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Faber polynomial coefficient bounds for a subclass of bi-univalent functions

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Abstract. In this work, considering a general subclass of bi-univalent functions and using the Faber polynomials, we obtain coefficient expansions for functions in this class. In certain cases, our estimates improve some of those existing coefficient bounds.

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1. Introduction

Let A denote the class of functions f which are analytic in the open unit disk $U = \{z : |z| < 1\}$ with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let S be the subclass of A consisting of the form (1.1) which are also univalent in U and let P be the class of functions $\varphi(z) = 1 + \sum\limits_{n=1}^{\infty} \varphi_n z^n$ that are analytic in U and satisfy the condition $\mathrm{Re}\,(\varphi(z)) > 0$ in U. By the Caratheodory's lemma (e.g., see [11]) we have $|\varphi_n| \leq 2$.

The Koebe one-quarter theorem [11] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \ , \ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \ , \ \left(|w| < r_0(f) \ , \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. For a brief history and interesting examples in the class Σ, see [27].

Lewin [20] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Netanyahu [22] showed that $max\,|a_2| = \frac{4}{3}$ if $f \in \Sigma$. Subsequently, Brannan and Clunie [7] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [8] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([5], [10], [13], [18], [19], [21], [24], [27], [28], [29]).

The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in geometric function theory. Grunsky [14] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalency of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [25] gave a differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (Schaeffer-Spencer [26]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([6], [9], [15], [16], [17]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, ...\}$) is still an open problem.

For $f(z)$ and $F(z)$ analytic in U, we say that f subordinate to F, written $f \prec F$, if there exists a Schwarz function $u(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ with $|u(z)| < 1$ in U, such that $f(z) = F(u(z))$. For the Schwarz function $u(z)$ we note that $|c_n| < 1$. (e.g. see Duren [11]).

A function $f \in \Sigma$ is said to be $B_\Sigma(\mu, \lambda, \varphi)$, $\lambda \geq 1$ and $\mu \geq 0$, if the following subordination hold

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \varphi(z) \tag{1.2}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec \varphi(w) \tag{1.3}$$

where $g(w) = f^{-1}(w)$.

In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $B_\Sigma(\mu, \lambda, \varphi)$ as well as providing estimates for the initial coefficients of these functions.

2. Main results

Using the Faber polynomial expansion of functions $f \in A$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [3],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{2.1}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$ [4]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2} K_1^{-2} &= -a_2, \\ \frac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4} K_3^{-4} &= -\left(5a_2^3 - 5a_2 a_3 + a_4\right). \end{aligned} \tag{2.2}$$

In general, for any $p \in \mathbb{N}$, an expansion of K_n^p is as, [3],

$$K_n^p = p a_n + \frac{p(p-1)}{2} E_n^2 + \frac{p!}{(p-3)!3!} E_n^3 + \ldots + \frac{p!}{(p-n)!n!} E_n^n, \tag{2.3}$$

where $E_n^p = E_n^p(a_2, a_3, \ldots)$ and by [1],

$$E_n^m(a_1, a_2, \ldots, a_n) = \sum_{m=1}^{\infty} \frac{m!(a_1)^{\mu_1} \ldots (a_n)^{\mu_n}}{\mu_1! \ldots \mu_n!}, \tag{2.4}$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ \ldots \ + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \ \ldots \ + n\mu_n &= n. \end{aligned} \tag{2.5}$$

Evidently, $E_n^n(a_1, a_2, \ldots, a_n) = a_1^n$, [2].

