Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 1, 203–211
DOI: 10.24193/subbmath.2023.1.15

Deficient quartic spline of Marsden type with minimal deviation by the data polygon

Diana Curilă (Popescu)

Abstract. In this work we construct the deficient quartic spline with the knots following the Marsden's scheme and prove the existence and uniqueness of the deficient quartic spline with minimal deviation by the data polygon. The interpolation error estimate of the obtained quartic spline is given in terms of the modulus of continuity. A numerical example is included in order to illustrate the geometrical behaviour of the quartic spline with minimal quadratic oscillation in average in comparison with the two times continuous differentiable deficient quartic spline.

Mathematics Subject Classification (2010): 65D07, 65D10.

Keywords: Marsden type deficient quartic splines, optimal properties, minimal quadratic oscillation in average.

1. Introduction

Motivated by the nice properties of complete cubic splines, Howell and Vorma extend in [7] the complete splines to quartic degree in such a manner that the tridiagonal shape of the matrix for computing the local derivatives is preserved. The obtained deficient complete quartic spline of Marsden type (see [10]) has in each interval $[x_{i-1}, x_i]$, $i = \overline{1, n}$, the expression:

$$S_i(x) = \frac{(x_i - x)^2 \cdot \left((x_i - x)^2 + 4 \cdot (x_i - x) \cdot (x - x_{i-1}) - 5 \cdot (x - x_{i-1})^2\right)}{h_i^4} \cdot y_{i-1}$$
$$+ \frac{16 \cdot (x - x_{i-1})^2 \cdot (x_i - x)^2}{h_i^4} \cdot y_{i/2}$$
$$+ \frac{(x - x_{i-1})^2 \cdot \left[(x - x_{i-1})^2 + 4 \cdot (x_i - x) \cdot (x - x_{i-1}) - 5 \cdot (x_i - x)^2\right]}{h_i^4} \cdot y_i$$

Received 12 July 2020; Revised 01 April 2021.

$$+\frac{(x_i-x)^2\cdot(x-x_{i-1})\cdot(x_{i-1}+x_i-2\cdot x)}{h_i^3}\cdot m_{i-1}$$

$$+\frac{(x_i-x)\cdot(x-x_{i-1})^2\cdot(x_{i-1}+x_i-2\cdot x)}{h_i^3}\cdot m_i$$

$$=A_i(x)\cdot y_{i-1}+B_i(x)\cdot y_{i-1/2}+C_i(x)\cdot y_i+D_i(x)\cdot m_{i-1}+E_i(x)\cdot m_i, \quad (1.1)$$

where

$$a=x_0<x_1<...<x_{n-1}<x_n=b$$

is a mesh of $[a,b]$, $h_i=x_i-x_{i-1}$, $i=\overline{1,n}$, and under traditional notations

$$m_i=S'(x_i),\ y_i=S(x_i),\ i=\overline{0,n},$$

$$y_{i-1/2}=S\left(x_{i-1/2}\right),\ i=\overline{1,n},$$

with

$$x_{i-1/2}=\frac{x_{i-1}+x_i}{2}.$$

Since $S\in C^2[a,b]$, the local derivatives $m_i, i=\overline{0,n}$, are obtained by the continuity condition $S''\in C[a,b]$ arriving to the tridiagonal dominant linear system

$$-\frac{1}{h_i}\cdot m_{i-1}+\left(\frac{4}{h_i}+\frac{4}{h_{i+1}}\right)\cdot m_i-\frac{1}{h_{i+1}}\cdot m_{i+1}=\frac{5}{h_i^2}\cdot y_{i-1}-\frac{5}{h_{i+1}^2}\cdot y_{i+1}$$

$$+\left(\frac{11}{h_i^2}-\frac{11}{h_{i+1}^2}\right)\cdot y_i+\frac{16}{h_{i+1}^2}\cdot y_{i+1/2}-\frac{16}{h_i^2}\cdot y_{i/2}\ ,\quad i=\overline{1,n-1} \quad (1.2)$$

With two endpoint conditions of complete type $m_0=f'(a)$, $m_n=f'(b)$, the local derivatives are uniquely determined, obtaining the existence and uniqueness of the complete C^2-smooth quartic spline (see Theorem 1 in [7]).

The interpolation error estimates in the case of interpolated functions $f\in C^5[a,b]$ were obtained in [7] (for estimating $\|S-f\|_\infty$) and in[13] (for estimating $\|S'-f'\|_\infty$), with sharp error bounds.

