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Sufficient conditions for Janowski starlike functions

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Abstract. Let p be an analytic function defined on the open unit disc $\mathbb{D}$ with $p(0) = 1$. The conditions on C, D, E, F are derived for $p(z)$ to be subordinate to $(1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$ when $C(z)z^2p''(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0$ or $C(z)p^2(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0$ or $|D(z)zp'(z) + E(z)p(z) + F(z)| < M$, $(M > 0)$, where C, D, E, F are complex-valued functions. Sufficient conditions for confluent (Kummer) hypergeometric function, generalized and normalized Bessel function of the first kind of complex order and integral operator to be subordinate to $(1 + Az)/(1 + Bz)$ are obtained as applications. Few more applications are discussed.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For a fixed positive integer n, let $\mathcal{H}[a, n]$ be the subset of $\mathcal{H}$ consisting of functions p of the form $p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}_n$ denote the class of analytic functions in $\mathbb{D}$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,$$

and let $\mathcal{A} := \mathcal{A}_1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent (one-to-one) functions. For $-1 \leq B < A \leq 1$, the class $\mathcal{S}^*[A, B]$ defined by

$$\mathcal{S}^*[A, B] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

is the class of Janowski starlike functions [9]. For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the usual class of starlike functions of order β;

$$\mathcal{S}^*[1 - \beta, 0] := \mathcal{S}^*_\beta = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta\} \text{ and}$$

$$\mathcal{S}^*[\beta,-\beta] := \mathcal{S}^*[\beta] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta|zf'(z)/f(z) + 1|\}.$$

These classes have been studied, for example, in [2, 3, 14, 16]. The class $\mathcal{S}^* := \mathcal{S}^*(0)$ is the class of starlike functions. Recently, the authors have investigated the sufficient conditions for a function to belong to various subclasses of $\mathcal{S}^*[A,B]$ in [20, 19, 15]. A function $f \in \mathcal{A}$ is said to be close-to-convex of order β [13, 8] if $\operatorname{Re}(zf'(z)/g(z)) > \beta$ for some $g \in \mathcal{S}^*$. More results regarding these classes can be found in [7, 10].

In Theorem 2.1 of this paper, we investigate the conditions on C, D, E, F so that

$$C(z)z^2p''(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0$$

implies that $p(z) \prec (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$, where C, D, E, F are complex-valued functions. Miller and Mocanu [11] have obtained the linear integral operators that preserve analytic function with positive real part. We extend this result by investigating the sufficient conditions for integral operator to be subordinate to $(1 + Az)/(1 + Bz)$ by applying Theorem 2.1. We also apply Theorem 2.1 to obtain sufficient conditions for generalized and normalized Bessel function of the first kind of complex order and confluent (Kummer) hypergeometric function to be subordinate to $(1 + Az)/(1 + Bz)$. For $A = 1, B = -1$, all these applications get reduced to some well-known results. As an application, we also get some conditions on functions $f \in \mathcal{A}$, $g \in \mathcal{H}[1, 1]$ so that their product $fg \in \mathcal{S}^*[A,B]$. Section 3 deals with the problem of finding conditions on C, D, E, F so that $C(z)p^2(z)+D(z)zp'(z)+E(z)p(z)+F(z) = 0$ or $|D(z)zp'(z)+E(z)p(z)+F(z)| < M$, $(M > 0)$ implies that $p(z) \prec (1+Az)/(1+Bz)$.

Let Q be the class of functions q that are analytic and injective in $\overline{\mathbb{D}} \setminus R(q)$, where

$$R(q) := \{y \in \partial\mathbb{D} : \lim_{z\to y} q(z) = \infty\},$$

and are such that $q'(y) \neq 0$ for $y \in \partial\mathbb{D} \setminus R(q)$. The following results are required in our investigation.

Lemma 1.1. [13, Theorem 2.2d, p.24] *Let $p \in \mathcal{H}[a, n]$ and $q \in Q$ with $p(z) \not\equiv a$ and $q(0) = a$. If $p \not\prec q$, then there points $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D} \setminus R(q)$ and an $m \geq n \geq 1$ such that $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{D})$,*
(i) $p(z_0) = q(\zeta_0)$,
(ii) $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$,
(iii) $\operatorname{Re}((z_0p''(z_0)/p'(z_0)) + 1) \geq m \operatorname{Re}((z_0q''(z_0)/q'(z_0)) + 1)$.

Lemma 1.2. [13, Theorem 2.3i, p.35] *Let $\Omega \subset \mathbb{C}$ and suppose that $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfies the condition $\psi(i\rho, \sigma, \mu + i\nu; z) \notin \Omega$ whenever ρ, σ, μ and ν are real numbers, $\sigma \leq -n(1 + \rho^2)/2$, $\mu + \sigma \leq 0$. If $p \in \mathcal{H}[1, n]$ and $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$.*

2. Main results

Theorem 2.1. *Let n be a positive integer, $-1 \leq B < A \leq 1, C(z) = C \geq 0$. Suppose that the functions $D, E, F : \mathbb{D} \to \mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z) \geq C$,
(ii) Either $\operatorname{Re} E(z) > 0$ *and* $\operatorname{Re} F(z) > 0$ *or more generally,*

$$(A - B)(\operatorname{Re} D(z) - C)n + (1 + A)(1 + B) \operatorname{Re} E(z) + (1 + B)^2 \operatorname{Re} F(z) > 0,$$

