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On a subclass of analytic functions for operator on a Hilbert space

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Abstract. In this paper we introduce and study a subclass of analytic functions for operators on a Hilbert space in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. We have established coefficient estimates, distortion theorem for this subclass, and also an application to operators based on fractional calculus for this class is investigated.

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1. Introduction

Let A denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n \tag{1.1}$$

in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of A, consisting of functions of the form (1.1) which are normalised and univalent in U.

A function $f \in A$ is said to be starlike of order δ $(0 \le \delta < 1)$ if and only if

$$\mathrm{Re}\,\frac{zf^{'}(z)}{f(z)} > \delta,\ z \in \mathrm{U}. \tag{1.2}$$

Also, a function $f \in A$ is said to be convex of order δ $(0 \le \delta < 1)$ if and only if

$$\mathrm{Re}\left[1 + \frac{zf^{''}(z)}{f'(z)}\right] > \delta,\ z \in \mathrm{U}. \tag{1.3}$$

We denote by $S^*(\delta)$ and $K(\delta)$ respectively the classes of functions in S, which are starlike and convex of order δ in U. The subclass $S^*(\delta)$ was introduced by Robertson [7] and studied further by Schild [8], MacGregor [4], and others.

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n\ z^n,\ a_n \geq 0. \tag{1.4}$$

We begin by setting

$$F_\lambda(z) = (1-\lambda)f(z) + \lambda z f'(z), \quad 0 \leq \lambda \leq 1, \quad f \in T, \tag{1.5}$$

so that

$$F_\lambda(z) = z - \sum_{n=2}^{\infty} [1+\lambda(n-1)] a_n\ z^n. \tag{1.6}$$

A function $f \in S$ is said to be in the class $S_\lambda(\alpha, \beta, \mu)$ if it satisfies

$$\left| \frac{\dfrac{zF'_\lambda(z)}{F_\lambda(z)} - 1}{\mu \dfrac{zF'_\lambda(z)}{F_\lambda(z)} + 1 - (1+\mu)\alpha} \right| < \beta\ ,\ z \in \mathrm{U}, \tag{1.7}$$

where $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $0 \leq \mu \leq 1$.

Let us define

$$S^*_\lambda(\alpha, \beta, \mu) = S_\lambda(\alpha, \beta, \mu) \cap T. \tag{1.8}$$

The study of various subclasses of S and other related work has been done by Silverman [9], Gupta and Jain [3], Owa and Aouf [6].

Let H be a complex Hilbert space and A be an operator on H. For an analytic function f defined on U, we denote by $f(A)$ the operator on H defined by the well known *Riesz-Dunford integral*

$$f(A) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(zI - A)^{-1} dz \quad , \tag{1.9}$$

where I is the identity operator on H, $\mathcal{C}$ is a positively oriented simple closed contour lying in U and containing the spectrum of A on the interior of the domain. The conjugate operator of A is denoted by A^*.

A function given by (1.4) is in the class $S^*_\lambda(\alpha, \beta, \mu; A)$ if it satisfies the condition

$$||AF'_\lambda(A) - F_\lambda(A)|| < \beta\, ||\mu\ A\ F'_\lambda(A) + F_\lambda(A) - (1+\mu)\alpha F_\lambda(A)|| \tag{1.10}$$

with the same constraints as α, β and μ, given in (1.7) and for all A with $||A|| < 1,\ A \neq \theta$, where θ is the zero operator on H. Such type of work was earlier done by Fan [2], Xiaopei [10], etc.

In the present paper we have established coefficient estimates, distortion theorem for $S^*_\lambda(\alpha, \beta, \mu; A)$ and further we consider application to a class of operators defined through fractional calculus.

2. Main Results

Theorem 2.1. *A function f be given by (1.4) is in the class $S^*_\lambda(\alpha,\beta,\mu;A)$ for all proper contraction A with $A\neq\theta$ if and only if*

$$\sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu n-(1+\mu)\alpha]\}a_n\leq\beta(1+\mu)(1-\alpha), \tag{2.1}$$

for $0\leq\alpha<1,\ 0<\beta\leq1,\ 0\leq\mu\leq1.$

The result is best possible for

$$f(z)=z-\frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]}\ z^n,\quad n\in\mathbb{N}\setminus\{1\} \tag{2.2}$$

Proof. Assuming that (2.1) holds, we deduce that

$$||AF'_\lambda(A)-F_\lambda(A)||-\beta\ ||\mu\ A\ F'_\lambda(A)+F_\lambda(A)-(1+\mu)\alpha\ F_\lambda(A)||$$

$$=\|\sum_{n=2}^{\infty}(n-1)a_nA^n\|-\beta\|(1+\mu)(1-\alpha)A^n-\sum_{n=2}^{\infty}\{1+\mu\ n-(1+\mu)\alpha\}a_n\ A^n\|$$

$$\leq\sum_{n=2}^{\infty}\{(n-1)+\beta\ [1+\mu\ n-(1+\mu)\alpha]\}a_n-\beta(1+\mu)(1-\alpha)\leq0\ ,$$

hence, f is in the class $S^*_\lambda(\alpha,\beta,\mu;A)$.

