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Explicit limit cycles of a cubic polynomial differential systems

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Abstract. In this paper, we determine sufficient conditions for a cubic polynomial differential systems of the form

$$\begin{cases} x' = x + ax^3 + bx^2y + cxy^2 + ny^3 \\ y' = y + sx^3 + ux^2y + vxy^2 + wy^3 \end{cases}$$

where a, b, c, n, s, u, v, w are real constants, to possess an algebraic, non-algebraic limit cycles, explicitly given. Concrete examples exhibiting the applicability of our result is introduced.

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1. Introduction

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of the form

$$\begin{cases} x' = \frac{dx}{dt} = P(x, y) \\ y' = \frac{dy}{dt} = Q(x, y) \end{cases} \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are coprime polynomials and we denote by $n = \max\{\deg P, \deg Q\}$ and we say that n is the degree of system (1.1). A limit cycle of system (1.1) is an isolated periodic solution in the set of all periodic solution of system (1.1) see [4, 6, 10], and it is said to be algebraic if it is contained in the zero level set of a polynomial function, see for example [1, 2, 8]. We usually only ask for the number of such limit cycles, but their location as orbits of the system is also an interesting problem. And an even more difficult problem is to give an explicit expression of them. We are able to solve this last problem for a given system of the form (1.1). Until recently, the only limit cycles known in an explicit way were algebraic. In [3, 5, 7] examples of explicit limit cycles which are not algebraic are given. For instance, the limit cycle appearing in van der Pol's system is not algebraic as it is proved in [9].

In this paper, we determine sufficient conditions for a planar systems of the form

$$\begin{cases} x' = x + ax^3 + bx^2y + cxy^2 + ny^3 \\ y' = y + sx^3 + ux^2y + vxy^2 + wy^3 \end{cases} \tag{1.2}$$

where a, b, c, n, s, u, v and w are real constants, to possess an explicit algebraic, non-algebraic limit cycles. Concrete examples exhibiting the applicability of our result is introduced.

We define the trigonometric functions

$$\begin{aligned} f(\theta) &= 3a + c + u + 3w + 4(a - w)(\cos 2\theta) + 2(b + n + s + v)(\sin 2\theta) \\ &\quad + (a - c - u + w)(\cos 4\theta) + (b - n + s - v)(\sin 4\theta) \end{aligned}$$

$$\begin{aligned} g(\theta) &= 3s - 3n - b + v + 4(n + s)(\cos 2\theta) + 2(u - c - a + w)(\sin 2\theta) \\ &\quad + (c - a + u - w)(\sin 4\theta) + (b - n + s - v)(\cos 4\theta) \end{aligned}$$

2. Main result

Our main result is contained in the following theorem.

Theorem 2.1. *Consider a multi-parameter cubic polynomial differential system (1.2), then the following statements hold.*

H1) if

$$3a + c + u + 3w + 4|a - w| + 2|b + n + s + v| + |a - c - u + w| + |b - n + s - v| < 0,$$

$$3s - 3n - b + v + 4|n + s| + 2|u - c - a + w| + |c - a + u - w| + |b - n + s - v| < 0,$$

then system (1.2) has limit cycle explicitly given in polar coordinates (r, θ), by

$$r(\theta, r_*) = \exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)} d\mu\right) \sqrt{r_*^2 + 16 \int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega}$$

where a, b, c, n, s, u, v, w are real constants, and

$$r_* = 4 \sqrt{\frac{\exp\left(2\int_0^{2\pi} \frac{f(\mu)}{g(\mu)} d\mu\right)}{1 - \exp\left(2\int_0^{2\pi} \frac{f(\mu)}{g(\mu)} d\mu\right)} \int_0^{2\pi} \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega}$$

H2) if $f(\theta)$, and $g(\theta)$ are not constant functions for all $\theta \in \mathbb{R}$, then this limit cycle is non algebraic limit cycle.

Moreover, this limit cycle is a stable hyperbolic limit cycle.

