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# On certain class of meromorphic univalent functions with positive coefficients defined by Dziok-Srivastava operator

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**Abstract.** In this paper, we introduce a new class of meromorphic univalent functions defined by using Dziok-Srivastava operator and obtain some results including coefficient inequality, growth and distortion theorems and modified Hadamard products.

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## 1. Introduction

Let $\Sigma_m$ denote the class of functions $f$ of the form:

$$f(z)=\frac{1}{z}+\sum_{k=m}^{\infty}a_kz^k \;\; \left(\, m\in\mathbb{N}=\{1,2,...\}\right), \tag{1.1}$$

which are analytic and univalent in the punctured unit disc $U^*=\{z:z\in\mathbb{C}$ and $0<|z|<1\}=U\backslash\{0\}$. For $g\in\Sigma_m$, given by

$$g\left(z\right)=\frac{1}{z}+\sum_{k=m}^{\infty}b_kz^k, \tag{1.2}$$

the Hadamard product (or convolution) of $f$ and $g$ is given by

$$\left(f*g\right)\left(z\right)=\frac{1}{z}+\sum_{k=m}^{\infty}a_kb_kz^k=\left(g*f\right)\left(z\right). \tag{1.3}$$

A function $f\in\Sigma_m$ is said to be meromorphically starlike of order $\lambda$ if

$$-\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\lambda \;\; \left(z\in U;\; 0\le\lambda<1\right). \tag{1.4}$$

Denote by $\Sigma S_m^*(\lambda)$ the class of all meromorphically starlike functions of order $\lambda$. A function $f \in \Sigma_m$ is said to be meromorphically convex of order $\lambda$ if

$$-\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \lambda \ (z \in U;\ 0 \le \lambda < 1). \tag{1.5}$$

Denote by $\Sigma K_m(\lambda)$ the class of all meromorphically convex functions of order $\lambda$. We note that

$$f(z) \in \Sigma K_m(\lambda) \Longleftrightarrow -zf'(z) \in \Sigma S_m^*(\lambda).$$

The classes $\Sigma S_m^*(\lambda)$ and $\Sigma K_m(\lambda)$ were introduced by Owa et al. [8]. Various subclasses of the class $\Sigma_m$ when $m = 1$ were considered earlier by Pommerenke [9], Miller [6] and others.

For complex parameters

$$\alpha_1, ..., \alpha_q \text{ and } \beta_1, ..., \beta_s \quad \left(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\};\ j = 1, 2, ..., s\right),$$

the generlized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is defined by

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, ..., (\alpha_q)_k}{(\beta_1)_k, ..., (\beta_s)_k} \cdot \frac{z^k}{k!}$$

$$(q \le s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in U), \tag{1.6}$$

where $(\theta)_v$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by

$$(\theta)_v = \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases} 1 & \text{if } (v = 0;\ \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \theta(\theta+1)(\theta+2)...(\theta+v-1) & \text{if } (v \in \mathbb{N};\ \theta \in \mathbb{C}). \end{cases} \tag{1.7}$$

Corresponding to the function $h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$, defined by

$$h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-1} \ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z), \tag{1.8}$$

we consider the linear operator

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \Sigma_m \to \Sigma_m,$$

which is defined by means of the following Hadamard product (or convolution):

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z). \tag{1.9}$$

We observe that, for a function $f$ of the form (1.1), we have

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = z^{-1} + \sum_{k=m}^{\infty} \frac{(\alpha_1)_{k+1}, ..., (\alpha_q)_{k+1}}{(\beta_1)_{k+1}, ..., (\beta_s)_{k+1}} \cdot \frac{a_k}{(k+1)!} z^k. \tag{1.10}$$

For convenience, we write

$$H_{q,s}(\alpha_1) = H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s). \tag{1.11}$$

The linear operator $H_{q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [5, with $p = 1$] and Aouf [2, with $p = 1$].