Conversely, if we suppose that f belongs to $S^*_\lambda(\alpha,\beta,\mu;A)$, then

$$||AF'_\lambda(A)-F_\lambda(A)||<\beta\ ||\mu\ A\ F'_\lambda(A)+F_\lambda(A)-(1+\mu)\alpha\ F_\lambda(A)||\ ,$$

therefore

$$||\sum_{n=2}^{\infty}(n-1)a_nA^n||\leq\beta\ ||(1+\mu)(1-\alpha)-\sum_{n=2}^{\infty}\{\mu\ n+1-(1+\mu)\alpha\}a_n\ A^n||\ .$$

Selecting $A=eI$ $(0<e<1)$ in the above inequality, we get

$$\frac{\sum_{n=2}^{\infty}(n-1)a_ne^n}{(1+\mu)(1-\alpha)-\sum_{n=2}^{\infty}\{\mu\ n+1-(1+\mu)\alpha\}}<\beta. \tag{2.3}$$

Upon clearing denominator in (2.3) and letting $e\to1$ $(0<e<1)$, we get

$$\sum_{n=2}^{\infty}(n-1)a_n\leq\beta(1+\mu)(1-\alpha)-\beta\sum_{n=2}^{\infty}\{\mu\ n+1-(1+\mu)\alpha\}a_n,$$

which implies that

$$\sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu n-(1+\mu)\alpha]\}a_n\leq\beta(1+\mu)(1-\alpha),$$

and this completes the proof of our theorem. □

Corollary 1.1. *If a function f given by (1.4) is in the class $S_\lambda^*(\alpha,\beta,\mu;A)$, then*

$$a_n \le \frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[1+\mu\, n-(1+\mu)\alpha]}\ , \quad n=2,3,4,\ldots \tag{2.4}$$

Theorem 2.2. *If the function f given by (1.4) is in the class $S_\lambda^*(\alpha,\beta,\mu;A)$ for $0\le \alpha<1,\ 0<\beta\le 1,\ 0\le\mu\le 1,\ ||A||<1$ and $A\ne\theta$, then*

$$||A|| - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ ||A||^2 \le ||f(A)||$$
$$\le ||A|| + \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ ||A||^2. \tag{2.5}$$

The result is sharp for the function

$$f(z) = z - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ z^n. \tag{2.6}$$

Proof. In view of Theorem 2.1, we have

$$1+\beta[(1+2\mu)-(1+2\mu)\alpha]\sum_{n=2}^{\infty} a_n$$
$$\le \sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu n-(1+2\mu)\alpha]\}a_n \le \beta(1+2\mu)(1-\alpha),$$

which gives us

$$\sum_{n=2}^{\infty} a_n \le \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ . \tag{2.7}$$

Hence, we have

$$||f(A)|| \ge ||A|| - ||A||^2\sum_{n=2}^{\infty} a_n$$
$$\ge ||A|| - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}||A||^2,$$

and

$$||f(A)|| \le ||A|| + ||A||^2\sum_{n=2}^{\infty} a_n$$
$$\le ||A|| + \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}||A||^2,$$

which completes our proof. □

Theorem 2.3. *Let $f_1(z)=z$, and*

$$f_n(z) = z - \frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[(1+\mu\ n)\ -(1+\mu)\alpha]}\ z^n, \quad n\ge 2. \tag{2.8}$$

Then, any function f *of the form* (1.4) *is in the class* $S^*_\lambda(\alpha,\beta,\mu;A)$ *if and only if it can be expressed as,*

$$f(z)=\sum_{n=1}^{\infty}\lambda_n\ f_n(z)\ ,\quad \text{with}\quad \lambda_n\geq 0\ ,\quad \sum_{n=1}^{\infty}\lambda_n=1. \tag{2.9}$$

Proof. First, let us assume that

$$f(z)=\sum_{n=1}^{\infty}\lambda_n\ f_n(z)=z-\sum_{n=2}^{\infty}\frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[(1+\mu\ n)\ -(1+\mu)\alpha]}\ \lambda_n\ z^n.$$

Then, we have

$$\sum_{n=2}^{\infty}\frac{(n-1)+\beta[(1+\mu\ n)\ -(1+\mu)\alpha]}{\beta(1+\mu)(1-\alpha)}\ \lambda_n\ \frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[(1+\mu\ n)\ -(1+\mu)\alpha]}$$

$$=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\leq 1\ ,$$

hence $f\in S^*_\lambda(\alpha,\beta,\mu;A)$.

