Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 95–108

Compact hypersurfaces in a locally symmetric manifold

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Abstract. Let M be an n-dimensional compact hypersurface in a locally symmetric manifold N^{n+1}. Denote by S and H the squared norm of the second fundamental form and the mean curvature of M. Let $|\Phi|^2$ be the nonnegative C^2-function on M defined by $|\Phi|^2 = S - nH^2$. In this paper, we prove that if M is oriented and has constant mean curvature and $|\Phi|$ satisfies $P_{n,H,\delta}(|\Phi|) \geq 0$, then (1) $|\Phi|^2 = 0$, (i) $H = 0$ and M is totally geodesic in N^{n+1}, (ii) $H \neq 0$ and M is totally umbilical in the unit sphere $S^{n+1}(1)$; or (2) $|\Phi|^2 = B_H$ if and only if (i) $H = 0$ and M is a Clifford torus, (ii) $H \neq 0$, $n \geq 3$, and M is an $H(r)$-torus with $r^2 < (n-1)/n$, (iii) $H \neq 0$, $n = 2$, and M is an $H(r)$-torus with $0 < r < 1$, $r^2 \neq \frac{1}{2}$. If M has constant normalized scalar curvature R, $\bar{R} = R - 1 \geq 0$, $\tilde{R} = R - \delta$ and S satisfies $\varphi_{n,\bar{R},\tilde{R},\delta}(S) \geq 0$, then (1) M is totally umbilical in $S^{n+1}(1)$; or (2) M is a product $S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $r = \sqrt{\frac{n-2}{n(R+1)}}$, where $P_{n,H,\delta}(x)$ and $\varphi_{n,\bar{R},\tilde{R},\delta}(x)$ are defined by (1.7) and (1.10).

Mathematics Subject Classification (2010): 53B20, 53A10.

Keywords: Locally symmetric, Riemannian manifolds, hypersurfaces, totally umbilical.

1. Introduction

If the ambient manifolds possess very nice symmetry, for example, the sphere, many important results had been obtained in the investigation of the minimal hypersurfaces and hypersurfaces with constant mean curvature or constant scalar curvature. One can see [1], [3], [5], [8], [9], [16] and [19]. For minimal hypersurfaces in a unit sphere, Simons [16], Chern-do Carmo-Kobayashi [5] and Lawson [8] obtained the following famous integral inequality and rigidity result:

Theorem 1.1. ([5, 8, 16]) *Let M be an n-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$. Then*

$$\int_M (S-n)S dv \geq 0. \tag{1.1}$$

In particular, if

$$0 \le S \le n,$$

then $S = 0$ *and* M *is totally geodesic, or* $S \le n$ *and* M *is the Clifford torus*

$$M_{m,n-m} = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right),$$

where S *is the squared norm of the second fundamental form of* M*.*

In the case of closed hypersurfaces with constant mean curvature H, H. Alencar and M. do Carmo [1] obtained the following integral inequality

$$\int_M |\Phi|^2 \left\{ n(1+H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| - |\Phi|^2 \right\} dv \le 0, \tag{1.2}$$

where $|\Phi|^2$ is a nonnegative C^2-function on M defined by $|\Phi|^2 = S - nH^2$.

In order to represent our theorem, we need some notation (one can see [1]). An $H(r)$-torus in $S^{n+1}(1)$ is the product immersion $S^{n-1}(r) \times S^1(\sqrt{1-r^2}) \hookrightarrow R^n \times R^2$, where $S^{n-1}(r) \subset R^n, S^1(\sqrt{1-r^2}) \subset R^2, 0 < r < 1$, are the standard immersions. In some orientation, $H(r)$-torus has principal curvatures given by

$$\lambda_1 = \cdots = \lambda_{n-1} = \sqrt{1-r^2}/r,\ \lambda_n = -r/\sqrt{1-r^2}.$$

For each $H \ge 0$, set

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(1+H^2),$$

and let B_H be the square of the positive root of $P_H(x) = 0$. By using (1.2), Alencar and do Carmo [1] also proved the following result:

Theorem 1.2. ([1]) *Let* M *be a closed and oriented hypersurface in a unit sphere* $S^{n+1}(1)$ *with constant mean curvature* H*. Assume that* $|\Phi|^2 \le B_H$ *, then*