Theorem 2.1. *For* $\lambda \geq 1$ *and* $\mu \geq 0$, *let* $f \in \mathrm{B}_{\Sigma}(\mu, \lambda, \varphi)$. *If* $a_m = 0$; $2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{2}{\mu + (n-1)\lambda}; \qquad n \geq 4 \tag{2.6}$$

Proof. Let functions f given by (1.1). We have

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, ..., a_n)\, a_n z^{n-1}, \tag{2.7}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = 1 + \sum_{n=1}^{\infty} F_{n-1}(A_2, A_3, ..., A_n)\, a_n w^{n-1}$$

where

$$\begin{aligned} F_1 &= (\mu+\lambda)a_2, \\ F_2 &= (\mu+2\lambda)\left[\frac{\mu-1}{2}a_2^2 + a_3\right], \\ F_3 &= (\mu+3\lambda)\left[\frac{(\mu-1)(\mu-2)}{3!}a_2^3 + (\mu-1)\,a_2a_3 + a_4\right], \end{aligned} \tag{2.8}$$

In general, (see [9]).

On the other hand, the inequalities (1.2) and (1.3) imply the existence of two positive real part functions $u(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $v(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$ where $\mathrm{Re}\, u(z) > 0$ and $\mathrm{Re}\, v(w) > 0$ in P so that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = \varphi(u(z)) \tag{2.9}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = \varphi(v(w)) \tag{2.10}$$

where

$$\varphi(u(z)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \varphi_k E_n^k(c_1, c_2, ..., c_n)\, z^n, \tag{2.11}$$

and

$$\varphi(v(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \varphi_k E_n^k(d_1, d_2, ..., d_n)\, w^n. \tag{2.12}$$

Comparing the corresponding coefficients of (2.9) and (2.11) yields

$$[\mu + (n-1)\lambda]\, a_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^k(c_1, c_2, ..., c_{n-1}),\ n \geq 2 \tag{2.13}$$

and similarly, from (2.10) and (2.12) we obtain

$$[\mu+(n-1)\lambda]\, b_n=\sum_{k=1}^{n-1}\varphi_k E_{n-1}^k\left(d_1, d_2, \ldots, d_{n-1}\right), \quad n \geq 2. \tag{2.14}$$

Note that for $a_m=0$; $2 \leq m \leq n-1$ we have $b_n=-a_n$ and so

$$\begin{aligned} {[\mu+(n-1)\lambda]\, a_n } & =\varphi_1 c_{n-1} \\ -[\mu+(n-1)\lambda]\, a_n & =\varphi_1 d_{n-1} \end{aligned}$$

Now taking the absolute values of either of the above two equations and using the facts that $|\varphi_1| \leq 2$, $|c_{n-1}| \leq 1$and $|d_{n-1}| \leq 1$, we obtain

$$\left|a_n\right| \leq \frac{\left|\varphi_1 c_{n-1}\right|}{\mu+(n-1)\lambda}=\frac{\left|\varphi_1 d_{n-1}\right|}{\mu+(n-1)\lambda} \leq \frac{2}{\mu+(n-1)\lambda}. \tag{2.15}$$

□

Theorem 2.2. *Let* $f \in \mathrm{B}_{\Sigma}(\mu, \lambda, \varphi)$, $\lambda \geq 1$ *and* $\mu \geq 0$. *Then*

$$\begin{aligned} &(i) \quad \left|a_2\right| \leq \min \left\{\frac{2}{\mu+\lambda}, \sqrt{\frac{8}{(\mu+2\lambda)(\mu+1)}}\right\} \\ &(ii) \quad \left|a_3\right| \leq \min \left\{\frac{4}{(\mu+\lambda)^2}+\frac{2}{\mu+2\lambda}, \frac{8}{(\mu+2\lambda)(\mu+1)}+\frac{2}{\mu+2\lambda}\right\} \end{aligned} \tag{2.16}$$

Proof. Replacing n by 2 and 3 in (2.13) and (2.14), respectively, we find that

$$(\mu+\lambda) a_2=\varphi_1 c_1, \tag{2.17}$$

$$(\mu+2\lambda)\left[\frac{\mu-1}{2} a_2^2+a_3\right]=\varphi_1 c_2+\varphi_2 c_1^2, \tag{2.18}$$

$$-(\mu++\lambda) a_2=\varphi_1 d_1, \tag{2.19}$$

$$(\mu+2\lambda)\left[\frac{\mu+3}{2} a_2^2-a_3\right]=\varphi_1 d_2+\varphi_2 d_1^2 \tag{2.20}$$