In this brief work we intend to find the local derivatives $m_i,\ i=\overline{0,n}$, in order to minimize the deviation of the quartic spline by the data polygon and preserving a less smooth condition $S\in C^1[a,b]$. The deviation of a parametric spline by the data polygon is described in [5] by using the Hausdorff distance. Another measure of the spline deviation by the data polygon is the quadratic oscillation in average (QOA) and was introduced in [2] obtaining the cubic spline of Hermite type with minimal QOA.

Since the request of interpolating the mid-points could introduce some oscillation of the quartic spline, in this paper we try to obtain the deficient quartic spline $S\in C^1[a,b]$, as in (1.1), with minimal QOA.

Considering the polygonal line $L:[a,b]\longrightarrow\mathbb{R}$ with the pieces

$$L_{|[x_{i-1},x_i]}=L_i, i=\overline{1,n},$$

$$L_i(x)=\frac{x_i-x}{h_i}\cdot y_{i-1}+\frac{x-x_i}{h_i}\cdot y_i,\ x\in[x_{i-1},x_i], i=\overline{1,n},$$

and according to [2], the quadratic oscillation in average is the functional

$$\rho_2(S) = \left(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [S(x) - L_i(x)]^2 \, dx \right)^{\frac{1}{2}},$$

which contains the local derivatives $m_i, \ i = \overline{0,n}$, as unknown parameters.

Concerning optimal properties for cubic splines, recently, in [6], the derivative oscillation was introduced by considering the functional

$$I_1(m_0, m_1, ..., m_n) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[S'(x) - L_i'(x) \right]^2 dx,$$

and the cubic spline with minimal derivative oscillation was obtained. In [6], $I_0(m_0, m_1, ..., m_n) = (\rho_2(S))^2$ and $I_1(m_0, m_1, ..., m_n)$ where considered together as the functionals

$$I_k(m_0, m_1, ..., m_n) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[S^{(k)}(x) - L_i^{(k)}(x) \right]^2 dx,$$

for $k = 0, 1, 2$, with $I_2(m_0, m_1, ..., m_n)$ being the well-known curvature of the cubic spline (see [11]). The minimal curvature of convex preserving cubic splines was considered in [4]. Cubic splines with minimal norms

$$J_k(S) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[S^{(k)}(x) \right]^2 dx, \ k = 0, 1, 2, 3,$$

where determined in [8]. Optimal properties for quartic splines were obtained in [9] and [12], concerning the minimization of the norms $J_k(S)$, $k = 0, 1, 2, 3$, (see [9]), and considering the derivative interpolating quartic splines (see [12]). The derivative interpolating splines of even degree and their optimal properties were investigated for the first time in [3].

In the next sections we prove the existence and uniqueness of the deficient quartic spline with minimal QOA and provide the corresponding interpolation error estimate in terms of the modulus of continuity, considering a numerical experiment as test example for the theoretical result.

2. Quartic spline with minimal quadratic oscillation in average

In order to obtain the quartic spline with minimal QOA we consider the residual type functional $I_0(m_0, m_1, ..., m_n)$ denoted here by $R(m_0, m_1, ..., m_n)$, as follows

$$\begin{aligned} R(m_0, m_1, ..., m_n) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \Big[& A_i(x) \cdot y_{i-1} + B_i(x) \cdot y_{i-1/2} \\ & + C_i(x) \cdot y_i + D_i(x) \cdot m_{i-1} \\ & + E_i(x) \cdot m_i - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i \Big]^2 dx. \end{aligned} \tag{2.1}$$

Theorem 2.1. *There exists unique deficient quartic spline (*1.1*) with minimal quadratic oscillation in average. If* $(m_0, m_1, ..., m_n)$ *are the local derivatives of this spline* S *and if* S *interpolates a continuous function* $f \in C[a,b]$*, then the corresponding interpolation error estimate is obtained:*

$$|S(x) - f(x)| \leq \left(\frac{9317}{8192} + \frac{14\sqrt{3}\beta^3}{9}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h), \ \forall x \in [a,b] \quad (2.2)$$

where

$$h = \max\left\{h_i : i = \overline{1,n}\right\}, \ \underline{h} = \min\left\{h_i : i = \overline{1,n}\right\}, \ \beta = \frac{h}{\underline{h}},$$