(1) *either* $|\Phi|^2 = 0$*,* M *is totally umbilical; or* $|\Phi|^2 = B_H$*.*

(2) $|\Phi|^2 = B_H$ *if and only if*

(*i*) $H = 0$ *and* M *is a Clifford torus;*

(*ii*) $H \ne 0$*,* $n \ge 3$*, and* M *is an* $H(r)$*-torus with* $r^2 < (n-1)/n$*;*

(*iii*) $H \ne 0$*,* $n = 2$*, and* M *is an* $H(r)$*-torus with* $0 < r < 1$*,* $r^2 \ne \frac{1}{2}$*.*

We should note that Zhong [19] also obtained the following important result:

Theorem 1.3. ([19]) *Let* M *be a closed hypersurface in a unit sphere* $S^{n+1}(1)$ *with constant mean curvature* H*. Then*

(1) *if* $S < 2\sqrt{n-1}$*,* M *is a small hypersphere* $S^n(r)$ *of radius* $r = \sqrt{\frac{n}{n+S}}$*;*

(2) *if* $S = 2\sqrt{n-1}$*,* M *is either a small hypersphere* $S^n(r_0)$ *or an* $H(r)$*-torus* $S^1(r) \times S^{n-1}(t)$*, where* $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$*,* $r^2 = \frac{1}{\sqrt{n-1}+1}$ *and* $t^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$*.*

In the case of hypersurfaces with constant scalar curvature, H. Li [9] obtained the following integral inequality

$$\int_M (S - n\bar{R})[n + 2(n-1)\bar{R} - \frac{n-2}{n}S \\ - \frac{n-2}{n}\sqrt{(S + n(n-1)\bar{R})(S - n\bar{R})}]dv \leq 0, \tag{1.3}$$

and the following important result:

Theorem 1.4. ([9]) *Let M be an n-dimensional ($n \geq 3$) compact hypersurface in a unit sphere $S^{n+1}(1)$ with constant normalized scalar curvature R and $\bar{R} = R - 1 \geq 0$. If*

$$n\bar{R} \leq S \leq \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}, \tag{1.4}$$

then either $S = n\bar{R}$ and M is totally umbilical, or

$$S = \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}$$

and M is a product

$$S^1(\sqrt{1-r^2}) \times S^{n-1}(r), \ r = \sqrt{\frac{n-2}{n(R+1)}}.$$

Recently, many researchers begin to study the ambient manifolds which do not possess symmetry in general, for example, the locally symmetric manifolds and the pinched Riemannian manifolds. One can see [6], [7], [12-15] and [17].

Let N^{n+1} denote the locally symmetric manifold whose sectional curvature K_N satisfies the following condition

$$1/2 < \delta \leq K_N \leq 1,$$

at all points $x \in M$. If M is a compact minimal hypersurface in N^{n+1}, Hineva and Belchev [7], Chen [2], Shui and Wu[15], obtained the following important rigidity theorems:

Theorem 1.5. ([7]) *Let M be an n-dimensional compact minimal hypersurface in a locally symmetric manifold N^{n+1}. If*

$$S \leq \frac{(2\delta - 1)n}{n-1}, \tag{1.5}$$

then S is constant.

Theorem 1.6. ([2], [15]) *Let M be an n-dimensional compact minimal hypersurface in a locally symmetric manifold N^{n+1}. If*

$$S \leq (2\delta - 1)n, \tag{1.6}$$

then

(1) *$S = 0$, M is totally geodesic and locally symmetric; or*

(2) *$S = n$, M is a product $V^m(\frac{n}{m}) \times V^{n-m}(\frac{n}{n-m})$, $m = 1, 2, \cdots, n-1$, where $V^r(c)$ denotes the r-demensional Riemannian manifold with constant sectional curvature c.*

By making use of the generalized maximal principle duo to Omori [11] and Yau [18], the author [13] and [14] obtained the following:

Theorem 1.7. ([13]) *Let M be an n-dimensional complete hypersurface with constant mean curvature H in N^{n+1}. Assume that the sectional curvature $K_{n+1in+1i}$ of N^{n+1} at point x of M satisfies $\sum_i \lambda_i K_{n+1in+1i} = nH$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the principal curvatures at point x of M, then*