H3) if $f(\theta) = \lambda, g(\theta) = \beta$ are constant functions for all $\theta \in \mathbb{R}$ where $\lambda, \beta \in \mathbb{R}_-^*$, then this limit cycle is algebraic limit cycle given by $r_*^2 = \frac{-8}{\lambda}$ i e: $x^2 + y^2 = \frac{-8}{\lambda}$ is the circle.

In short, since it is well known that the polynomial differential systems of degree 1 have no limit cycles, it remains the following open question:
Open question. Are there or not polynomial differential systems of degree 2 exhibiting explicit non-algebraic limit cycles.

Proof. In order to prove our results we write the polynomial differential system (1.2) in polar coordinates (r,θ), defined by $x = r\cos\theta$, and $y = r\sin\theta$, then system becomes

$$\begin{cases} r' = r + f(\theta)\, r^3 \\ \theta' = g(\theta)\, r^2 \end{cases} \tag{2.1}$$

where $\theta' = \frac{d\theta}{dt}$, $r' = \frac{dr}{dt}$.
According to

$$3s - 3n - b + v + 4|n+s| + 2|u-c-a+w| + |c-a+u-w| + |b-n+s-v| < 0$$

hence $g(\theta) < 0$ for all $\theta \in \mathbb{R}$, then θ' is negative for all t, which means that the orbits $(r(t), \theta(t))$ of system (2.1) have the opposite orientation with respect to those $(x(t), y(t))$ of system (1.2).

Taking as new independent variable the coordinate θ, this differential system writes

$$\frac{dr}{d\theta} = \frac{f(\theta)}{g(\theta)} r + \frac{8}{g(\theta)} \frac{1}{r} \tag{2.2}$$

which is a Bernoulli equation.

By introducing the standard change of variables $\rho = r^2$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = \frac{16}{g(\theta)} + \frac{2f(\theta)}{g(\theta)} \rho \tag{2.3}$$

The general solution of linear equation (2.3) is

$$\rho(\theta) = \exp\left(\int_0^\theta \frac{2f(\mu)}{g(\mu)} d\mu\right) \left(k + 16\int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega\right) \tag{2.4}$$

where $k \in \mathbb{R}$

Then the general solution of Bernoulli equation (2.2) is

$$r(\theta) = \exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)} d\mu\right) \sqrt{k + 16\int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega} \tag{2.5}$$

where $k \in \mathbb{R}$

Notice that system (1.2) has a periodic orbit if and only if equation (2.5) has a strictly positive 2π periodic solution.

It is easy to check that the solution $r(\theta; r_0)$ of the differential equation (2.2) such that $r(0, r_0) = r_0$ is

$$r(\theta; r_0) = \exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)} d\mu\right) \sqrt{r_0^2 + 16 \int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega} \tag{2.6}$$

where $r_0 = r(0)$.
A periodic solution of system (2.1) must satisfy the condition $r(2\pi, r_0) = r(0, r_0)$, which leads to a unique value $r_0 = r_*$, given by

$$r_* = 4\sqrt{\frac{\exp\left(\int_0^{2\pi} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{1 - \exp\left(\int_0^{2\pi} \frac{2f(\mu)}{g(\mu)} d\mu\right)} \int_0^{2\pi} \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega} \tag{2.7}$$

Since

$$3a + c + u + 3w + 4|a - w| + 2|b + n + s + v| + |a - c - u + w| + |b - n + s - v| < 0$$

and

$$3s - 3n - b + v + 4|n + s| + 2|u - c - a + w| + |c - a + u - w| + |b - n + s - v| < 0$$

we have $f(\mu) < 0$, $g(\mu) < 0$ for all $\mu \in [0, 2\pi]$ hence $r_* > 0$.
Injecting this value of r_* in (2.6), we get the candidate solution

$$r(\theta, r_*) = 4\exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)} d\mu\right) \sqrt{\begin{array}{c} \frac{\exp\left(\int_0^{2\pi} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{1-\exp\left(\int_0^{2\pi} \frac{2f(\mu)}{g(\mu)} d\mu\right)} \int_0^{2\pi} \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega \\ + \int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega \end{array}}$$