and

$$\omega(f,h) = \sup\{|f(x) - f(y)| : |x - y| \leq h\}$$

is the modulus of continuity

Proof. The system of normal equations

$$\frac{\partial R}{\partial m_i} = 0, \ i = \overline{0,n}$$

is

$$\begin{cases} \frac{h_1^3}{630} \cdot m_0 + \frac{h_1^3}{1260} \cdot m_1 = \\ = -\frac{h_1^2}{63} \cdot y_0 - \frac{2 \cdot h_1^2}{315} \cdot y_{1-1/2} + \frac{h_1^2}{180} \cdot y_1 + \frac{h_1^2}{60} \cdot y_0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{h_i^3}{1260} \cdot m_{i-1} + \left(\frac{h_i^3}{630} + \frac{h_{i+1}^3}{630}\right) \cdot m_i + \frac{h_{i+1}^3}{1260} \cdot m_{i+1} = \\ = -\frac{h_i^2}{180} \cdot y_{i-1} + \frac{2 \cdot h_i^2}{315} \cdot y_{i-1/2} + \frac{h_i^2}{63} \cdot y_i - \frac{h_i^2}{60} \cdot y_{i-1} - \frac{h_{i+1}^2}{63} \cdot y_i - \\ -\frac{2 \cdot h_{i+1}^2}{315} \cdot y_{i+1/2} + \frac{h_{i+1}^2}{180} \cdot y_{i+1} + \frac{h_{i+1}^2}{60} \cdot y_i \ , \ i = \overline{1, n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{h_n^3}{1260} \cdot m_{n-1} + \frac{h_n^3}{630} \cdot m_n = \\ = -\frac{h_n^2}{180} \cdot y_{n-1} + \frac{2 \cdot h_n^2}{315} \cdot y_{n-1/2} + \frac{h_n^2}{63} \cdot y_n - \frac{h_n^2}{60} \cdot y_n \end{cases} \quad (2.3)$$

which can be written in tridiagonal dominant form

$$\begin{cases} m_0 + \frac{1}{2} \cdot m_1 = d_0 \\ \cdots\cdots\cdots \\ \frac{h_i^3}{2 \cdot \left(h_i^3 + h_{i+1}^3\right)} \cdot m_{i-1} + m_i + \frac{h_{i+1}^3}{2 \cdot \left(h_i^3 + h_{i+1}^3\right)} \cdot m_{i+1} = d_i \ , \ i = \overline{1, n-1} \\ \cdots\cdots\cdots \\ \frac{1}{2} \cdot m_{n-1} + m_n = d_n \end{cases} \quad (2.4)$$

where

$$d_0 = \frac{1}{2\cdot h_1}\cdot\left(y_0 - y_{1/2}\right) + \frac{7}{2\cdot h_1}\cdot\left(y_1 - y_{1/2}\right)$$

. . . .

$$d_i = \frac{h_{i+1}^2}{2\cdot\left(h_i^3+h_{i+1}^3\right)}\cdot\left(y_i - y_{i+1/2}\right) + \frac{7\cdot h_{i+1}^2}{2\cdot\left(h_i^3+h_{i+1}^3\right)}\cdot\left(y_{i+1} - y_{i+1/2}\right) + \qquad (2.5)$$

$$+\frac{14\cdot h_i^2}{h_i^3+h_{i+1}^3}\cdot\left(y_{i-1/2} - y_{i-1}\right) + \frac{10\cdot h_i^2}{h_i^3+h_{i+1}^3}\cdot\left(y_i - y_{i-1/2}\right)\ ,\ i=\overline{1,n-1}$$

. . . .

$$d_n = \frac{7}{2\cdot h_n}\cdot\left(y_{n-1/2} - y_{n-1}\right) + \frac{1}{2\cdot h_n}\cdot\left(y_{n-1/2} - y_n\right)$$

Since the matrix A of this system is diagonally dominant we have unique solution $(m_0, m_1, ..., m_n)$ and $\left\|A^{-1}\right\| \leq 2$. The Hessian matrix $\left(\frac{\partial^2 R}{\partial m_i \partial m_j}\right)_{i,j=\overline{0,n}}$ has all the diagonal minors positive and therefore $(m_0, m_1, ..., m_n)$ is the unique minimum point of R. So, the local derivatives m_i , $i = \overline{0,n}$ which minimize the functional R are uniquely determined as the solution of the linear system (2.4), and the quartic spline S with minimal QOA is uniquely determined. When S interpolates $f \in C[a,b]$, since

$$|d_0| \leq \frac{\left|y_0-y_{1/2}\right|+7\cdot\left|y_1-y_{1/2}\right|}{2\cdot h_1} \leq \frac{4}{h_1}\cdot\omega\left(f,\frac{h}{2}\right)$$