(iii) $((AB-1)\operatorname{Im}E(z)-(B^2-1)\operatorname{Im}F(z))^2$

$$< \big((A-B)(\operatorname{Re}D(z)-C)n+(1+A)(1+B)\operatorname{Re}E(z)+(1+B)^2\operatorname{Re}F(z)\big)$$
$$\big((A-B)(\operatorname{Re}D(z)-C)n-(1-A)(1-B)\operatorname{Re}E(z)-(1-B)^2\operatorname{Re}F(z)\big).$$

If $p \in \mathcal{H}[1,n]$, $(1+B)p(z) \neq (1+A)$ *and satisfy*

$$Cz^2p''(z)+D(z)zp'(z)+E(z)p(z)+F(z)=0, \tag{2.1}$$

then $p(z) \prec (1+Az)/(1+Bz)$.

Proof. For $p \in \mathcal{H}[1,n]$, define the function $q:\mathbb{D}\to\mathbb{C}$ by

$$q(z)=\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)}. \tag{2.2}$$

Then q is analytic in $\mathbb{D}$ and $q(0)=1$. A simple computation shows that

$$p(z)=\frac{(1-A)+(1+A)q(z)}{(1-B)+(1+B)q(z)}, \tag{2.3}$$

$$p'(z)=\frac{2(A-B)q'(z)}{((1-B)+(1+B)q(z))^2} \tag{2.4}$$

and

$$p''(z)=\frac{2(A-B)((1-B)+(1+B)q(z))q''(z)-4(A-B)(1+B)(q'(z))^2}{((1-B)+(1+B)q(z))^3}. \tag{2.5}$$

Using (2.3), (2.4) and (2.5) in (2.1), a calculation shows that q satisfies the following equation

$$\begin{aligned} Cz^2q''(z)-&\frac{2C(1+B)}{(1-B)+(1+B)q(z)}(zq'(z))^2+D(z)zq'(z)\\ &+\frac{E(z)((1-A)+(1+A)q(z))((1-B)+(1+B)q(z))}{2(A-B)}\\ &+\frac{F(z)((1-B)+(1+B)q(z))^2}{2(A-B)}=0. \end{aligned} \tag{2.6}$$

Let $\psi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ be defined by

$$\begin{aligned} \psi(r,s,t;z)=Ct-&\frac{2C(1+B)}{(1-B)+(1+B)r}s^2+D(z)s\\ &+\frac{E(z)((1-A)+(1+A)r)((1-B)+(1+B)r)}{2(A-B)}\\ &+\frac{F(z)((1-B)+(1+B)r)^2}{2(A-B)}. \end{aligned} \tag{2.7}$$

Then the condition (2.6) is equivalent to $\psi(q(z),zq'(z),z^2q''(z);z)\in\Omega=\{0\}$.
To show that $\operatorname{Re}q(z)>0$ for $z\in\mathbb{D}$, from Lemma 1.2, it is sufficient to prove that

$\operatorname{Re}\psi(i\rho,\sigma,\mu+i\nu;z)<0$ in $\mathbb{D}$ for any real ρ, σ, μ and ν satisfying $\sigma\leq -n(1+\rho^2)/2$, $\mu+\sigma\leq 0$. For $z\in\mathbb{D}$, it follows from (2.7) that

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma,\mu+i\nu;z) = C\mu &- \frac{2C(1-B^2)\sigma^2}{(1-B)^2+(1+B)^2\rho^2} + \sigma\operatorname{Re}D(z)\\ &+ \frac{\operatorname{Re}E(z)\Big((1-A)(1-B)-(1+A)(1+B)\rho^2\Big)}{2(A-B)}\\ &+ \frac{((1-B)^2-(1+B)^2\rho^2)\operatorname{Re}F(z)}{2(A-B)}\\ &+ \frac{(B^2-1)\rho\operatorname{Im}F(z)}{A-B} + \frac{(AB-1)\rho\operatorname{Im}E(z)}{A-B}.\end{aligned} \tag{2.8}$$

Using conditions $\operatorname{Re}D(z)\geq C\geq 0$, $\mu+\sigma\leq 0$ and $\sigma\leq -n(1+\rho^2)/2$, we get

$$C\mu+\sigma\operatorname{Re}D(z)\leq -C\sigma+\sigma\operatorname{Re}D(z)\leq -n(1+\rho^2)(\operatorname{Re}D(z)-C)/2$$

and

$$-\frac{2C(1-B^2)\sigma^2}{(1-B)^2+(1+B)^2\rho^2}\leq 0.$$

Thus from (2.8), we have

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma,\mu+i\nu;z) \leq \frac{-n}{2}(1+\rho^2)(\operatorname{Re}D(z)-C) &+ \frac{(AB-1)\rho\operatorname{Im}E(z)}{A-B}\\ &+ \frac{\operatorname{Re}E(z)\Big((1-A)(1-B)-(1+A)(1+B)\rho^2\Big)}{2(A-B)}\\ &+ \frac{((1-B)^2-(1+B)^2\rho^2)\operatorname{Re}F(z)}{2(A-B)}\\ &+ \frac{(B^2-1)\rho\operatorname{Im}F(z)}{A-B} =: a\rho^2+b\rho+c,\end{aligned}$$