For fixed parameters $A, B, \beta$ and $\lambda$ $(0 < \beta \leq 1,\ -1 \leq A < B \leq 1,\ 0 \leq \lambda < 1)$, we say that a function $f \in \Sigma_m$ is in the class $\Sigma_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$ of meromorphically univalent functions in if it satisfies the inequality:

$$\left| \frac{\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + 1}{B\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + [B + (A - B)(1 - \lambda)]} \right| < \beta \quad (z \in U^*). \tag{1.12}$$

A function $f$ in $\Sigma_m$ is said to belong to the class $C_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$ if and only if $-zf'(z) \in \Sigma_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$ that is

$$f \in C_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta) \Longleftrightarrow -zf' \in \Sigma_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta). \tag{1.13}$$

We note that:

(i) $\Sigma_{2,1}^{m}(1; -1, 1, \lambda, 1) = \Sigma S_m^*(\lambda)$ and
$C_{2,1}^{m}(1; -1, 1, \lambda, 1) = \Sigma K_m(\lambda)$ $(0 \leq \lambda < 1, m \in \mathbb{N})$.

(ii) $\Sigma_{2,1}^{1}(1; A, B, \lambda, \beta) = \Sigma^*(A, B, \lambda, \beta)$ was studied by Aouf [1];

(iii) $\Sigma_{2,1}^{1}(1; -1, 1, \lambda, \beta) = \Sigma^*(\lambda, \beta)$ and $C_{2,1}^{1}(1; -1, 1, \lambda, \beta) = C(\lambda, \beta)$ (Mogra et al.[7]);

(iv) $\Sigma_{2,1}^{m}(1; A, B, \lambda, \beta) = \Sigma_m(A, B, \lambda, \beta)$ (Aouf et al. [6]).

We note also that:

$$\Sigma_{q,s}^{1}(\alpha_1; \beta, -\beta, \lambda, 1) = \Sigma_{q,s}^{+}(\alpha_1; \lambda, \beta)$$

$$= \left\{ f(z) \in \Sigma_m : \left| \frac{\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + 1}{\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - 1 + 2\lambda} \right| < \beta \ (z \in U,\ 0 < \beta \leq 1,\ 0 \leq \lambda < 1) \right\}.$$

## 2. Coefficient inequality

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ are positive real numbers, $0 < \beta \leq 1,\ -1 \leq A < B \leq 1,\ 0 \leq \lambda < 1, m \in \mathbb{N},\ \Gamma_{k+1}(\alpha_1)$ is defined by (2.2) and $z \in U^*$.

In order to prove our results we need the following lemma for the class $\Sigma_{q,s}^{m}(\alpha_1; A, B, \lambda, 1)$ given by Aouf [3, with $p = 1$].

**Lemma 2.1.** Let a function $f$ defined by (1.1) be in the class $\Sigma_m$. If

$$\sum_{k=m}^{\infty} \{(k+1) + \beta[(Bk + A) + (B - A)\lambda]\} \Gamma_{k+1}(\alpha_1) |a_k| \leq (B - A)\beta(1 - \lambda) \tag{2.1}$$

then $f \in \Sigma_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$, where

$$\Gamma_{k+1}(\alpha_1) = \frac{(\alpha_1)_{k+1}, ..., (\alpha_q)_{k+1}}{(\beta_1)_{k+1}, ..., (\beta_s)_{k+1}} \cdot \frac{1}{(k+1)!}. \tag{2.2}$$

From Lemma 2.1 and (1.13), we have the following lemma.

**Lemma 2.2.** Let a function $f$ defined by (1.1) be in the class $\Sigma_m$. If

$$\sum_{k=m}^{\infty} k\{(k+1) + \beta[(Bk + A) + (B - A)\lambda]\} \Gamma_{k+1}(\alpha_1) |a_k| \leq (B - A)\beta(1 - \lambda) \tag{2.3}$$

then $f \in C_{q,s}^{m}\left(\alpha_1; A, B, \lambda, \beta\right)$.

## 3. Growth and distortion theorems

**Theorem 3.1.** If the function $f$ defined by (1.1) is in the class $\Sigma_{q,s}^{m}\left(\alpha_1; A, B, \lambda, \beta\right)$, then

$$\frac{1}{|z|} - \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^m \le |f(z)|$$

$$\le \frac{1}{|z|} + \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^m, \tag{3.1}$$

and

$$\frac{1}{|z|^2} - \frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1} \le \left|f'(z)\right|$$

$$\le \frac{1}{|z|^2} + \frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1}. \tag{3.2}$$