(1) *if $S < 2\sqrt{n-1}(2\delta-1)$, M is totally umbilical;*

(2) *if $S = 2\sqrt{n-1}(2\delta-1)$, $(n \geq 3)$, M is locally a piece of an $H(r)$-torus $S^1(r) \times S^{n-1}(t)$, where $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $t^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

Theorem 1.8. ([14]) *Let M be an n-dimensional complete hypersurface with constant mean curvature H in N^{n+1}. Assume that the sectional curvature $K_{n+1in+1i}$ of N^{n+1} at point x of M satisfies $\sum_i \lambda_i K_{n+1in+1i} = nH$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the principal curvatures at point x of M, then*

(1) *if $S < D(n,H)$, then M is totally umbilical;*

(2) *if $S = D(n,H)$, then*

(*i*) *$H = 0$ and M is locally a piece of a Clifford torus;*

(*ii*) *$H \neq 0$, $n \geq 3$ and M is locally a piece of an $H(r)$-torus with $r^2 < (n-1)/n$;*

(*iii*) *$H \neq 0$, $n = 2$ and M is locally a piece of an $H(r)$-torus with $r^2 \neq 1/2$, $0 < r < 1$, where*

$$D(n,H) = (2\delta-1)n + \frac{n^3H^2}{2(n-1)} - \frac{(n-2)nH}{2(n-1)}[n^2H^2 + 4(n-1)(2\delta-1)]^{1/2}.$$

Remark 1.1. We should note that in theorem 1.7 and theorem 1.8, the condition $\sum_i \lambda_i K_{n+1in+1i} = nH$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the principal curvatures at point x of M, is needed. But if $\delta = 1$, N^{n+1} is a unit sphere $S^{n+1}(1)$, then $K_{n+1in+1i} = 1$, $\sum_i \lambda_i K_{n+1in+1i} = nH$, by theorem 1.7 and theorem 1.8, we obtain some important results for complete hypersurfaces in a unit sphere $S^{n+1}(1)$ (see [13] and [14]).

In this paper, we shall study the compact hypersurfaces with constant mean curvature and constant scalar curvature in a locally symmetric manifold N^{n+1}. In order to present our theorems, we denote by H the mean curvature of M and S the squared norm of the second fundamental form of M. We define a polynomial $P_{n,H,\delta}(x)$ by

$$P_{n,H,\delta}(x) = \left(\frac{5\delta-3}{2} + H^2\right)nx^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}Hx^3 - x^4 - \frac{1}{2}(1-\delta)n^2H^2. \tag{1.7}$$

We shall prove the following:

Main Theorem 1.1. *Let M be an n-dimensional compact and oriented hypersurface with constant mean curvature in a locally symmetric manifold N^{n+1}. Then*

$$\int_M \left\{\left(\frac{5\delta-3}{2} + H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2\right\}dv \leq 0. \tag{1.8}$$

In particular, if $|\Phi|$ satisfies

$$P_{n,H,\delta}(|\Phi|) \geq 0, \tag{1.9}$$

then either

(1) $|\Phi|^2 = 0$ *and*
(*i*) $H = 0$ *and* M *is totally geodesic in* N^{n+1},
(*ii*) $H \neq 0$ *and* M *is totally umbilical in the unit sphere* $S^{n+1}(1)$; *or*
(2) $|\Phi|^2 = B_H$, *if and only if*
(*i*) $H = 0$ *and* M *is a Clifford torus.*
(*ii*) $H \neq 0$, $n \geq 3$, *and* M *is an* $H(r)$*-torus with* $r^2 < (n-1)/n$.
(*iii*) $H \neq 0$, $n = 2$, *and* M *is an* $H(r)$*-torus with* $0 < r < 1$, $r^2 \neq \frac{1}{2}$.