where

$$\begin{aligned}a &= -\frac{1}{2(A-B)}\big((A-B)(\operatorname{Re}D(z)-C)n+(1+A)(1+B)\operatorname{Re}E(z)\\ &\quad+(1+B)^2\operatorname{Re}F(z)\big),\\ b &= -\frac{1}{2(A-B)}\big(2(AB-1)\operatorname{Im}E(z)-2(B^2-1)\operatorname{Im}F(z)\big),\\ c &= -\frac{1}{2(A-B)}\big((A-B)(\operatorname{Re}D(z)-C)n-(1-A)(1-B)\operatorname{Re}E(z)\\ &\quad-(1-B)^2\operatorname{Re}F(z)\big).\end{aligned}$$

In view of the conditions (*ii*) and (*iii*) of Theorem 2.1, we see that $a<0$ and $b^2-4ac<0$ respectively. So, $\operatorname{Re}\psi(i\rho,\sigma,\mu+i\nu;z)<0$ in $\mathbb{D}$. Hence by Lemma 1.2, we deduce that $\operatorname{Re}q(z)>0$, that is, by using (2.2), we get

$$\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)}\prec\frac{1+z}{1-z}.$$

Therefore, there exist an analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)} = \frac{1+w(z)}{1-w(z)}$$

which gives that $p(z) = (1+Aw)/(1+Bw)$ and thus, $p(z) \prec (1+Az)/(1+Bz)$. □

By taking $A = 1$ and $B = -1$ in Theorem 2.1, we get the following result.

Corollary 2.2. *Let n be a positive integer, $C(z) = C \geq 0$. Suppose that the functions $D, E, F : \mathbb{D} \to \mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z) \geq C$,
(ii) $(\operatorname{Im} E(z))^2 < \big((\operatorname{Re} D(z) - C)n\big)\big((\operatorname{Re} D(z) - C)n - 2\operatorname{Re} F(z)\big)$.
If $p \in \mathcal{H}[1, n]$ and satisfy $Cz^2p''(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0$, then

$$\operatorname{Re} p(z) > 0.$$

By taking $C(z) = 0$ and $F(z) = 0$ in Theorem 2.1, we get the following result for first order differential subordination.

Corollary 2.3. *Let n be a positive integer, $-1 \leq B < A \leq 1$. Suppose that the functions $D, E : \mathbb{D} \to \mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z) \geq 0$,
(ii) $\operatorname{Re} E(z) > (-n(A-B)\operatorname{Re} D(z))/((1+A)(1+B))$,
(iii) $((AB-1)\operatorname{Im} E(z))^2 < \big((A-B)n\operatorname{Re} D(z) + (1+A)(1+B)\operatorname{Re} E(z)\big)$

$$\big((A-B)n\operatorname{Re} D(z) - (1-A)(1-B)\operatorname{Re} E(z)\big).$$

If $p \in \mathcal{H}[1, n]$, $(1+B)p(z) \neq (1+A)$ and satisfy $D(z)zp'(z) + E(z)p(z) = 0$, then $p(z) \prec (1+Az)/(1+Bz)$.

Next, we study the confluent (Kummer) hypergeometric function $\Phi(a, c; z)$ given by

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}, \tag{2.9}$$

where $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \cdots$, and $(\lambda)_n$ denotes the Pochhammer symbol given by $(\lambda)_0 = 1$, $(\lambda)_n = \lambda(\lambda+1)_{n-1}$. The function $\Phi \in \mathcal{H}[1, 1]$ is a solution of the differential equation

$$z\Phi''(a, c; z) + (c - z)\Phi'(a, c; z) - a\Phi(a, c; z) = 0 \tag{2.10}$$

introduced by Kummer in 1837 [21]. The function $\Phi(a, c; z)$ satisfies the following recursive relation

$$c\Phi'(a; c; z) = a\Phi(a+1; c+1; z).$$

When $\operatorname{Re} c > \operatorname{Re} a > 0$, Φ can be expressed in the integral form

$$\Phi(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}dt.$$

There has been several works [1, 11, 17, 18] studying geometric properties of the function $\Phi(a;c;z)$, such as on its close-to-convexity, starlikeness and convexity. By the use of Theorem 2.1, we obtain the following sufficient conditions for

$$\Phi(a,c;z) \prec (1+Az)/(1+Bz).$$

Corollary 2.4. *Let n be a positive integer and $-1 \le B < A \le 1$.*
If $(1+B)\Phi(a,c;z) \neq (1+A)$ and $a,c \in \mathbb{R}$ satisfy
(i) $(A-B)(c-2) - (1+A)(1+B)|a| > 0$,
(ii) $(a-1)^2B - (1+a)^2A < 0$,
(iii) $a^2(A-B)(AB-1)^2 + 2a(A+B)(AB-1)^2 + (A-B)(AB(AB+4c^2-8c+2)+1) < 0$,
then $\Phi(a;c;z) \prec (1+Az)/(1+Bz)$.