From (2.17) or (2.19) we obtain

$$\left|a_2\right| \leq \frac{\left|\varphi_1 c_1\right|}{\mu+\lambda}=\frac{\left|\varphi_1 d_1\right|}{\mu+\lambda} \leq \frac{2}{\mu+\lambda}. \tag{2.21}$$

Adding (2.18) to (2.20) implies

$$(\mu+2\lambda)(\mu+1)\, a_2^2=\varphi_1\left(c_2+d_2\right)+\varphi_2\left(c_1^2+d_1^2\right)$$

or, equivalently,

$$\left|a_2\right| \leq \sqrt{\frac{8}{(\mu+2\lambda)(\mu+1)}}. \tag{2.22}$$

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.20) from (2.18). We thus get

$$2(\mu+2\lambda)\left(a_3-a_2^2\right)=\varphi_1\left(c_2-d_2\right)+\varphi_2\left(c_1^2-d_1^2\right) \tag{2.23}$$

or

$$\left|a_3\right|=\left|a_2\right|^2+\frac{\left|\varphi_1\left(c_2-d_2\right)\right|}{2(\mu+2\lambda)} \leq\left|a_2\right|^2+\frac{2}{\mu+2\lambda} \tag{2.24}$$

Upon substituting the value of a_2^2 from (2.21) and (2.22) into (2.24), it follows that

$$|a_3| \leq \frac{4}{(\mu+\lambda)^2} + \frac{2}{\mu+2\lambda}$$

and

$$|a_3| \leq \frac{8}{(\mu+2\lambda)(\mu+1)} + \frac{2}{\mu+2\lambda}.$$

□

If we put $\lambda = 1$ in Theorem 2.2, we obtain the following consequence.

Corollary 2.3. *Let* $f \in \mathrm{B}_\Sigma(\mu, \varphi)$, $\mu \geq 0$. *Then*

$$|a_2| \leq \frac{2}{\mu+1}$$

and

$$|a_3| \leq \frac{4}{(\mu+1)^2} + \frac{2}{\mu+2}$$

Remark 2.4. The above estimates for $|a_2|$ and $|a_3|$ show that Corollary 2.3 is an improvement of the estimates given in Prema and Keerthi ([23], Theorem 3.2) and Bulut ([9], Corollary 3).

If we put $\mu = 1$ in Theorem 2.2, we obtain the following consequence.

Corollary 2.5. *Let* $f \in \mathrm{B}_\Sigma(\lambda, \varphi)$, $\lambda \geq 1$. *Then*

$$|a_2| \leq \frac{2}{\lambda+1}$$

and

$$|a_3| \leq \frac{4}{(\lambda+1)^2} + \frac{2}{1+2\lambda}$$

Remark 2.6. The above estimates for $|a_2|$ and $|a_3|$ show that Corollary 2.5 is an improvement of the estimates given in Bulut ([9], Corollary 2).

References

[1] Airault, H., *Symmetric sums associated to the factorization of Grunsky coefficients*, in Conference, Groups and Symmetries, Montreal, Canada, 2007.

[2] Airault, H., *Remarks on Faber polynomials*, Int. Math. Forum, **3**(2008), no. 9, 449-456.

[3] Airault, H., Bouali, H., *Differential calculus on the Faber polynomials*, Bulletin des Sciences Mathematiques, **130**(2006), no. 3, 179-222.

[4] Airault, H., Ren, J., *An algebra of differential operators and generating functions on the set of univalent functions*, Bulletin des Sciences Mathematiques, **126**(2002), no. 5, 343-367.

[5] Altinkaya, Ş., Yalcin, S., *Initial coefficient bounds for a general class of bi-univalent functions*, International Journal of Analysis, Article ID 867871, (2014), 4 pp.