. . . .

$$|d_i| \leq \frac{h_{i+1}^2\cdot\left(\left|y_i-y_{i+1/2}\right|+7\cdot\left|y_{i+1}-y_{i+1/2}\right|\right)}{2\cdot\left(h_i^3+h_{i+1}^3\right)} + \frac{h_i^2\cdot\left(14\cdot\left|y_{i-1/2}-y_{i-1}\right|+10\cdot\left|y_i-y_{i-1/2}\right|\right)}{h_i^3+h_{i+1}^3} \leq \qquad (2.6)$$

$$\leq \frac{28\cdot h^2}{h_i^3+h_{i+1}^3}\cdot\omega\left(f,\frac{h}{2}\right)\ ,\ i=\overline{1,n-1}$$

. . . .

$$|d_n| \leq \frac{7\cdot\left|y_{n-1/2}-y_{n-1}\right|+\left|y_{n-1/2}-y_n\right|}{2\cdot h_n} \leq \frac{4}{h_n}\cdot\omega\left(f,\frac{h}{2}\right)$$

we get

$$\|d\|_\infty = \max\left\{|d_i| : i=\overline{0,n}\right\} \leq \frac{14\cdot h^2}{\underline{h}^3}\cdot\omega\left(f,\frac{h}{2}\right).$$

The linear system (2.4) has the vectorial form

$$A\cdot m = d$$

and thus

$$m = A^{-1}\cdot d,$$

with

$$m = (m_0, m_1, ..., m_n)^T,\ d = (d_0, d_1, ..., d_n)^T.$$

Then

$$\|m\| = \max\left\{|m_i| : i=\overline{0,n}\right\} \leq \left\|A^{-1}\right\|\cdot\|d\| \leq \frac{28\cdot h^2}{\underline{h}^3}\cdot\omega\left(f,\frac{h}{2}\right).$$

Since

$$A_i(x) \geq 0,\ B_i(x) \geq 0,\ C_i(x) \leq 0,\ D_i(x) \geq 0,\ E_i(x) \geq 0,\ \forall x \in \left[x_{i-1}, x_{i-1/2}\right]$$

and

$$A_i(x) \leq 0,\ B_i(x) \geq 0,\ C_i(x) \geq 0,\ D_i(x) \leq 0,\ E_i(x) \leq 0,\ \forall x \in \left[x_{i-1/2}, x_i\right],$$

we estimate $|S(x) - f(x)|$ separately on $\left[x_{i-1}, x_{i-1/2}\right]$ and $\left[x_{i-1/2}, x_i\right]$.
On $\left[x_{i-1}, x_{i-1/2}\right]$ we have

$$\begin{aligned}|S(x) - f(x)| &\leq |A_i(x) + B_i(x)| \cdot \max\left\{|y_{i-1} - f(x)|, \left|y_{i-1/2} - f(x)\right|\right\} \\ &+ |C_i(x)| \cdot |y_i - f(x)| + |D_i(x) + E_i(x)| \cdot \max\{|m_{i-1}|, |m_i|\} \end{aligned} \tag{2.7}$$

because $A_i(x) + B_i(x) + C_i(x) = 1, \forall x \in [x_{i-1}, x_i]$, and on $\left[x_{i-1/2}, x_i\right]$ we get

$$\begin{aligned}|S(x) - f(x)| &\leq |A_i(x)| \cdot |y_{i-1} - f(x)| \\ &+ |B_i(x) + C_i(x)| \cdot \max\left\{|y_i - f(x)|, \left|y_{i-1/2} - f(x)\right|\right\} \\ &+ |D_i(x) + E_i(x)| \cdot \max\{|m_{i-1}|, |m_i|\}\end{aligned} \tag{2.8}$$

with

$$|D_i(x) + E_i(x)| = t \cdot (1-t) \cdot |1 - 2 \cdot t| \cdot h,$$

where

$$t = \frac{x - x_{i-1}}{h} \in [0,1],\ i = \overline{1,n}.$$

Elementary calculus lead as to

$$\max_{t \in \left[0, \frac{1}{2}\right]} |A_i(x) + B_i(x)| = \max_{t \in \left[\frac{1}{2}, 1\right]} |B_i(x) + C_i(x)| = \frac{9317}{8192},$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} |C_i(x)| = \max_{t \in \left[\frac{1}{2}, 1\right]} |A_i(x)| = \frac{1125}{8192},$$

and

$$\max_{t \in [0,1]} |D_i(x) + E_i(x)| = \frac{\sqrt{3}}{18} \cdot h_i,\ i = \overline{1,n}.$$