The bounds in (3.1) and (3.2) are attained for the function $f$ given by

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}z^m. \tag{3.3}$$

*Proof.* First of all, for $\Sigma_{q,s}^{m}\left(\alpha_1; A, B, \lambda, \beta\right)$, it follows from (2.1) that

$$\sum_{k=m}^{\infty} a_k \le \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}, \tag{3.4}$$

which, in view of (1.1), yields

$$\begin{aligned} |f(z)| &\ge \frac{1}{|z|} - |z|^m \sum_{k=m}^{\infty} |a_k| \qquad (3.5) \\ &\ge \frac{1}{|z|} - \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^m, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\le \frac{1}{|z|} + |z|^m \sum_{k=m}^{\infty} |a_k| \qquad (3.6) \\ &\le \frac{1}{|z|} + \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^m. \end{aligned}$$

Next, we see from (2.1) that

$$\frac{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}{m}\sum_{k=m}^{\infty}k\left|a_k\right| \tag{3.7}$$

$$\leq \sum_{k=m}^{\infty}\{(k+1)+\beta\left[(Bk+A)+(B-A)\lambda\right]\}\Gamma_{k+1}(\alpha_1)\left|a_k\right|$$

$$\leq (B-A)\beta(1-\lambda)$$

then

$$\sum_{k=m}^{\infty}k\left|a_k\right| \leq \frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}.$$

which, again in view of (1.1) , yields

$$\left|f'(z)\right| \geq \frac{1}{|z|^2}-|z|^{m-1}\sum_{k=m}^{\infty}k\left|a_k\right| \tag{3.8}$$

$$\geq \frac{1}{|z|^2}-\frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1},$$

and

$$\left|f'(z)\right| \leq \frac{1}{|z|^2}+|z|^{m-1}\sum_{k=m}^{\infty}k\left|a_k\right| \tag{3.9}$$

$$\leq \frac{1}{|z|^2}+\frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1}.$$

Finally, it is easy to see that the bounds in (3.1) and (3.2) are attained for the function $f$ given by (3.3) .

**Corollary 3.1.** If the function $f$ defined by (1.1) is in the class $C_{q,s}^{m}(\alpha_1;A,B,\lambda,\beta)$, then

$$\frac{1}{|z|}-\frac{(B-A)\beta(1-\lambda)}{m\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m} \leq |f(z)|$$

$$\leq \frac{1}{|z|}+\frac{(B-A)\beta(1-\lambda)}{m\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m}, \tag{3.10}$$

and

$$\frac{1}{|z|^2}-\frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1} \leq \left|f'(z)\right|$$

$$\leq \frac{1}{|z|^2}+\frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1}. \tag{3.11}$$

The bounds in (3.1) and (3.2) are attained for the function $f$ given by

$$f(z)=\frac{1}{z}+\frac{(B-A)\beta(1-\lambda)}{m\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}z^{m}. \tag{3.12}$$

## 4. Modified Hadamard product

Let each of the functions $f_1$ and $f_1$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{k=m}^{\infty} a_{k,j} z^k \quad (j = 1,\ 2) \tag{4.1}$$

belong to the class $\Sigma_m$. We denote by $(f_1 * f_2)$ the modified Hadamard product (or convolution) of the functions $f_1$ and $f_{2,}$ that is,

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=m}^{\infty} a_{k,1} a_{k,2} z^k. \tag{4.2}$$

**Theorem 4.1**. Let the functions $f_j$ $(j = 1,\ 2)$ defined by (4.1) be in the class $\Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$. Then $(f_1 * f_2)(z) \in \Sigma_{q,s}^m(\alpha_1; A, B, \gamma, \beta)$ ,where

$$\gamma = 1 - \frac{(B-A)\beta(1-\lambda)^2(1+\beta B)(m+1)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2 \Gamma_{m+1}(\alpha_1) + (B-A)^2 \beta^2 (1-\lambda)^2}. \tag{4.3}$$

The result is sharp for the functions $f_j$ $(j = 1,\ 2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)} z^m \quad (j = 1, 2). \tag{4.4}$$

