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Finite time blow-up for quasilinear wave equations with nonlinear dissipation

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Abstract. In this paper we consider a class of quasilinear wave equations

$$u_{tt} - \Delta_\alpha u - \omega_1 \Delta u_t - \omega_2 \Delta_\beta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u,$$

associated with initial and Dirichlet boundary conditions. Under certain conditions on α, β, m, p, we show that any solution with positive initial energy, blows up in finite time. Furthermore, a lower bound for the blow-up time will be given.

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1. Introduction

In this paper, we would like to study the blow-up of solutions of the following initial boundary value problem of a quasilinear wave equation

$$\begin{cases} u_{tt} - \Delta_\alpha u - \omega_1 \Delta u_t - \omega_2 \Delta_\beta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Additionally, we assume that

$$u_0 \in W_0^{1,\alpha}(\Omega), \quad u_1 \in L^2(\Omega), \tag{1.2}$$

and $\alpha, \beta, \omega_1, \omega_2, \mu, m, p$ are positive constants, with

$$\begin{cases} 2 < p \le \alpha^* = \frac{\alpha n}{n-\alpha}, & \text{for } n > \alpha, \\ 2 < p < \infty, & \text{for } n = \alpha. \end{cases} \tag{1.3}$$

The operator Δ_α is the classical α-Laplacian given by:

$$\Delta_\alpha u = \operatorname{div}\left(|\nabla u|^{\alpha-2} \nabla u\right).$$

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Notice that $\Delta_\beta u_t$ is a quasilinear strong damping term, and it is degenerate when $\beta > 2$.

Nonlinear hyperbolic equations of the type (1.1) have been investigated in the papers [2, 5, 7, 9, 15], and the references therein. Several examples of this type arise in physics, for example, the problem (1.1) represents a longitudinal motion of a viscoelastic rod obeying the nonlinear Voight model.

Zhijiang [14] proved a blow up result for the problem (1.1) when the initial energy is sufficiently negative. This result was extended by Messaoudi and Houari [8] to a situation when the solution has negative initial energy. Liu and Wang [6] studied a more general model including (1.1), and by improving the arguments in [14] and [8] they established a blow-up result in the subcritical initial energy case, i.e. $E(0) < d$, where $E(0)$ is the initial energy and d is the depth of the potential well.

For $\alpha = \beta = m = 2$, equation in (1.1) reduces to the linearly damped wave equation

$$u_{tt} - \Delta u + \omega \Delta u_t + \mu u_t = |u|^{p-2}u. \tag{1.4}$$

Gazzola and Squassina [3] studied (1.4) and gave a necessary and sufficient conditions for blow-up if $E(0) < d$. Recently, Yang and Xu [13] gave a sufficient condition for blow-up if $E(0) > d$. Sun et al. [12] obtained, for (1.4), an estimate of the lower bound for the blow-up time when $2 < p \leq \frac{2(n-1)}{n-2}$. This work was extended by Guo and Liu [4] to the case when the exponent $p \in \left(\frac{2(n-1)}{n-2}, \frac{2(n^2-2)}{n-2}\right]$. Later, in the case of $\omega > 0$, Baghaei [1] improved the results in [12] and [4] by enlarging the upper bound for p to 2^*.

In related work, Song and Xue [11] studied the following nonlinear wave equation with strong damping

$$u_{tt} - \Delta u + \int_0^t g((t-\tau)\Delta u(\tau)d\tau - \Delta u_t = |u|^{p-2}u. \tag{1.5}$$

They introduced a new technique to obtain a finite time blow-up result with arbitrary high initial energy in the case of linear strong damping. By applying the technique similar to that in [11], Song [10] extended the result in [11] to the case of nonlinear weak damping $\mu|u_t|^{m-2}u_t$ in place of $-\Delta u_t$ in (1.5).

In this paper, by using the technique in [10], we give sufficient conditions for finite time blow-up of solutions of (1.1), in the case $E(0) \geq d$. Furthermore, by using the techniques in [4], we obtain a lower bound for the blow-up time.