We also define a function $\varphi_{n,\bar{R},\tilde{R},\delta}(x)$ by

$$\begin{aligned}\varphi_{n,\bar{R},\tilde{R},\delta}(x) = & \frac{n-1}{n}(x - n\bar{R})[\frac{5\delta-3}{2}n + 2(n-1)\bar{R} - \frac{n-2}{n}x \\ & - \frac{n-2}{n}\sqrt{(x + n(n-1)\tilde{R})(x - n\bar{R})}] \\ & - \frac{1}{2}(n-1)(1-\delta)\left[(5\delta-3)n + 2(n-1)\bar{R} + n\tilde{R} + \frac{3n-2}{n(n-1)}x\right].\end{aligned} \tag{1.10}$$

We shall prove the following:

Main Theorem 1.2. *Let* M *be an* n*-dimensional compact hypersurface in a locally symmetric manifold* N^{n+1} *with constant normalized scalar curvature* R *and* $\bar{R} = R - 1 \geq 0$, $\tilde{R} = R - \delta$. *Then*

$$\begin{aligned}\int_M \Big\{ & \frac{n-1}{n}(S - n\bar{R})\Big[\frac{5\delta-3}{2}n + 2(n-1)\bar{R} - \frac{n-2}{n}S \\ & - \frac{n-2}{n}\sqrt{(S + n(n-1)\tilde{R})(S - n\bar{R})}\Big] \\ & - \frac{1}{2}(n-1)(1-\delta)\Big[(5\delta-3)n + 2(n-1)\bar{R} + n\tilde{R} + \frac{3n-2}{n(n-1)}S\Big]\Big\}dv \leq 0.\end{aligned} \tag{1.11}$$

In particular, if S *satisfies*

$$\varphi_{n,\bar{R},\tilde{R},\delta}(S) \geq 0, \tag{1.12}$$

then either
(1) $S = n\bar{R}$ *and* M *is totally umbilical in the unit sphere* $S^{n+1}(1)$; *or*
(2) $S = \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}$ *and* M *is a product*

$$S^1(\sqrt{1-r^2}) \times S^{n-1}(r),\ r = \sqrt{\frac{n-2}{n(R+1)}}.$$

Remark 1.2. If $\delta = 1$, that is, N^{n+1} is the unit sphere $S^{n+1}(1)$, (1.8) reduces to (1.1) if $H = 0$ and (1.2). Main theorem 1.1 reduces to the theorem 1.1, if $H = 0$, of Simons, Chern-do Carmo-Kobayashi and Lawson [16, 5, 8] and theorem 1.2 of Alencar and do Carmo [1]. We should note that (1.11) reduces to (1.3) and Main theorem 1.2 reduces to the theorem 1.4 of H. Li [9].

2. Preliminaries

Let N^{n+1} be the locally symmetric manifold with sectional curvature K_N satisfies the condition $1/2 < \delta \le K_N \le 1$ at all points $x \in M$, M be the compact oriented hypersurface in N^{n+1}. Let $\{e_1, e_2, \dots, e_n, e_{n+1}\}$ be a local frame of orthonormal vector fields in N^{n+1} such that, restricted to M the vectors $\{e_1, e_2, \dots, e_n\}$ are tangent to M, the vector e_{n+1} is normal to M. We shall make use of the following convention on the ranges of indies:

$$1 \le i, j, k, \dots \le n, \quad 1 \le A, B, C, \dots \le n+1.$$

Let $\{\omega_{ij}\}$ be the connection 1-form of M, $h = \{h_{ij}\}$ be the second fundamental form of M. The squared norm of h is denoted by $S = \sum_{i,j=1}^{n}(h_{ij})^2$. Let $\{K_{ABCD}\}$ and $\{R_{ijkl}\}$ be the components of the curvature tensors of N^{n+1} and M, respectively. Since N^{n+1} is a locally symmetric manifold, we have

$$K_{ABCD,E} = 0. \tag{2.1}$$

Let $\{h_{ijk}\}$ and $\{h_{ijkl}\}$ be the covariant derivative of $\{h_{ij}\}$ and $\{h_{ijk}\}$, respectively. We call $\xi = \frac{1}{n}\sum_{i=1}^{n} h_{ii}e_{n+1}$ the mean curvature vector of M. The mean curvature of M is given by $H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}$.

It is well known that for an arbitrary hypersurface M of N^{n+1}, we have

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.2}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + (1/2)\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.3}$$

$$R_{ijkl} = K_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.4}$$

$$n(n-1)R = \sum_{i,j} K_{ijij} + n^2H^2 - S, \tag{2.5}$$

where R is the normalized scalar curvature of M.