So, if $r(\theta; r_*) > 0$ for all $\theta \in \mathbb{R}$, we shall have $r(\theta; r_*) > 0$ would be periodic orbit, and consequently a limit cycle. In what follows it is proved that $r(\theta; r_*) > 0$ for all $\theta \in \mathbb{R}$. Indeed

$$r(\theta, r_*) = 4\exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)} d\mu\right) \sqrt{\begin{array}{c} \frac{\exp\left(\int_0^{2\pi} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{1-\exp\left(\int_0^{2\pi} \frac{2f(\mu)}{g(\mu)} d\mu\right)} \int_0^{2\pi} \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega \\ + \int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega \end{array}}$$

$$> 4\exp\left(\int_0^{\theta}\frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{\begin{array}{l}\displaystyle\int_0^{2\pi}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{-g(\omega)}\right)d\omega\\ \displaystyle+\int_0^{\theta}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\end{array}}$$

$$= 4\exp\left(\int_0^{\theta}\frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{\int_{2\pi}^{\theta}\left(\frac{e^{-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu}}{g(\omega)}\right)d\omega}$$

$$= 4\exp\left(\int_0^{\theta}\frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{\int_{\theta}^{2\pi}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{-g(\omega)}\right)d\omega} > 0$$

because $f(\mu) < 0, g(\mu) < 0$ for all $\mu \in \mathbb{R}$, hence $\frac{f(\mu)}{g(\mu)} > 0$ for all $\mu \in \mathbb{R}$
Consequently, this is a limit cycle for the differential system (1.2).
This completes the proof of statement H1 of Theorem 2.1.

If $f(\theta)$ and $g(\theta)$ are not constant functions for all $\theta \in \mathbb{R}$, the curve $(r(\theta)\cos\theta, r(\theta)\sin(\theta))$ in the (x, y) plane with

$$r(\theta; r_*)^2 = \exp\left(\int_0^{\theta}\frac{2f(\mu)}{g(\mu)}d\mu\right)\left(r_*^2 + 16\int_0^{\theta}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\right)$$

is not algebraic. More precisely, in Cartesian coordinates $r(\theta; r_*)^2 = x^2 + y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$, the curve defined by this limit cycle is

$$f(x, y) = x^2 + y^2 - \exp\left(\int_0^{\arctan\left(\frac{y}{x}\right)}\frac{2f(\mu)}{g(\mu)}d\mu\right)$$

$$\times\left(r_*^2 + 16\int_0^{\arctan\left(\frac{y}{x}\right)}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\right) = 0.$$

But there is no integer n for which both $\frac{\partial^{(n)} f}{\partial x^n}$ and $\frac{\partial^{(n)} f}{\partial y^n}$ vanish identically.
To be convinced by this fact, one has to compute for example $\frac{\partial f}{\partial x}$, that is

$$\frac{\partial f}{\partial x}(x,y) = 2x + \frac{y \exp\left(\frac{2f\left(\arctan\left(\frac{y}{x}\right)\right)}{g\left(\arctan\left(\frac{y}{x}\right)\right)}\right) \exp\left(\int_0^{\arctan\left(\frac{y}{x}\right)} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{x^2+y^2} r_*^2$$
$$+16 \frac{y \exp\left(\frac{2f\left(\arctan\left(\frac{y}{x}\right)\right)}{g\left(\arctan\left(\frac{y}{x}\right)\right)}\right) \exp\left(\int_0^{\arctan\left(\frac{y}{x}\right)} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{x^2+y^2}$$
$$\times \int_0^{\arctan\left(\frac{y}{x}\right)} \left(\frac{\exp\left(-\int_0^{\omega} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega + \frac{16y}{(x^2+y^2)\, g\left(\arctan\left(\frac{y}{x}\right)\right)}$$

Since $f(x,y)$ appears again, it will remains in any order of derivation, therefore the curve $f(x,y)=0$ is non-algebraic and the limit cycle will also be non-algebraic.