Proof. To begin with, note that in view of (2.10), the function $\Phi(a,c;z)$ satisfies (2.1) with $C(z)=1$, $D(z)=c-z$, $E(z)=-az$ and $F(z)=0$. Since by the given condition (i), $c>2$, we get $\operatorname{Re} D(z) = c-x > C$ for $z \in \mathbb{D}$. The given condition (i) yields

$$\begin{aligned}(A-B)(\operatorname{Re} D(z) - C)n + (1+A)(1+B)\operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z)\\ > (A-B)(c-2) - (1+A)(1+B)ax\\ > (A-B)(c-2) - (1+A)(1+B)|a| > 0.\end{aligned}$$

For $z = x+iy \in \mathbb{D}$, we have

$$\begin{aligned}&((AB-1)\operatorname{Im} E(z) - (B^2-1)\operatorname{Im} F(z))^2 - \big((A-B)(\operatorname{Re} D(z) - C)n\\ &+(1+A)(1+B)\operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z)\big)\big((A-B)\\ &(\operatorname{Re} D(z) - C)n - (1-A)(1-B)\operatorname{Re} E(z) - (1-B)^2 \operatorname{Re} F(z)\big)\\ &= (AB-1)^2a^2y^2 - ((A-B)(c-x-1) - (1+A)(1+B)ax)\\ &((A-B)(c-x-1) + (1-A)(1-B)ax)\\ &< (AB-1)^2a^2(1-x^2) - ((A-B)(c-x-1)\\ &-(1+A)(1+B)ax)((A-B)(c-x-1)\\ &+(1-A)(1-B)ax) =: G(x) = px^2+qx+r,\end{aligned}$$

where

$$p = (A-B)\left((a-1)^2B - (a+1)^2A\right),$$
$$q = 2(c-1)(A-B)((a+1)A + (a-1)B)$$

and

$$r = a^2(AB-1)^2 - (c-1)^2(A-B)^2.$$

Using (ii) and (iii), we get $p<0$ and $q^2-4pr<0$ respectively. So, $G(x)<0$ and thus, all the conditions of the Theorem 2.1 are satisfied.
Hence, $\Phi(a;c;z) \prec (1+Az)/(1+Bz)$. □

Remark 2.5. Taking $A=1$ and $B=-1$ in Corollary 2.4, we get the following well known result:

If $a,c \in \mathbb{R}$ such that $c > 1+\sqrt{1+a^2}$, then $\operatorname{Re} \Phi(a;c;z) > 0$.

Miller and Mocanu [11] have obtained the linear integral operators I such that $I[\mathcal{P}_n] \subseteq \mathcal{P}_n$, where $\mathcal{P}_n = \{f \in \mathcal{H}[1,n] : \operatorname{Re} f(z) > 0 \text{ for } z \in \mathbb{D}\}$. We extend this result by investigating the sufficient conditions for $I[f](z) \prec (1+Az)/(1+Bz)$ for $f \in \mathcal{P}_n$.

Corollary 2.6. *Let $\gamma \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Re}\gamma \geq 0$, n be a positive integer, $-1 \leq B < A \leq 1$. Suppose that ϕ, $\psi \in \mathcal{H}[1,n]$ such that $\phi(z) \neq 0$ and $\psi(z) \neq 0$ for $z \in \mathbb{D}$. Define the integral operator I as*

$$I[f](z) = \frac{\gamma}{z^\gamma \phi(z)} \int_0^z f(t) t^{\gamma-1} \psi(t) dt.$$

If for $f \in \mathcal{P}_n$, the following conditions hold:

$$\operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right) \geq 0, \tag{2.11}$$

$$\begin{aligned} n(A-B)\operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right) + (1+A)(1+B)\operatorname{Re}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\psi(z)}\right) \\ - (1+B)^2 \operatorname{Re} f(z) > 0, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\left((AB-1)\operatorname{Im}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\psi(z)}\right) + (B^2-1)\operatorname{Im} f(z)\right)^2 \\ &< \left(n(A-B)\operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right) + (1+A)(1+B)\operatorname{Re}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\psi(z)}\right)\right. \\ &\quad \left. - (1+B)^2 \operatorname{Re} f(z)\right)\left(n(A-B)\operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right)\right. \\ &\quad \left. - (1-A)(1-B)\operatorname{Re}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\psi(z)}\right) + (1-B)^2 \operatorname{Re} f(z)\right), \end{aligned} \tag{2.13}$$

then $I[f](z) \prec (1+Az)/(1+Bz)$.

Proof. Suppose that the function $f \in \mathcal{P}_n$ satisfy (2.11)– (2.13). Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{\gamma}{z^\gamma \phi(z)} \int_0^z f(t) t^{\gamma-1} \psi(t) dt. \tag{2.14}$$

Result [13, Lemma 1.2c, p. 11] together with some calculations show that p is well defined and $p \in \mathcal{H}[1,n]$. On differentiating (2.14), we see that p satisfies the differential equation

$$D(z)zp'(z) + E(z)p(z) - f(z) = 0,$$

where $D(z) = \phi(z)/\gamma\psi(z)$ and $E(z) = (\gamma\phi(z) + z\phi'(z))/\gamma\psi(z)$. It is easy to verify that (2.11), (2.12) and (2.13) respectively shows that the conditions (i), (ii) and (iii) of Theorem 2.1 are satisfied with $C = 0$, $F(z) = -f(z)$. Therefore, by Theorem 2.1, it follows that $p(z) \prec (1+Az)/(1+Bz)$. ☐

Taking $A = 1$ and $B = -1$ in Corollary 2.6, we get the following result.