*Proof.* Employing the technique used ealier by Schild and Silverman [10], we need to find the largest $\gamma$ such that

$$\sum_{k=m}^{\infty} \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\gamma]\}\Gamma_{k+1}(\alpha_1)}{(B-A)\beta(1-\gamma)} |a_{k,1}||a_{k,2}| \le 1 \tag{4.5}$$

for $(f_1 * f_2)(z) \in \Sigma_{q,s}^m(\alpha_1; A, B, \gamma, \beta)$. Indeed, since each of the functions $f_j$ $(j = 1, 2)$ belongs to the class $\Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$, then

$$\sum_{k=m}^{\infty} \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}\Gamma_{k+1}(\alpha_1)}{(B-A)\beta(1-\lambda)} |a_{k,j}| \le 1 \quad (j = 1, 2). \tag{4.6}$$

Now, by the Cauchy-Schwarz inequality, we find from (4.6) that

$$\sum_{k=m}^{\infty} \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}\Gamma_{k+1}(\alpha_1)}{(B-A)\beta(1-\lambda)} \sqrt{|a_{k,1}||a_{k,2}|} \le 1. \tag{4.7}$$

Equation (4.7) implies that we need only to show that

$$\begin{aligned} &\frac{\{(k+1)+\beta[(Bk+A)+(B-A)\gamma]\}}{(1-\gamma)} |a_{k,1}||a_{k,2}| \\ &\le \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}}{(1-\lambda)} \sqrt{|a_{k,1}||a_{k,2}|} \quad (k \ge m), \end{aligned} \tag{4.8}$$

that is, that

$$\sqrt{|a_{k,1}||a_{k,2}|} \le \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}(1-\gamma)}{\{(k+1)+\beta[(Bk+A)+(B-A)\gamma]\}(1-\lambda)} \quad (k \ge m). \tag{4.9}$$

Hence, by the inequality (4.7) it is sufficient to prove that

$$\frac{(B-A)\,\beta\,(1-\lambda)}{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\lambda]\}\,\Gamma_{k+1}(\alpha_1)} \leq \frac{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\lambda]\}\,(1-\gamma)}{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\gamma]\}\,(1-\lambda)} \quad (k\geq m)\,. \tag{4.10}$$

It follows from (4.10) that

$$\gamma \leq 1-\frac{(B-A)\beta(1+\beta B)(k+1)(1-\lambda)^2}{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}^2\Gamma_{k+1}(\alpha_1)+(B-A)^2\beta^2(1-\lambda)^2} \quad (k\geq m)\,. \tag{4.11}$$

Defining the function $\Phi\,(k)$ by

$$\Phi\,(k)=1-\frac{(B-A)\beta(1+\beta B)(k+1)(1-\lambda)^2}{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}^2\Gamma_{k+1}(\alpha_1)+(B-A)^2\beta^2(1-\lambda)^2} \quad (k\geq m)\,, \tag{4.12}$$

we see that $\Phi\,(k)$ is an increasing function of $k$ $(k\geq m)$. Therefore, we conclude from (4.11) that

$$\gamma\leq\Phi\,(m)=1-\frac{(B-A)\beta(1+\beta B)(m+1)(1-\lambda)^2}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2\Gamma_{m+1}(\alpha_1)+(B-A)^2\beta^2(1-\lambda)^2}\,, \tag{4.13}$$

which completes the proof of the main assertion of Theorem 4.1.

**Corollary 4.1**. Let the functions $f_j$ $(j=1,\ 2)$ defined by (4.1) be in the class $C^m_{q,s}\,(\alpha_1;A,B,\lambda,\beta)\,.$ Then $(f_1*f_2)\,(z)\in C^m_{q,s}\,(\alpha_1;A,B,\mu,\beta)\,,$ where

$$\mu=1-\frac{(B-A)\beta(1-\lambda)^2(1+\beta B)(m+1)}{m\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2\Gamma_{m+1}(\alpha_1)+(B-A)^2\beta^2(1-\lambda)^2}\,. \tag{4.14}$$

The result is sharp for the functions $f_j$ $(j=1,\ 2)$ given by

$$f_j\,(z)=\frac{1}{z}+\frac{(B-A)\,\beta\,(1-\lambda)}{m\,\{(m+1)+\beta\,[(Bm+A)+(B-A)\,\lambda]\}\,\Gamma_{m+1}(\alpha_1)}z^m \quad (j=1,2)\,. \tag{4.15}$$