Consequently,

$$|S(x) - f(x)| \leq \frac{9317}{8192} \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h) + \frac{\sqrt{3}}{18} \cdot h,$$

$$\|m\|_\infty \leq \left(\frac{9317}{8192} + \frac{14\sqrt{3} \cdot h^3}{9 \cdot \underline{h}^3}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h),$$

$\forall\ x \in [x_{i-1}, x_i],\ i = \overline{1,n}$, obtaining (2.2). □

If f is L-Lipchitz function, then the error estimate becomes

$$|S(x) - f(x)| \leq \left(\frac{11567}{16384} + \frac{7\sqrt{3}}{9} \cdot \beta^3\right) \cdot Lh,\ \forall x \in [a,b].$$

For uniform partitions, $\beta = 1$ and the error estimate will be

$$|S(x) - f(x)| \leq \left(\frac{11567}{16384} + \frac{7\sqrt{3}}{9} \right) \cdot Lh \simeq 2.0532 \cdot Lh, \ \forall x \in [a,b].$$

Remark 2.2. The diagonally dominant linear system (2.4) can be solved easily by using the iterative algorithm provided by the Gaussian elimination technique applied to tridiagonal systems (see [1], pages 14-15).

3. Numerical experiment

In order to illustrate the theoretical result we consider a numerical example where the given data are presented in the following table, with $n = 5$:

TABLE 1. The input data

i :	0	1	2	3	4	5
x_i :	0	2	4	6	8	10
y_i :	16	20	28	21	24	28
$y_{i-1/2}$:		12	23	32	18	30

In the context of Theorem 2.1 we will make a comparison of the geometrical performances for the following three quartic splines: the C^2-smooth deficient quartic spline proposed in [7], the C^1-smooth deficient quartic spline with minimal QOA obtained before, and the C^1-smooth deficient quartic spline that minimize the functional $I_2(m_0, m_1, ..., m_n)$. For the C^2-smooth deficient quartic spline $\overline{S}$ introduced in [7] the computed local derivatives m_i , $i = \overline{0,5}$, are:

$$m_0 = -8.7018, \ m_1 = 7.1929, \ m_2 = 8.2452,$$

$$m_3 = -10.731, \ m_4 = 7.9057, \ m_5 = -4.5236.$$

The C^1-smooth deficient quartic spline S with minimal QOA has the local derivatives

$$m_0 = 13.61, \ m_1 = 2.7799, \ m_2 = 15.27,$$

$$m_3 = -12.361, \ m_4 = 2.6746, \ m_5 = 9.6627.$$

In Figure 1 are represented the C^2-smooth quartic spline with dots line, the quartic spline having minimal QOA with solid line, and the polygonal line joining the data points with dashed line. The graphs and the figure were obtained by using the Matlab application.

Computing for comparison the quadratic oscillation in average (QOA) of the above presented two quartic splines S and $\overline{S}$, and the QOA of the C^1-smooth deficient quartic spline $\widetilde{S}$ that has the local derivatives m_i , $i = \overline{0,5}$, obtained by minimizing the curvature $I_2(m_0, m_1, ..., m_n)$ we get the following results:

	S	$\overline{S}$	$\widetilde{S}$
ρ_2 :	11.173	11.359	11.284
$\mathcal{L}$:	64.703	68.676	68.237

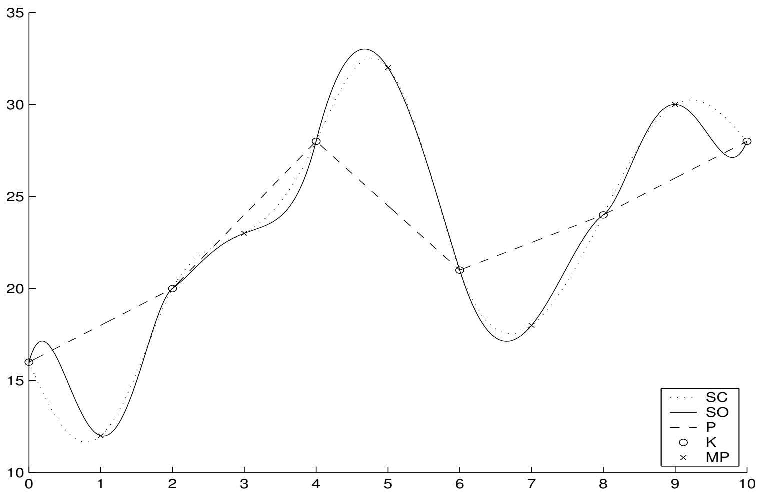

FIGURE 1. The graph of the C^2-smooth quartic spline SC represented by dotted line(....), the graph of the quartic spline SO with minimal QOA represented by solid line (—), the data polygon P represented by dashed line (- -), the knots represented with o and the midpoints represented with x