Corollary 2.7. *Let* $\gamma \in \mathbb{C} \setminus \{0\}$ *such that* $\operatorname{Re}\gamma \geq 0$, n *be a positive integer. Suppose that* ϕ, $\psi \in \mathcal{H}[1,n]$ *such that* $\phi(z) \neq 0$ *and* $\psi(z) \neq 0$ *for* $z \in \mathbb{D}$. *Define the integral operator* I *as*

$$I[f](z) = \frac{\gamma}{z^{\gamma}\phi(z)} \int_0^z f(t)t^{\gamma-1}\psi(t)dt.$$

If for $f \in \mathcal{P}_n$, *the following conditions hold:*

$$\Big(\operatorname{Im}\Big(\frac{\gamma\phi(z) + z\phi'(z)}{\gamma\psi(z)}\Big)\Big)^2 < \Big(n\operatorname{Re}\Big(\frac{\phi(z)}{\gamma\psi(z)}\Big)\Big)\Big(n\operatorname{Re}\Big(\frac{\phi(z)}{\gamma\psi(z)}\Big) + 2\operatorname{Re} f(z)\Big), \tag{2.15}$$

then $\operatorname{Re}(I[f](z)) > 0$.

Remark 2.8. [13, Lemma 4.2a, p. 202] proves that if

$$\left|\operatorname{Im}\Big(\frac{\gamma\phi(z) + z\phi'(z)}{\gamma\psi(z)}\Big)\right| \leq \Big(n\operatorname{Re}\Big(\frac{\phi(z)}{\gamma\psi(z)}\Big)\Big),$$

then $I[\mathcal{P}_n] \subset \mathcal{P}_n$. Since for any $f \in \mathcal{P}_n$, we have

$$\Big(n\operatorname{Re}\Big(\frac{\phi(z)}{\gamma\psi(z)}\Big)\Big) < \Big(n\operatorname{Re}\Big(\frac{\phi(z)}{\gamma\psi(z)}\Big) + 2\operatorname{Re} f(z)\Big).$$

Therefore, Corollary 2.7 can be regarded as one of the particular case of [13, Lemma 4.2a, p. 202].

Next, we study the generalized and normalized Bessel function of the first kind of order p, $u_p(z) = u_{p,b,c}(z)$ given by the power series

$$u_p(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n} \frac{z^n}{n!},$$

where $b, p, c \in \mathbb{C}$ such that $k = p + (b+1)/2$ and $k \neq 0, -1, -2, \cdots$. The function $u_p \in \mathcal{H}[1,1]$ is a solution of the differential equation

$$4z^2 u_p''(z) + 4kzu_p'(z) + czu_p(z) = 0. \tag{2.16}$$

The function $u_p(z)$ also satisfy the following recursive relation

$$4ku_p(z) = cu_{p+1}(z),$$

which is useful for studying its various geometric properties. More results regarding this function can be found in [6, 5, 4]. By the use of Theorem 2.1, we obtain the following sufficient conditions for $u_p(z) \prec (1+Az)/(1+Bz)$.

Corollary 2.9. *Suppose that* $-1 \leq B < A \leq 1$ *and* $(1+B)u_p(z) \neq 1+A$. *If* $b, p, c \in \mathbb{R}$ *satisfy the following conditions*
(i) $4(A-B)(k-1) - (1+A)(1+B)|c| > 0$,
(ii) $c^2 < AB((2-AB)c^2 - 64(k-1)^2)$,
then $u_p(z) \prec (1+Az)/(1+Bz)$.

Proof. In view of (2.16), the function $u_p(z)$ satisfies (2.1) with $C(z) = 4$, $D(z) = 4k$, $E(z) = cz$ and $F(z) = 0$. Since by the given condition (i), $k > 1$, we get $\operatorname{Re} D(z) = 4k > C$. The given condition (i) yields

$$(A-B)(\operatorname{Re} D(z) - C)n + (1+A)(1+B)\operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z) > 4(A-B)(k-1) - (1+A)(1+B)|c| > 0.$$