Conversely, let us assume that the function f given by (1.4) is in the class $S^*_\lambda(\alpha,\beta,\mu;A)$. Then, from Corollary 1.1 we get

$$a_n\leq\frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]}\ .$$

We may set

$$\lambda_n=\frac{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]}{\beta(1+\mu)(1-\alpha)}\ a_n\ ,$$

and

$$\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n\ ,$$

hence it is easy to check that f can be expressed by (2.9), and this completes the proof of Theorem 2.3. □

3. Distortion Theorem involving Fractional Calculus

In this section we shall prove distortion theorem for function belonging to the class $S^*_\lambda(\alpha,\beta,\mu;A)$, and each of these results would involve operators of fractional calculus which are defined as follows (for details, see [5]).

Definition 3.1. *The fractional integral operator of order* k *associated with a function* f *is defined by*

$$D_A^{-k}f(A)=\frac{1}{\Gamma(k)}\int_0^1 A^k\ f(tA)\ (1-t)^{k-1}dt\ ,$$

where $k>0$ *and* f *is an analytic function in a simply connected region of the complex plane containing the origin.*

Definition 3.2. *The fractional derivative operator of order k associated with a function f is defined by*

$$D_A^k f(A) = \frac{1}{\Gamma(1-k)} g'(A),$$

where

$$g(A) = \int_0^1 A^{(1-k)}\ f(tA)\ (1-t)^{-k} dt\ ,\quad 0 < k < 1\ ,$$

and f is an analytic function in a simply connected region of the complex plane containing the origin.

Theorem 3.3. *If the function f given by (1.4) is in the class $S_\lambda^*(\alpha,\beta,\mu;A)$ for $0 \le \alpha < 1,\ 0 < \beta \le 1,\ 0 \le \mu \le 1$, then*

$$||D_A^{-k} f(A)|| \ge \frac{||A||^k}{\Gamma(k+2)} - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \frac{||A||^{k+2}}{\Gamma(k+2)}\ ,$$

and

$$||D_A^{-k} f(A)|| \le \frac{||A||^k}{\Gamma(k+2)} + \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \frac{||A||^{k+2}}{\Gamma(k+2)}\ .$$

Proof. If we consider

$$F(A) = \Gamma(k+2) A^{-k} D_A^{-k} f(A)$$

$$= A - \sum_{n=1}^{\infty} \frac{\Gamma(n+2)\ \Gamma(k+2)}{\Gamma(n+k+2)}\ a_{n+1} A^{n+1} = A - \sum_{n=2}^{\infty} B_n A^n\ ,$$

where $B_n = \dfrac{\Gamma(n+1)\ \Gamma(k+2)}{\Gamma(n+k+1)}\ a_n$, then we obtain that

$$\sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]\}B_n$$

$$\le \sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]\}a_n \le \beta(1+\mu)(1-\alpha),$$

as $0 < \dfrac{\Gamma(n+1)\ \Gamma(k+2)}{\Gamma(n+k+1)} < 1$, hence F belongs to $S_\lambda^*(\alpha,\beta,\mu;A)$.

Therefore, by Theorem 2.2 we deduce that

$$||D_A^{-k} f(A)|| \le \frac{||A^{k+1}||}{\Gamma(k+2)} + \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \frac{||A^{k+2}||}{\Gamma(k+2)}\ .$$

and

$$||D_A^{-k} f(A)|| \ge \frac{||A^{k+1}||}{\Gamma(k+2)} - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \frac{||A^{k+2}||}{\Gamma(k+2)}.$$

Note that $(A^{\frac{1}{q}}) * A^{\frac{1}{q}} = A^{\frac{1}{q}}(A^{\frac{1}{q*}}); q \in N$ and by Corollary 3.8 [11] we have $||A^m|| = ||A||^m$, where m is rational number and ‘ * ’ is the *Hadamard product* or convolution product of two analytic functions. When s is any irrational number, we choose a single-valued branch of z^s and a single valued branch of z^{k_n} (k_n is a sequence of rational numbers) such that $k_n \to s$, as $||A^{k_n}|| = ||A||^{k_n}$, and Lemma 13 [1] allows us to have $||A^{k_n}|| \to ||A^s||$, $||A^{k_n}|| = ||A||^{k_n} \to ||A^s||,\ k_n \to s$.

That is $||A^s|| = ||A||^s$, hence $||A^k|| = ||A||^k,\ k > 0$. □

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