2. Preliminaries

We denote by $\|.\|_p$ the $L^p(\Omega)$ norm $(2 \leq p < \infty)$, and by $(.,.)$ the L^2 inner product. We introduce the following functional space

$$\mathcal{H} := L^\infty([0,T), W_0^{1,\alpha}(\Omega)) \cap W^{1,\infty}([0,T), L^2(\Omega)) \\ \cap W^{1,\beta}([0,T), W^{1,\beta}(\Omega)) \cap W^{1,m}([0,T), L^m(\Omega)),$$

for $T > 0$, and the energy functional

$$E(t) := \frac{1}{2}\|\nabla u\|_\alpha^\alpha + \frac{1}{2}\|u_t\|_2^2 - \frac{1}{p}\|u\|_p^p.$$

We define also the following constant

$$\lambda = B_*^{-\frac{p}{p-\alpha}},$$

where B_* is the best constant of the Sobolev embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$. Finally, we characterize the depth of the potential well d as follows:

$$d = \left(\frac{1}{\alpha} - \frac{1}{p}\right)\lambda^2.$$

Lemma 2.1. *Let u be a global solution to problem (1.1). Then we have*

$$E'(t) = -\omega_1\|\nabla u_t\|_2^2 - \omega_2\|\nabla u_t\|_\beta^\beta - \mu\|u_t\|_m^m, \quad \forall t \geq 0.$$

As a consequence, we have the following inequalities:

$$E(t) \leq E(0), \quad \forall t \geq 0, \tag{2.1}$$

and

$$-E'(t) \geq \omega_1\|\nabla u_t\|_2^2, \quad -E'(t) \geq \omega_2\|\nabla u_t\|_\beta^\beta, \quad -E'(t) \geq \mu\|u_t\|_m^m. \tag{2.2}$$

Subsequently, we state the following theorems (see [6]).

Theorem 2.2 (Local existence). *Assume that conditions (1.2) and (1.3) hold. Then problem (1.1) has a unique local solution $u \in \mathcal{H}$.*

Theorem 2.3 (Blow-up for $E(0) < d$). *Assume (1.2) and (1.3) hold. Assume further that $\alpha, \beta, m \geq 2$ and $p > \alpha > \max\{m, \beta\}$. Suppose $E(0) < d$ and*

$$\|\nabla u_0\|_\alpha > \lambda. \tag{2.3}$$

Then u blows up in finite time.

3. Finite time blow-up

In this section we extend the blow-up result in [8] to the case $E(0) \geq d$. Here is our main result:

Theorem 3.1 (Blow-up for $E(0) \geq d$). *Assume (1.2), (2.3) and (1.3) hold. Assume further that $\alpha, \beta, m > 2$, $\alpha > \beta$ and $p > \max\{m, \alpha\}$. Suppose $E(0) \geq d$ and*

$$(u_t(0), u(0)) > ME(0), \tag{3.1}$$

where $M > 0$ is defined in (3.7), then the solution $u \in \mathcal{H}$ of (1.1) blows up in finite time.

Proof. Assume by contradiction that $u(t)$ is a global solution of (1.1). Setting

$$F(t) := \frac{1}{2}\|u(t)\|_2^2,$$

it follows from (1.1) that

$$\begin{aligned} F''(t) &= \|u_t\|_2^2 + \|u\|_p^p - \|\nabla u\|_\alpha^\alpha \\ &\quad - \omega_1(\nabla u_t, \nabla u) - \omega_2(|\nabla u_t|^{\beta-2}\nabla u_t, u) - \mu(|u_t|^{m-2}u_t, u). \end{aligned} \tag{3.2}$$

By using Hölder's inequality and Young's inequality, we estimate the two last terms in the right-hand side of the previous equation, as follows

$$(\nabla u_t, \nabla u) \leq \eta\|\nabla u\|_2^2 + \tfrac{1}{4\eta}\|\nabla u_t\|_2^2, \quad \eta > 0,$$

$$(|\nabla u_t|^{\beta-2}\nabla u_t, u) \leq \tfrac{1}{\beta}\sigma^\beta\|\nabla u\|_\beta^\beta + \tfrac{\beta-1}{\beta}\sigma^{\beta/(1-\beta)}\|\nabla u_t\|_\beta^\beta, \quad \sigma > 0,$$

$$(|u_t|^{m-2}u_t, u) \leq \tfrac{1}{m}\delta^m\|u\|_m^m + \tfrac{m-1}{m}\delta^{m/(1-m)}\|u_t\|_m^m, \quad \delta > 0.$$