For $z = x + iy \in \mathbb{D}$, we have

$$\begin{aligned}
&((AB-1)\operatorname{Im} E(z) - (B^2-1)\operatorname{Im} F(z))^2 - \big((A-B)(\operatorname{Re} D(z) - C)n \\
&+(1+A)(1+B)\operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z)\big)\big((A-B) \\
&(\operatorname{Re} D(z) - C)n - (1-A)(1-B)\operatorname{Re} E(z) - (1-B)^2 \operatorname{Re} F(z)\big) \\
&= (AB-1)^2 c^2 y^2 - (4(A-B)(k-1) + (1+A)(1+B)cx) \\
&(4(A-B)(k-1) - (1-A)(1-B)cx) \\
&< (AB-1)^2 c^2 (1-x^2) - (4(A-B)(k-1) \\
&+(1+A)(1+B)cx)(4(A-B)(k-1) - (1-A)(1-B)cx) \\
&=: H(x) = px^2 + qx + r,
\end{aligned}$$

where

$$p = -(A-B)^2 c^2, \; q = -8c(k-1)\left(A^2 - B^2\right)$$

and

$$r = c^2(AB-1)^2 - 16(k-1)^2(A-B)^2.$$

From the hypothesis, we obtain $p < 0$ and

$$q^2 - 4pr = 4c^2(A-B)^2\left(AB\left(c^2(AB-2) + 64(k-1)^2\right) + c^2\right) < 0.$$

So, $H(x) < 0$. Therefore, by applying Theorem 2.1, we conclude that

$$u_p(z) \prec (1+Az)/(1+Bz). \qquad \square$$

Remark 2.10. If $A = 1$ and $B = -1$, then Corollary 2.9 reduces to [4, Theorem 2.2, p. 29]. So, Corollary 2.9 generalises [4, Theorem 2.2, p. 29].

The following result gives the sufficient conditions for functions $h \in \mathcal{A}_n$ to belong to the class of Janowski starlike functions.

Corollary 2.11. *Let n be a positive integer, $-1 \le B < A \le 1, C(z) = C \ge 0$. Suppose that the functions $D, E, F : \mathbb{D} \to \mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z) \ge C$,
(ii) Either $\operatorname{Re} E(z) > 0$ *and* $\operatorname{Re} F(z) > 0$, *or more generally,*

$$(A-B)(\operatorname{Re} D(z) - C)n + (1+A)(1+B)\operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z) > 0,$$

(iii) $((AB-1)\operatorname{Im} E(z) - (B^2-1)\operatorname{Im} F(z))^2$

$$\begin{aligned}
&< \big((A-B)(\operatorname{Re} D(z) - C)n + (1+A)(1+B)\operatorname{Re} E(z) \\
&+(1+B)^2 \operatorname{Re} F(z)\big)\big((A-B)(\operatorname{Re} D(z) - C)n \\
&-(1-A)(1-B)\operatorname{Re} E(z) - (1-B)^2 \operatorname{Re} F(z)\big).
\end{aligned}$$

If $h \in \mathcal{A}_n$*,* $(1+B)zh'(z)/h(z) \neq (1+A)$ *and satisfy*

$$Cz^3\Big(2\Big(\frac{h'(z)}{h(z)}\Big)^3 - \frac{3h'(z)h''(z)}{h^2(z)} + \frac{h'''(z)}{h(z)}\Big) + (2C + D(z))$$
$$z^2\Big(\frac{h''(z)}{h(z)} - \Big(\frac{h'(z)}{h(z)}\Big)^2\Big) + (D(z)+E(z))\frac{zh'(z)}{h(z)} + F(z) = 0,$$

then $h \in \mathcal{S}^*[A, B]$*.*

Proof. Let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = zh'(z)/h(z)$. Then p is analytic in $\mathbb{D}$ with $p(0) = 1$. A calculation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{zh''(z)}{h'(z)} - \frac{zh'(z)}{h(z)}.$$

The result now follows from Theorem 2.1. □

We obtain our next application by taking $n = 1$, $C(z) = F(z) = 0$, $D(z) = 1$, $h(z) = f(z)g(z)$ with $f \in \mathcal{A}$, $g \in \mathcal{H}[1,1]$ and $E(z) = -1 - zh''(z)/h'(z) + zh'(z)/h(z)$ in Corollary 2.11.

Corollary 2.12. *Let* $-1 \leq B < A \leq 1$*. Suppose that the functions* $f \in \mathcal{A}$*,* $g \in \mathcal{H}[1,1]$ *and* $K(z) = zf'(z)/f(z) + zg'(z)/g(z) - ((2zf'g' + zf''g + zg''f)/(f'g + g'f))$ *satisfy*
(i) $(1+B)z(f'(z)g(z) + g'(z)f(z)) \neq (1+A)f(z)g(z)$,
(ii) $\operatorname{Re} K(z) > 1 - (A-B)/((1+A)(1+B))$,
(iii) $((AB-1)(\operatorname{Im} K(z) - 1))^2 < \big((A-B) + (1+A)(1+B)(\operatorname{Re} K(z) - 1)\big)$

$$\big((A-B) - (1-A)(1-B)(\operatorname{Re} K(z) - 1)\big)$$

then $fg \in \mathcal{S}^*[A, B]$*.*

3. Two more sufficient conditions for Janowski starlikeness

For $p \in \mathcal{H}[1, n]$, Miller and Mocanu [13, Example 2.4m, p. 43] obtained the conditions on C, D, E, F so that

$$\operatorname{Re}(C(z)p^2(z) + D(z)zp'(z) + E(z)p(z) + F(z)) > 0 \Rightarrow \operatorname{Re} p(z) > 0, z \in \mathbb{D}.$$

In contrast to the above result, in this section, for $-1 \leq B < A \leq 1$, we investigate conditions on C, D, E, F so that for $z \in \mathbb{D}$,

$$C(z)p^2(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0 \Rightarrow p(z) \prec (1+Az)/(1+Bz)$$

and then give an application.