The Codazzi equation and Ricci identities are

$$h_{ijk} - h_{ikj} = K_{n+1ikj} = -K_{n+1ijk}, \tag{2.6}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}. \tag{2.7}$$

Let f be a smooth function on M. The first, second covariant derivatives f_i, f_{ij} and Laplacian of f are defined by

$$df = \sum_i f_i\theta_i, \quad \sum_j f_{ij}\theta_j = df_i + \sum_j f_j\theta_{ji}, \quad \Delta f = \sum_i f_{ii}. \tag{2.8}$$

We introduce an operator $\Box$ due to Cheng-Yau [4] by

$$\Box f = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{2.9}$$

Since M is compact, the operator $\Box$ is self-adjoint (see [4]) if and only if

$$\int_M (\Box f)g\,dv = \int_M f(\Box g)dv, \tag{2.10}$$

where f and g are smooth functions on M.

Setting $f = nH$ in (2.9), from (2.5), we have

$$\begin{aligned}\Box(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \sum_i (nH)(nH)_{ii} - \sum_{i,j} h_{ij}(nH)_{ij} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH_i)^2 - \sum_{i,j} h_{ij}(nH)_{ij} \\ &= \frac{1}{2}n(n-1)\Delta R - \frac{1}{2}\Delta(\sum_{i,j} K_{ijij}) + \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}(nH)_{ij}.\end{aligned} \tag{2.11}$$

The Laplacian Δh_{ij} of the second fundamental form h of M is defined by $\Delta h_{ij} = \sum_{k=1}^n h_{iikk}$. From Chern-do Carmo-Kobayashi [5], by a simple and direct calculation, we have

$$\begin{aligned}\Delta h_{ij} =& nHK_{n+1in+1j} - \sum_k K_{n+1kn+1k}h_{ij} + nH\sum_k h_{ik}h_{kj} \\ & - Sh_{ij} + \sum_{k,l}\{K_{lkik}h_{lj} + K_{lkjk}h_{li} + 2K_{lijk}h_{lk}\} + (nH)_{ij}.\end{aligned} \tag{2.12}$$

Choose a local frame of orthonormal vector fields $\{e_i\}$ such that at arbitrary point x of M

$$h_{ij} = \lambda_i \delta_{ij}, \tag{2.13}$$

then at point x we have

$$\begin{aligned}\sum_{i,j} h_{ij}\Delta h_{ij} =& nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i K_{n+1in+1i} \\ & + \sum_{i,j}(\lambda_i - \lambda_j)^2 K_{ijij} - S^2 + nH\sum_i \lambda_i^3 + \sum_i \lambda_i (nH)_{ii}.\end{aligned} \tag{2.14}$$

The following result due to Okumura [10], Alencar and do Carmo [1] will be very important to us.

Lemma 2.1. ([10], [1]) *Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$, and $\sum_i \mu_i^2 = \beta^2$, where $\beta = const. \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.15}$$

and equality holds in the right-hand (left-hand) side if and only if $(n-1)$ of the $\mu_i's$ are non-positive and equal ($(n-1)$ of the $\mu_i's$ are nonnegative and equal).

3. Proof of Main Theorem 1.1

Proof. We suppose that the mean curvature of M is constant and put a nonnegative C^2-function $|\Phi|^2$ by

$$|\Phi|^2 = S - nH^2, \tag{3.1}$$

then M is totally umbilical if and only if $|\Phi|^2 = 0$. Since $\frac{1}{2} < \delta \leq K_N \leq 1$, we have

$$\begin{aligned} nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i K_{n+1in+1i} \\ &= -\frac{1}{2}\sum_{i,j}(\lambda_i-\lambda_j)^2 K_{n+1in+1i} - \frac{1}{2}S\sum_i K_{n+1in+1i} + \frac{n}{2}\sum_i \lambda_i^2 K_{n+1in+1i} \\ &\geq -\frac{1}{2}\sum_{i,j}(\lambda_i-\lambda_j)^2 - \frac{1}{2}Sn + \frac{n}{2}\delta\sum_i \lambda_i^2 \\ &= -\frac{1}{2}[nS - 2n^2H^2 + nS] - \frac{1}{2}nS + \frac{\delta}{2}nS \\ &= -\frac{n}{2}(3-\delta)|\Phi|^2 - \frac{n^2}{2}(1-\delta)H^2, \end{aligned} \tag{3.2}$$

$$\sum_{i,j}(\lambda_i-\lambda_j)^2 K_{ijij} \geq \delta\sum_{i,j}(\lambda_i-\lambda_j)^2 = 2n\delta(S-nH^2) = 2n\delta|\Phi|^2. \tag{3.3}$$