In order to prove the hyperbolicity of the limit cycle notice that the Poincaré return map is $\Pi(\rho_0) = \rho(2\pi, \rho_0)$, for more details see [5, section 1.6].

An easy computation shows that

$$\left.\frac{dr(2\pi; r_0)}{dr_0}\right|_{r_0=r_*} = \exp\left(\int_0^{2\pi} \frac{2f(\mu)}{g(\mu)} d\mu\right) > 1.$$

because $f(\mu)\, g(\mu) > 0$ for all $\mu \in \mathbb{R}$

Therefore the limit cycle of the differential equation (2.2) is unstable and hyperbolic. Consequently, this is a stable and hyperbolic limit cycle for the differential system (1.2). This completes the proof of statement H2 of Theorem 2.1.

Suppose now that $f(\theta) = \lambda$, $g(\theta) = \beta$ are constant functions for all $\theta \in \mathbb{R}$.

According to

$$3a + c + u + 3w + 4|a-w| + 2|b+n+s+v| + |a-c-u+w| + |b-n+s-v| < 0$$
$$3s - 3n - b + v + 4|n+s| + 2|u-c-a+w| + |c-a+u-w| + |b-n+s-v| < 0$$

hence $f(\theta) = \lambda < 0$, $g(\theta) = \beta < 0$ for all $\theta \in \mathbb{R}$. then

$$r_* = \sqrt{\frac{16}{\beta} \frac{\exp\left(2\int_0^{2\pi} \frac{\lambda}{\beta} d\mu\right)}{1-\exp\left(2\int_0^{2\pi} \frac{\lambda}{\beta} d\mu\right)} \left(\int_0^{2\pi} \left(\exp\left(-2\int_0^{\omega} \frac{\lambda}{\beta} d\mu\right)\right) d\omega\right)} = \sqrt{-\frac{8}{\lambda}} > 0,$$

Injecting this value of r_* in (2.6), we get the solution

$$r(\theta, r_*) = \exp\left(\frac{\lambda}{\beta}\theta\right) \sqrt{\frac{-8}{\lambda} + \frac{16}{\beta} \int_0^{\theta} \left(\exp\left(-2\int_0^{\omega} \frac{\lambda}{\beta} d\mu\right)\right) d\omega}$$

$$r(\theta, r_*) = \sqrt{-\frac{8}{\lambda}} > 0,$$

for all $\theta \in \mathbb{R}$

In Cartesian coordinates

$$r(\theta; r_*)^2 = x^2 + y^2 = -\frac{8}{\lambda}$$

this limit cycle is algebraic (is the circle).
This completes the proof of statement H3 of Theorem 2.1. □

The following examples are given to illustrate our result.

Example 2.2. If we take $a = s = w = -1, b = 2, c = -2, n = 1$, and $u = v = 0$ then system (1.2) reads

$$\begin{cases} x' = x - x^3 + 2x^2y - 2xy^2 + y^3 \\ y' = y - x^3 - y^3 \end{cases}$$

equivalent to

$$\begin{cases} x' = x + (y - x)\left(x^2 - xy + y^2\right) \\ y' = y - (y + x)\left(x^2 - xy + y^2\right) \end{cases}$$

has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is,

$$r(\theta, r_*) = e^{\theta}\sqrt{r_*^2 - 4\int_0^{\theta}\left(\frac{e^{-2\omega}}{2 - \sin 2\omega}\right)d\omega}$$

where $\theta \in \mathbb{R}$, with $f(\theta) = g(\theta) = -8 + 4(\sin 2\theta)$, and the intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = \sqrt{\frac{2e^{4\pi}}{e^{4\pi} - 1}\int_0^{2\pi}\left(\frac{2}{2 - \sin 2\omega}e^{-2\omega}\right)d\omega} \simeq 1.191\,2$$

Moreover

$$\left.\frac{dr(2\pi; r_0)}{dr_0}\right|_{r_0 = r_*} = e^{4\pi} > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

Is the results presented by Jaume Llibre and Benterki Rebiha in [3] .