Theorem 3.1. *Let* n *be a positive integer,* $-1 \leq B < A \leq 1$*. Suppose that the functions* $C, D, E, F : \mathbb{D} \to \mathbb{C}$ *satisfy*
(i) $\operatorname{Re} D(z) \geq 0$,
(ii) $(A-B)n \operatorname{Re} D(z) + (1+A)(1+B) \operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z) + (1+A)^2 \operatorname{Re} C(z) > 0$,
(iii) $((1-AB) \operatorname{Im} E(z) + (1-B^2) \operatorname{Im} F(z) + (1-A^2) \operatorname{Im} C(z))^2$

$$< \Big((A-B)n \operatorname{Re} D(z) + (1+A)(1+B) \operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z)$$

$$+(1+A)^2 \operatorname{Re} C(z)\Big)\Big((A-B)n \operatorname{Re} D(z)$$
$$-(1-A)(1-B)\operatorname{Re} E(z) - (1-B)^2 \operatorname{Re} F(z) - (1-A)^2 \operatorname{Re} C(z)\Big).$$

If $p \in \mathcal{H}[1,n]$, $(1+B)p(z) \neq (1+A)$ *and satisfy*

$$C(z)p^2(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0, \tag{3.1}$$

then $p(z) \prec (1+Az)/(1+Bz)$.

Proof. For $p \in \mathcal{H}[1,n]$, define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)}. \tag{3.2}$$

Then q is analytic in $\mathbb{D}$ and $q(0)=1$. Proceeding as in Theorem 2.1, the differential equation (3.1) takes the following form

$$\begin{aligned} D(z)zq'(z) &+ \frac{C(z)((1-A)+(1+A)q(z))^2}{2(A-B)} \\ &+ \frac{E(z)((1-A)+(1+A)q(z))((1-B)+(1+B)q(z))}{2(A-B)} \\ &+ \frac{F(z)((1-B)+(1+B)q(z))^2}{2(A-B)} = 0. \end{aligned} \tag{3.3}$$

Let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined by

$$\begin{aligned} \psi(r,s;z) = D(z)s &+ \frac{C(z)((1-A)+(1+A)r)^2}{2(A-B)} \\ &+ \frac{E(z)((1-A)+(1+A)r)((1-B)+(1+B)r)}{2(A-B)} \\ &+ \frac{F(z)((1-B)+(1+B)r)^2}{2(A-B)}. \end{aligned} \tag{3.4}$$

It follows from (3.3) that $\psi(q(z), zq'(z); z) \in \Omega = \{0\}$. Now to ensure that $\operatorname{Re} q(z) > 0$ for $z \in \mathbb{D}$, from Lemma 1.2, it is enough to establish that $\operatorname{Re}\psi(i\rho,\sigma;z) < 0$ in $\mathbb{D}$ for any real ρ, σ, satisfying $\sigma \leq -n(1+\rho^2)/2$. For $z \in \mathbb{D}$ in (3.4), a computation using condition (i) yields that

$$\begin{aligned} \operatorname{Re}\psi(i\rho,\sigma;z) &= \sigma \operatorname{Re} D(z) + \frac{((1-A)^2-(1+A)^2\rho^2)\operatorname{Re} C(z)}{2(A-B)} \\ &+ \frac{(A^2-1)\rho \operatorname{Im} C(z)}{A-B} + \frac{(AB-1)\rho \operatorname{Im} E(z)}{A-B} \\ &+ \frac{\operatorname{Re} E(z)\Big((1-A)(1-B)-(1+A)(1+B)\rho^2\Big)}{2(A-B)} \\ &+ \frac{((1-B)^2-(1+B)^2\rho^2)\operatorname{Re} F(z)}{2(A-B)} + \frac{(B^2-1)\rho \operatorname{Im} F(z)}{A-B} \\ &\leq -\frac{n}{2}(1+\rho^2)\operatorname{Re} D(z) + \frac{((1-A)^2-(1+A)^2\rho^2)\operatorname{Re} C(z)}{2(A-B)} \end{aligned}$$

$$+\frac{\operatorname{Re} E(z)\Big((1-A)(1-B)-(1+A)(1+B)\rho^2\Big)}{2(A-B)}$$
$$+\frac{((1-B)^2-(1+B)^2\rho^2)\operatorname{Re} F(z)}{2(A-B)}+\frac{(B^2-1)\rho \operatorname{Im} F(z)}{A-B}$$
$$+\frac{(A^2-1)\rho \operatorname{Im} C(z)}{A-B}+\frac{(AB-1)\rho \operatorname{Im} E(z)}{A-B} =: a\rho^2+b\rho+c, \tag{3.5}$$

where

$$a=-\frac{1}{2(A-B)}\big((A-B)n \operatorname{Re} D(z)+(1+A)(1+B)\operatorname{Re} E(z)$$
$$+(1+B)^2\operatorname{Re} F(z)+(1+A)^2\operatorname{Re} C(z)\big),$$
$$b=-\frac{1}{(A-B)}\big((1-AB)\operatorname{Im} E(z)+(1-B^2)\operatorname{Im} F(z)+(1-A^2)\operatorname{Im} C(z)\big),$$
$$c=-\frac{1}{2(A-B)}\big((A-B)n \operatorname{Re} D(z)-(1-A)(1-B)\operatorname{Re} E(z)$$
$$-(1-B)^2\operatorname{Re} F(z)-(1-A)^2\operatorname{Re} C(z)\big).$$