Since $\sum_i(H-\lambda_i)=0$, $\sum_i(H-\lambda_i)^2 = S-nH^2 = |\Phi|^2$, by Lemma 2.1, we have

$$\left|\sum(H-\lambda_i)^3\right| \leq \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3.$$

Thus

$$\begin{aligned} nH\sum_i \lambda_i^3 &= 3nH^2S - 2n^2H^4 - nH\sum_i(H-\lambda_i)^3 \\ &\geq 3nH^2(|\Phi|^2+nH^2) - 2n^2H^4 - n|H|\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3 \\ &= 3nH^2|\Phi|^2 + n^2H^4 - n|H|\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3. \end{aligned} \tag{3.4}$$

From (3.1)-(3.4), (2.14) and $H = const.$, we have

$$\sum_{i,j} h_{ij}\Delta h_{ij} \geq \left(\frac{5\delta-3}{2}+H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2. \tag{3.5}$$

Therefore, we have

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij} \\ &\geq \left(\frac{5\delta-3}{2}+H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2. \end{aligned} \tag{3.6}$$

Since M is compact and oriented, we can choose an orientation for M such that $H \geq 0$. From (3.6), we have

$$\int_M \left\{\left(\frac{5\delta-3}{2}+H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2\right\} dv \leq 0. \tag{3.7}$$

From (1.9) and (3.7), we have

$$\left(\frac{5\delta-3}{2}+H^2\right)n|\Phi|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3-|\Phi|^4-\frac{1}{2}(1-\delta)n^2H^2=0. \tag{3.8}$$

(1) If $|\Phi|^2=0$, from (3.8), we have $(1-\delta)n^2H^2=0$. This implies that $H=0$ and M is totally geodesic in N^{n+1}, or $\delta=1$, that is, N^{n+1} is the unit sphere $S^{n+1}(1)$ and M is totally umbilical in $S^{n+1}(1)$.

(2) If $|\Phi|^2\neq 0$, we have the equalities in (3.2)-(3.7) hold. Therefore, we have for any i,j,k

$$h_{ijk}=0. \tag{3.9}$$

Putting

$$\vartheta=-\sum_{i,j,k}h_{ij}(K_{n+1kikj}+K_{n+1ijkk}).$$

From (2.17) of Chern-do Carmo-Kobayashi [5], we have

$$K_{n+1ijk,l}=K_{n+1ijkl}-K_{n+1in+1k}h_{jl}-K_{n+1ijn+1}h_{kl}+\sum_m K_{mijk}h_{ml}, \tag{3.10}$$

where $K_{n+1ijk,l}$ is the restriction to M of the covariant derivative $K_{ABCD,E}$ of K_{ABCD} as a curvature tensor of N^{n+1}. Since we suppose that N^{n+1} is a locally symmetric one, we have $K_{ABCD,E}=0$. From (3.10), we obtain that

$$K_{n+1ijkl}=K_{n+1in+1k}h_{jl}+K_{n+1ijn+1}h_{kl}-\sum_m K_{mijk}h_{ml}. \tag{3.11}$$

From (3.11) and the equalities of (3.2) and (3.3), we have

$$\begin{aligned}\vartheta=&nH\sum_i\lambda_iK_{n+1in+1i}-S\sum_k K_{n+1kn+1k}\\&+\sum_{i,j,k,m}h_{ij}(h_{mj}K_{mkik}+h_{mk}K_{mijk})\\=&-\frac{n}{2}(3-\delta)|\Phi|^2-\frac{n^2}{2}(1-\delta)H^2+\frac{1}{2}\sum_{i,k}(\lambda_i-\lambda_k)^2K_{ikik}\\=&\frac{3}{2}n(\delta-1)|\Phi|^2-\frac{1}{2}(1-\delta)n^2H^2\leq\frac{3}{2}n(\delta-1)|\Phi|^2.\end{aligned} \tag{3.12}$$