So, thatnks to the convexity of the function y^x/x for $y \geq 0$ and $x > 0$, we have

$$\begin{aligned} \frac{\delta^m}{m}\|u\|_m^m &\leq \frac{s}{2}\delta^m\|u\|_2^2 + \frac{1-s}{p}\delta^m\|u\|_p^p, \qquad s = \frac{p-m}{p-2}, \\ \frac{1}{\beta}\sigma^\beta\|\nabla u\|_\beta^\beta &\leq \frac{\theta}{2}\sigma^\beta\|\nabla u\|_2^2 + \frac{1-\theta}{\alpha}\sigma^\beta\|\nabla u\|_\alpha^\alpha, \qquad \theta = \frac{\alpha-\beta}{\alpha-2}. \end{aligned}$$

Hence, (3.2) becomes

$$\begin{aligned} F''(t) &\geq \|u_t\|_2^2 - \left[1 + \frac{\omega_2(1-\theta)}{\alpha}\sigma^\beta\right]\|\nabla u\|_\alpha^\alpha - \frac{\mu s}{2}\delta^m\|u\|_2^2 \\ &\quad - \left(\omega_1\eta + \frac{\omega_2\theta}{2}\sigma^\beta\right)\|\nabla u\|_2^2 + \left[1 - \frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p \\ &\quad - \frac{\omega_1}{4\eta}\|\nabla u_t\|_2^2 - \omega_2\frac{\beta-1}{\beta}\sigma^{\beta/(1-\beta)}\|\nabla u_t\|_\beta^\beta - \mu\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t\|_m^m. \end{aligned} \tag{3.3}$$

Next, since $u(t)$ is global and $E(0) \geq d$, then by Theorem 2.3, $E(t) \geq d$, $\forall t \geq 0$. Thus, using the embedding $L^\alpha(\Omega) \hookrightarrow L^2(\Omega)$ and the inequality

$$z^b \leq (z+a)\left(z + \frac{1}{a}\right), \quad z \geq 0,\ 0 < b \leq 1,\ a > 0,$$

we obtain

$$\begin{aligned} \|\nabla u\|_2^2 &\leq c\|\nabla u\|_\alpha^2 \\ &= c\left[\|\nabla u\|_\alpha^\alpha\right]^{2/\alpha} \\ &\leq c\left(1 + \frac{1}{d}\right)\left[\|\nabla u\|_\alpha^\alpha + d\right] \\ &\leq C\left[\|\nabla u\|_\alpha^\alpha + E(t)\right], \quad \forall t \geq 0. \end{aligned} \tag{3.4}$$

By using Lemma 2.1 and (2.2), we get

$$\frac{d}{dt}\left\{F'(t)-\left[\frac{1}{4\eta}+\frac{\beta-1}{\beta}\sigma^{\frac{-\beta}{\beta-1}}+\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right]E(t)\right\}$$
$$\geq F''(t)+\frac{\omega_1}{4\eta}\|\nabla u_t\|_2^2+\omega_2\frac{\beta-1}{\beta}\sigma^{-\frac{\beta}{\beta-1}}\|\nabla u_t\|_\beta^\beta+\mu\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t\|_m^m.$$