In view of the conditions (ii) and (iii), we see that $a<0$ and $b^2-4ac<0$ respectively. So, $\operatorname{Re}\psi(i\rho,\sigma,\mu+i\nu;z)<0$ in $\mathbb{D}$. Hence, by Lemma 1.2, we deduce that $\operatorname{Re} q(z)>0$, that is, by using (3.2), we get

$$\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)} \prec \frac{1+z}{1-z}.$$

Therefore, there exist an analytic function w in $\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$ such that

$$\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)}=\frac{1+w(z)}{1-w(z)}$$

which gives that $p(z)=(1+Aw)/(1+Bw)$ and thus, $p(z)\prec(1+Az)/(1+Bz)$. □

The next result follows by taking $p(z)=zf'(z)/f(z)$ in Theorem 3.1.

Corollary 3.2. *Let n be a positive integer, $-1\le B<A\le 1, C(z)=C\ge 0$. Suppose that the functions $D,E,F:\mathbb{D}\to\mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z)\ge 0$,
(ii) $(A-B)n\operatorname{Re} D(z)+(1+A)(1+B)\operatorname{Re} E(z)+(1+B)^2\operatorname{Re} F(z)+(1+A)^2\operatorname{Re} C(z)>0$,
(iii) $((1-AB)\operatorname{Im} E(z)+(1-B^2)\operatorname{Im} F(z)+(1-A^2)\operatorname{Im} C(z))^2$

$$<\Big((A-B)n\operatorname{Re} D(z)+(1+A)(1+B)\operatorname{Re} E(z)+(1+B)^2\operatorname{Re} F(z)$$
$$+(1+A)^2\operatorname{Re} C(z)\Big)\Big((A-B)n\operatorname{Re} D(z)-(1-A)(1-B)\operatorname{Re} E(z)$$
$$-(1-B)^2\operatorname{Re} F(z)-(1-A)^2\operatorname{Re} C(z)\Big).$$

If $f\in\mathcal{A}_n$, $(1+B)zf'(z)/f(z)\ne(1+A)$ and satisfy

$$C(z)\Big(\frac{zf'(z)}{f(z)}\Big)^2+D(z)\Big(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\Big)\frac{zf'(z)}{f(z)}+E(z)\Big(\frac{zf'(z)}{f(z)}\Big)+F(z)=0,$$

then $f \in \mathcal{S}^*[A,B]$.

We close this section by finding conditions on D, E, F so that $p(z) \prec (1+Az)/(1+Bz)$ when $|D(z)zp'(z) + E(z)p(z) + F(z)| < M$, $(M > 0)$.

Theorem 3.3. *Let n be a positive integer, $M > 0$ and $-1 \le B < A \le 1$. Suppose that the functions $D, E, F : \mathbb{D} \to \mathbb{C}$ satisfy*

$$n(A-B)|D(z)| - (1+|A|)(1+|B|)|E(z)| \ge (1+|B|)^2(M + |F(z)|). \tag{3.6}$$

If $p \in \mathcal{H}[1,n]$ and satisfy

$$|D(z)zp'(z) + E(z)p(z) + F(z)| < M, \tag{3.7}$$

then $p(z) \prec (1+Az)/(1+Bz)$.

Proof. In view of condition (3.7), we must have $|E(0) + F(0)| < M$. Suppose that $G(z) = D(z)zp'(z) + E(z)p(z) + F(z)$. If we assume that p is not subordinate to $(1+Az)/(1+Bz) =: q(z)$, then by Lemma 1.1, there exist points $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D}$ and an $m \ge n$ such that

$$p(z_0) = q(\zeta_0) = \frac{1+A\zeta_0}{1+B\zeta_0} \tag{3.8}$$

and

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = \frac{m(A-B)\zeta_0}{(1+B\zeta_0)^2}. \tag{3.9}$$

Using (3.8), (3.9) , (3.6) and the fact that $|\zeta_0| = 1$ and $m \ge n$, we get

$$|G(z_0)| = \frac{1}{|1+B\zeta_0|^2}|m(A-B)\zeta_0 D(z_0) + (1+A\zeta_0)(1+B\zeta_0)E(z_0) + (1+B\zeta_0)^2F(z_0)|$$

$$\ge \frac{1}{(1+|B|)^2}\big(m(A-B)|D(z_0)| - |1+A\zeta_0||1+B\zeta_0||E(z_0)| - |1+B\zeta_0|^2|F(z_0)|\big)$$

$$\ge \frac{1}{(1+|B|)^2}\big(n(A-B)|D(z_0)| - (1+|A|)(1+|B|)|E(z_0)| - (|1+|B|)^2|F(z_0)|\big) \ge M.$$

Since this contradicts (3.7), we get our required result. □

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