On the other hand, we define the globally vector field ϖ by

$$\varpi=\sum_{i,j,k}(h_{ik}K_{n+1jij}+h_{ij}K_{n+1ijk})e_k.$$

The divergence of ϖ can be written by

$$\mathtt{div}\varpi=\sum_{i,j,k}\nabla_k(h_{ik}K_{n+1jij}+h_{ij}K_{n+1ijk}).$$

From (3.9), we obtain that

$$\vartheta=\sum_{i,j,k}(h_{ikk}K_{n+1jij}+h_{ijk}K_{n+1ijk})-\mathtt{div}\varpi=-\mathtt{div}\varpi. \tag{3.13}$$

From (3.12) and (3.13), we have $\mathtt{div}\varpi \geq \frac{3}{2}n(1-\delta)|\Phi|^2 \geq 0$. Since M is compact, by the Green's divergence theorem, we have $\int_M \frac{3}{2}n(1-\delta)|\Phi|^2 = 0$. Since we suppose that $|\Phi|^2 \neq 0$, we have $\delta = 1$. We infer that N^{n+1} is the unit sphere $S^{n+1}(1)$ and (3.8) reduces to

$$|\Phi|^2\{n(1+H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - |\Phi|^2\} = 0. \tag{3.14}$$

From (3.14), we can get $|\Phi|^2 = B_H$. Therefore, from theorem 1.2 of Alencar and do Carmo [1], we know that Main theorem 1.1 is true. This completes the proof of Main Theorem 1.1. □

4. Proof of Main Theorem 1.2

In this section, we shall suppose that the normalized scalar curvature R of M is constant. We first need the following Lemma:

Lemma 4.1. *Let M be an n-dimensional hypersurface in a locally symmetric manifold N^{n+1} with constant normalized scalar curvature R and $\bar{R} = R - 1 \geq 0$. Then*

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2. \tag{4.1}$$

Proof. Taking the covariant derivative of (2.5), and using the fact $K_{ABCD,E} = 0$ and $R = const.$, we get

$$n^2HH_k = \sum_{i,j} h_{ij}h_{ijk}.$$

It follows that

$$\sum_k n^4H^2(H_k)^2 = \sum_k \left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 \leq \left(\sum_{i,j} h_{ij}^2\right)\sum_{i,j,k} h_{ijk}^2, \tag{4.2}$$

that is

$$n^4H^2|\nabla H|^2 \leq S\sum_{i,j,k} h_{ijk}^2. \tag{4.3}$$

On the other hand, from (2.5), $K_{ijij} \leq 1$ and $R-1 \geq 0$, we have $n^2H^2 - S \geq 0$. From (4.3), we know that lemma 4.1 follows.

Now we shall prove Main theorem 1.2. From (2.1), (2.11), (2.13) and (3.6), we have

$$\begin{aligned}\Box(nH) \geq &\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 \\ &+ \left(\frac{5\delta-3}{2} + H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2.\end{aligned} \tag{4.4}$$

Putting $\bar{R} = R - 1$, $\tilde{R} = R - \delta$, by (2.5), we know that

$$\frac{1}{n^2}[n(n-1)\bar{R} + S] \leq H^2 \leq \frac{1}{n^2}[n(n-1)\tilde{R} + S], \tag{4.5}$$

$$\frac{n-1}{n}(S-n\tilde{R}) \le |\Phi|^2 = S-nH^2 \le \frac{n-1}{n}(S-n\bar{R}). \tag{4.6}$$