Example 2.3. If we take $a = s = w = -2, b = 5, c = -5, n = 2$, and $u = v = 1$ then system (1.2) reads

$$\begin{cases} x' = x - 2x^3 + 5x^2y - 5xy^2 + 2y^3 \\ y' = y - 2x^3 + x^2y + xy^2 - 2y^3 \end{cases}$$

has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is,

$$r(\theta, r_*) = \exp(\theta)\sqrt{r_*^2 + 4\int_0^{\theta}\left(\frac{\exp(-2\omega)}{-4 + 3(\sin 2\omega)}\right)d\omega}$$

where $\theta \in \mathbb{R}$, with $f(\theta) = -16 + 12(\sin 2\theta), g(\theta) = -16 + 12(\sin 2\theta)$, and the intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = 4\sqrt{\frac{\exp(4\pi)}{1 - \exp(4\pi)}\left(\int_0^{2\pi}\left(\frac{\exp(-2\omega)}{-16 + 12(\sin 2\omega)}\right)d\omega\right)} \simeq 1.001\,0$$

Moreover

$$\left.\frac{dr\,(2\pi;r_0)}{dr_0}\right|_{r_0=r_*} = e^{4\pi} > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

Example 2.4. If we take $a = c = u = s = v = w = -1$ and $b = n = 1$, the system (1.2) reads

$$\begin{cases} x' = x - x^3 + x^2y - xy^2 + y^3 \\ y' = y - x^3 - x^2y - xy^2 - y^3 \end{cases}$$

in polar coordinates (r,θ) we obtained $f(\theta) = \lambda = -8$, $g(\theta) = \beta = -8$, and $r_* = \sqrt{\frac{-8}{\lambda}} = 1$ hence

$$r(\theta,r_*) = r(\theta,1) = \exp\left(\int_0^\theta d\mu\right)\sqrt{1 + 16\int_0^\theta \left(\frac{\exp\left(-\int_0^\omega 2d\mu\right)}{-8}\right) d\omega} = 1$$

for all $\theta \in \mathbb{R}$.

The system has a algebraic limit cycle whose expression in Cartesian coordinates (x,y) becomes

$$r(\theta;r_*)^2 = x^2 + y^2 = 1$$

this limit cycle is the circle.

Example 2.5. If we take $a = c = u = w = -\frac{1}{2}, b = n = \frac{1}{4}$, and $s = v = -\frac{1}{4}$, the system (1.2) reads

$$\begin{cases} x' = x - \frac{1}{2}x^3 + \frac{1}{4}x^2y - \frac{1}{2}xy^2 + \frac{1}{4}y^3 \\ y' = y - \frac{1}{4}x^3 - \frac{1}{2}x^2y - \frac{1}{4}xy^2 - \frac{1}{2}y^3 \end{cases}$$

in polar coordinates (r,θ) we obtained $f(\theta) = \lambda = -4, g(\theta) = \beta = -2$ and $r_* = \sqrt{\frac{-8}{\lambda}} = \sqrt{2}$ hence

$$r(\theta,r_*) = r\left(\theta,\sqrt{2}\right) = \exp\left(\int_0^\theta 2ds\right)\sqrt{2 + 16\int_0^\theta \left(\frac{\exp\left(-\int_0^\omega 4ds\right)}{-2}\right) d\omega} = \sqrt{2}$$

for all $\theta \in \mathbb{R}$.

The system has a algebraic limit cycle whose expression in Cartesian coordinates (x,y) becomes

$$r(\theta;r_*)^2 = x^2 + y^2 = 2$$

this limit cycle is the circle.

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