Here, we have included the length of graph ($\mathcal{L}$) for the three quartic splines, too. The computed local derivatives of $\widetilde{S}$ are

$$m_0 = -7.8476,\ m_1 = 6.9145,\ m_2 = 7.488,$$

$$m_3 = -10.225,\ m_4 = 7.8167,\ m_5 = -4.1417.$$

The geometric properties of the C^1-smooth quartic spline with minimal QOA are illustrated by considering in addition the length of graph,

$$\mathcal{L}(S) = \int_a^b \left[1 + (S'(x))^2\right]^{\frac{1}{2}} dx \tag{3.1}$$

the results for $\mathcal{L}(S)$, $\mathcal{L}\left(\overline{S}\right)$, and $\mathcal{L}\left(\widetilde{S}\right)$ being presented above. We see that better results were obtained for the C^1-smooth deficient quartic spline with minimal QOA because smaller QOA and smaller length of graph can be observed for this quartic spline. So, the theoretical result stated in Theorem 2.1 is confirmed.

4. Conclusions

The present work shows us how could be avoided possible wild oscillations induced by the interpolation at midpoints in the case of deficient quartic splines that follows the Marsden scheme of interpolation nodes. In this context we have obtained the unique C^1-smooth deficient quartic spline with minimal quadratic oscillation in average. The tridiagonal dominant linear system of normal equations which provides its local derivatives has the index of diagonal dominance $\frac{1}{2}$, and the corresponding matrix has the condition number $cond\,(A) \leq 3$, that ensures the stability of the procedure for solving this system. The numerical experiment confirm the obtained theoretical result and point out another nice geometric property of the deficient quartic spline with minimal QOA: a smaller length of the graph.

References

[1] Ahlberg, J.H., Nilson, E.H., Walsh, J.L., *The Theory of Splines and Their Applications*, Academic Press, New York, London, 1967.

[2] Bica, A.M., *Fitting data using optimal Hermite type cubic interpolating splines*, Appl. Math. Lett., **25**(2012), 2047-2051.

[3] Blaga, P., Micula, G., *Natural spline functions of even degree*, Studia Univ. Babeş-Bolyai, Mathematica, **38**(1993), no. 2, 31-40.

[4] Burmeister, W., Heß, W., Schmidt, J.W., *Convex splines interpolants with minimal curvature*, Computing, **35**(1985), 219-229.

[5] Floater, M., *On the deviation of a parametric cubic spline interpolant from its data polygon*, Comput. Aided Geom. Des., **23**(2008), 148-156.

[6] Han, X., Guo, X., *Cubic Hermite interpolation with minimal derivative oscillation*, J. Comput. Appl. Math., **33**(2018), 82-87.

[7] Howell, G., Varma, A.K., *Best error bounds for quartic spline interpolation*, J. Approx. Theory, **58**(1989), 58-67.

[8] Kobza, J., *Cubic splines with minimal norm*, Appl. Math., **47**(2002), 285-295.

[9] Kobza, J., *Quartic splines with minimal norms*, Acta Univ. Palacki. Olomuc, Fac. Res. Nat., Mathematica, **40**(2001), 103-124.

[10] Marsden, M., *Quadratic spline interpolation*, Bull. Amer. Math. Soc., **80**(1974), no. 5, 903-906.

[11] Micula, G., Micula, S., *Handbook of Splines*, Mathematics and Its Applications, vol.462, Kluwer Academic Publishers, Dordrecht, 1999.

[12] Micula, Gh., Santi, E., Cimoroni, M.G., *A class of even degree splines obtained through a minimum condition*, Studia Univ. Babeş-Bolyai, Mathematica, **48**(2003), no. 3, 93-104.

[13] Volkov, Yu. S., *Best error bounds for the derivative of a quartic interpolation spline*, (in Russian), Mat. Trud., **1**(1998), no. 2, 68-78.

Diana Curilă (Popescu)
University of Oradea,
Department of Mathematics and Informatics,
1, Universităţii Street,
410087 Oradea, Romania
e-mail: curila_diana@yahoo.com