By (2.5), (4.5) and (4.6), we get from (4.4)

$$\begin{aligned}
\Box(nH) \ge & \frac{n-1}{n}(S-n\tilde{R})\left[\frac{5\delta-3}{2}n+(n-1)\bar{R}+\frac{1}{n}S\right] \\
& -\frac{n-1}{n}(S-n\bar{R})\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})} \\
& -\left[\frac{n-1}{n}(S-n\bar{R})\right]^2-\frac{1}{2}(1-\delta)(S+n(n-1)\tilde{R}) \\
= & \frac{n-1}{n}(S-n\bar{R})\left[\frac{5\delta-3}{2}n+(n-1)\bar{R}+\frac{1}{n}S\right] \\
& +\frac{n-1}{n}(n\bar{R}-n\tilde{R})\left[\frac{5\delta-3}{2}n+(n-1)\bar{R}+\frac{1}{n}S\right] \\
& -\frac{n-1}{n}(S-n\bar{R})\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})} \\
& -\left[\frac{n-1}{n}(S-n\bar{R})\right]^2-\frac{1}{2}(1-\delta)(S+n(n-1)\tilde{R}) \\
= & \frac{n-1}{n}(S-n\bar{R})\left[\frac{5\delta-3}{2}n+2(n-1)\bar{R}-\frac{n-2}{n}S\right. \\
& \left.-\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})}\right] \\
& -\frac{1}{2}(n-1)(1-\delta)\left[(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S\right],
\end{aligned} \tag{4.7}$$

where $\bar{R}-\tilde{R}=\delta-1$ is used. Since M is compact, from (2.10), we have

$$\int_M \Box(nH)dv = 0.$$

By (4.7), we get

$$\begin{aligned}
\int_M \Bigg\{ & \frac{n-1}{n}(S-n\bar{R})\Big[\frac{5\delta-3}{2}n+2(n-1)\bar{R}-\frac{n-2}{n}S \\
& -\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})}\Big] \\
& -\frac{1}{2}(n-1)(1-\delta)\Big[(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S\Big]\Bigg\}dv \le 0.
\end{aligned} \tag{4.8}$$

From (1.12) and (4.8), we have

$$\begin{aligned}&\frac{n-1}{n}(S-n\bar{R})\Big[\frac{5\delta-3}{2}n+2(n-1)\bar{R}-\frac{n-2}{n}S\\&-\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})}\Big]\\&-\frac{1}{2}(n-1)(1-\delta)\Big[(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S\Big]=0.\end{aligned} \tag{4.9}$$

(1) If $S=n\bar{R}$, from (4.9), we have

$$-\frac{1}{2}(n-1)(1-\delta)\left[(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S\right]=0. \tag{4.10}$$

Since $S=n\bar{R}$ and $\tilde{R}=\bar{R}+1-\delta$, from $\delta>\frac{1}{2}$ and $\bar{R}\geq 0$, we have

$$(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S=2(2\delta-1)n+\frac{3n-2}{n-1}n\bar{R}>0.$$

Thus, from (4.10), we have $\delta=1$, that is, N^{n+1} is the unit sphere $S^{n+1}(1)$ and $S=n\bar{R}$. From (4.6), we have $|\Phi|^2=0$ and M is totally umbilical in $S^{n+1}(1)$.

(2) If $S\neq n\bar{R}$, we know that the equalities in (4.8), (4.7), (4.4), (3.2) and (3.3) hold. We infer that

$$K_{ijij}=\delta,\quad \sum_{i,j,k}h_{ijk}^2=n^2|\nabla H|^{\cdot} \tag{4.11}$$

From (4.9), we have $S=const.$, (2.5) and (4.11) imply that $H=const.$ and $h_{ijk}=0$, for any i,j,k. Putting

$$\vartheta=-\sum_{i,j,k}h_{ij}(K_{n+1kikj}+K_{n+1ijkk}),$$

and making use of the same assertion as in the proof of Main theorem 1.1, we conclude that $\delta=1$, that is, N^{n+1} is the unit sphere $S^{n+1}(1)$, and (4.9) reduces to

$$\begin{aligned}&\frac{n-1}{n}(S-n\bar{R})[n+2(n-1)\bar{R}-\frac{n-2}{n}S\\&-\frac{n-2}{n}\sqrt{(S+n(n-1)\bar{R})(S-n\bar{R})}]=0.\end{aligned} \tag{4.12}$$

From (4.12), we have

$$S=\frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2+4(n-1)\bar{R}+n\}.$$

Therefore, from theorem 1.4 of H. Li [9], we know that Main theorem 1.2 is true. This completes the proof of Main Theorem 1.2. □

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