**Theorem 4.2.** Let the functions $f_j$ $(j=1,\ 2)$ defined by (4.1) be in the class $\Sigma^m_{q,s}\,(\alpha_1;A,B,\lambda,\beta)\,.$ Then the function $h(z)$ defined by

$$h\,(z)=\frac{1}{z}+\sum_{k=m}^{\infty}\left(a^2_{k,1}+a^2_{k,2}\right)z^k \tag{4.16}$$

belongs to the class $\Sigma^m_{q,s}\,(\alpha_1;A,B,\xi,\beta)\,,$ where

$$\xi=1-\frac{2\,(B-A)\,\beta\,(1-\lambda)^2\,(1+\beta B)\,(m+1)}{\{(m+1)+\beta\,[(Bm+A)+(B-A)\,\lambda]\}^2\,\Gamma_{m+1}(\alpha_1)+2\,(B-A)^2\,\beta^2\,(1-\lambda)^2}\,. \tag{4.17}$$

The result is sharp for the functions $f_j$ $(j=1,\ 2)$ given by $(4.4)\,.$

*Proof.* Noting that

$$\sum_{k=m}^{\infty}\left[\frac{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\lambda]\}\,(\Gamma_{k+1}(\alpha_1))}{(B-A)\,\beta\,(1-\lambda)}\right]^2|a_{k,j}|^2 \leq \left[\sum_{k=m}^{\infty}\frac{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\lambda]\}\,(\Gamma_{k+1}(\alpha_1))}{(B-A)\,\beta\,(1-\lambda)}|a_{k,j}|\right]^2\leq 1, \tag{4.18}$$

for $f_j \in \Sigma_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$ $(j = 1, 2)$, we have

$$\sum_{k=m}^{\infty} \frac{\{(k+1) + \beta[(Bk + A) + (B - A)\lambda]\}^2 (\Gamma_{k+1}(\alpha_1))^2}{2(B - A)^2 \beta^2 (1 - \lambda)^2} \left(|a_{k,1}|^2 + |a_{k,2}|^2\right) \le 1. \tag{4.19}$$

Thus we need to find the largest $\xi$ such that

$$\frac{\{(k+1) + \beta[(Bk + A) + (B - A)\xi]\}}{(1 - \xi)} \tag{4.20}$$

$$\le \frac{\{(k+1) + \beta[(Bk + A) + (B - A)\lambda]\}^2 (\Gamma_{k+1}(\alpha_1))}{2(B - A)\beta(1 - \lambda)^2} \quad (k \ge m),$$

that is, that

$$\xi \le 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(k+1)}{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}^2\Gamma_{k+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2} \quad (k \ge m). \tag{4.21}$$

Defining the function $\Theta(k)$ by

$$\Theta(k) = 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(k+1)}{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}^2\Gamma_{k+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2} \quad (k \ge m), \tag{4.22}$$

we observe that $\Theta(k)$ is an increasing function of $k$ $(k \ge m)$. Therefore, we conclude from (4.21) that

$$\xi \le \Theta(m) = 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(m+1)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2\Gamma_{m+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2}, \tag{4.23}$$

which completes the proof of Theorem 4.2.

**Corollary 4.2**. Let the functions $f_j$ $(j = 1, 2)$ defined by (4.1) be in the class $C_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$. Then the function $h(z)$ defined by (4.18) belongs to the class $C_{q,s}^{m}(\alpha_1; A, B, \rho, \beta)$, where

$$\rho = 1 - \frac{2(B - A)\beta(1 - \lambda)^2 (1 + \beta B)(m + 1)}{m\{(m+1) + \beta[(Bm + A) + (B - A)\lambda]\}^2 \Gamma_{m+1}(\alpha_1) + 2(B - A)^2 \beta^2 (1 - \lambda)^2}. \tag{4.24}$$

The result is sharp for the functions $f_1$ and $f_2$ given by (4.15).

**Remarks.** $(i)$ Putting $q = 2$ and $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in the above results, we get the results obtained by Aouf et al. [4, Lemmas 1 and 2 and Corollaries 1, 2, 3, 4, 7 and 8, respectively];

$(ii)$ Putting $q = 2$, $s = \alpha_1 = \alpha_2 = \beta_1 = B = 1$ and $A = -1$, in Theorems 4.1, 4.2 and Corollaries 4.1, 4.2, we get the results obtained by Aouf et al. [4, Corollaries 5, 9, 6 and 10, respectively].

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