, 2008, 267-321.

[11] Simon, L., *Semilinear hyperbolic functional equations*, Banach Center Publications, **101**(2014), 205-222.

[12] Wu, J., *Theory and Applications of Partial Functional Differential Equations*, Springer, 1996.

[13] Zeidler, E., *Nonlinear functional analysis and its applications II A and II B*, Springer, 1990.

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