[6] Altinkaya, Ş., Yalcin, S., *Coefficient bounds for a subclass of bi-univalent functions*, TWMS Journal of Pure and Applied Mathematics, **6**(2015), no. 2, 180-185.

[7] Brannan, D.A., Clunie, J., *Aspects of contemporary complex analysis*, Proceedings of the NATO Advanced Study Instute Held at University of Durham, New York: Academic Press, 1979.

[8] Brannan, D.A., Taha, T.S., *On some classes of bi-univalent functions*, Stud. Univ. Babeş-Bolyai Math., **31**(1986), no. 2, 70-77.

[9] Bulut, S., *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions*, C.R. Acad. Sci. Paris, Ser. I, **352**(2014), no. 2, 479-484.

[10] Crişan, O., *Coefficient estimates certain subclasses of bi-univalent functions*, Gen. Math. Notes, **16**(2013), no. 2, 93-1002.

[11] Duren, P.L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, **259**(1983).

[12] Faber, G., *Über polynomische entwickelungen*, Math. Ann., **57**(1903), no. 3, 1569-1573.

[13] Frasin, B.A., Aouf, M.K., *New subclasses of bi-univalent functions,* Applied Mathematics Letters, **24**(2011), 1569-1573.

[14] Grunsky, H., *Koeffizientenbedingungen für schlict abbildende meromorphe funktionen*, Math. Zeit., **45**(1939), 29-61.

[15] Hamidi, S.G., Halim, S.A., Jahangiri, J.M., *Faber polynomial coefficient estimates for meromorphic bi-starlike functions*, International Journal of Mathematics and Mathematical Sciences, 2013, Article ID 498159, 4 p.

[16] Hamidi, S.G., Jahangiri, J.M., *Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C.R. Acad. Sci. Paris, Ser.I, **352**(2014), no. 1, 17-20.

[17] Jahangiri, J.M., Hamidi, S.G., *Coefficient estimates for certain classes of bi-univalent functions*, International Journal of Mathematics and Mathematical Sciences, 2013, Article ID 190560, 4 p.

[18] Keerthi, B.S., Raja, B., *Coefficient inequality for certain new subclasses of analytic bi-univalent functions*, Theoretical Mathematics and Applications, **3**(2013), no. 1, 1-10.

[19] Li, X.F., Wang, A.P., *Two new subclasses of bi-univalent functions*, Internat. Math. Forum, **7**(2012), 1495-1504.

[20] Lewin, M., *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., **18**(1967), 63-68.

[21] Magesh, N., Yamini, J., *Coefficient bounds for a certain subclass of bi-univalent functions*, International Mathematical Forum, **8**(2013), no. 27, 1337-1344.

[22] Netanyahu, E., *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Archive for Rational Mechanics and Analysis, **32**(1969), 100-112.

[23] Prema, S., Keerthi, B.S., *Coefficient bounds for certain subclasses of analytic functions*, J. Math. Anal., **4**(2013), no. 1, 22-27.

[24] Porwal, S., Darus, M., *On a new subclass of bi-univalent functions*, J. Egypt. Math. Soc., **21**(2013), no. 3, 190-193.

[25] Schiffer, M., *A method of variation within the family of simple functions*, Proc. London Math. Soc., **44**(1938), no. 2, 432-449.

[26] Schaeffer, A.C., Spencer, D.C., *The coefficients of schlict functions*, Duke Math. J., **10**(1943), 611-635.

[27] Srivastava, H.M., Mishra, A.K., Gochhayat, P., *Certain subclasses of analytic and bi-univalent functions,* Applied Mathematics Letters, **23**(2010), no. 10, 1188-1192.

[28] Srivastava, H.M., Bulut, S., Çağlar, M., Yağmur, N., *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat, **27**(2013), no. 5, 831-842.

[29] Xu, Q.H., Gui, Y.C., Srivastava, H.M., *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Applied Mathematics Letters, **25**(2012), 990-994.

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