Adding and subtracting $p(1-\varepsilon)E(t)$, for $\varepsilon\in(0,1)$, in the right-hand side of the last inequality, and using (3.4) and the Poincaré inequality we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{F'(t)-\left[\frac{1}{4\eta}+\frac{\beta-1}{\beta}\sigma^{-\frac{\beta}{\beta-1}}+\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right]E(t)\right\}\\
&\geq \|u_t\|_2^2-\frac{\mu s}{2}\delta^m\|u\|_2^2-\left[1+\frac{\omega_2(1-\theta)}{\alpha}\sigma^\beta\right]\|\nabla u\|_\alpha^\alpha\\
&-\left(\omega_1\eta+\frac{\omega_2\theta}{2}\sigma^\beta\right)\|\nabla u\|_2^2+\left[1-\frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p\\
&\geq\left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2-\frac{\mu s}{2}\delta^m\|u\|_2^2+k(\varepsilon)\|\nabla u\|_\alpha^\alpha\\
&-\left(\omega_1\eta+\frac{\omega_2\theta}{2}\sigma^\beta\right)\|\nabla u\|_2^2+\left[\varepsilon-\frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p-p(1-\varepsilon)E(t)\\
&\geq\left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2-\frac{\mu s}{2}\delta^m\|u\|_2^2+\gamma(\varepsilon)\|\nabla u\|_2^2\\
&+\left[\varepsilon-\frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p-\left[k(\varepsilon)+p(1-\varepsilon)\right]E(t)\\
&\geq\left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2+\left\{\gamma(\varepsilon)B-\frac{\mu s}{2}\delta^m\right\}\|u\|_2^2\\
&+\left[\varepsilon-\frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p-\left[k(\varepsilon)+p(1-\varepsilon)\right]E(t),
\end{aligned}\tag{3.5}$$

where

$$k(\varepsilon)=\frac{1}{\alpha}\left[p(1-\varepsilon)-\alpha-\omega_2(1-\theta)\sigma^\beta\right],$$
$$\gamma(\varepsilon)=\frac{k(\varepsilon)}{C}-\omega_1\eta-\frac{\omega_2\theta}{2}\sigma^\beta,$$

and B is the best constant of Poincaré inequality

$$\|\nabla u\|_2^2\geq B\|u\|_2^2.$$

Therefore, taking $\eta=\varepsilon$, $\sigma=\varepsilon$,

$$\delta=\left[\frac{p\varepsilon}{\mu(1-s)}\right]^{1/m},$$

setting

$$\gamma_1(\varepsilon)=\frac{1}{4\varepsilon}+\frac{\beta-1}{\beta}\varepsilon^{-\frac{\beta}{\beta-1}}+\frac{m-1}{m}\left(\frac{1-s}{p\varepsilon}\right)^{-\frac{1}{m-1}},$$

and substituting in (3.5), we arrive at

$$\frac{d}{dt}\left[F'(t)-\gamma_1(\varepsilon)E(t)\right] \geq \left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2 + \left[\gamma(\varepsilon)B-\frac{ps}{2(1-s)}\varepsilon\right]\|u\|_2^2 - \left[k(\varepsilon)+p(1-\varepsilon)\right]E(t).$$

By using the Schwarz inequality, we have

$$2\left[1+\tfrac{p}{2}(1-\varepsilon)\right]^{1/2}\left[\gamma(\varepsilon)B-\tfrac{ps}{2(1-s)}\varepsilon\right]^{1/2}(u_t,u) \leq \left[1+\tfrac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2+\left[\gamma(\varepsilon)B-\tfrac{ps}{2(1-s)}\varepsilon\right]\|u\|_2^2.$$

Consequently, we obtain

$$\begin{aligned}\frac{d}{dt}\left[F'(t)-\gamma_1(\varepsilon)E(t)\right] &\geq a(\varepsilon)(u_t,u)-\left[k(\varepsilon)+p(1-\varepsilon)\right]E(t)\\ &= a(\varepsilon)\left[F'(t)-\gamma_2(\varepsilon)E(t)\right],\end{aligned} \tag{3.6}$$

where

$$\begin{aligned} a(\varepsilon) &= 2\left[1+\tfrac{p}{2}(1-\varepsilon)\right]^{1/2}\left[\gamma(\varepsilon)B-\tfrac{ps}{2(1-s)}\varepsilon\right]^{1/2},\\ \gamma_2(\varepsilon) &= \tfrac{k(\varepsilon)+p(1-\varepsilon)}{a(\varepsilon)}.\end{aligned}$$

Since

$$\gamma(\varepsilon)B-\frac{ps}{2(1-s)}\varepsilon \to \begin{cases} \frac{B(p-\alpha)}{\alpha C}>0 & \text{as } \varepsilon\to 0^+\\ -\left[\frac{\alpha+\omega_2(1-\theta)}{\alpha C}+\omega_1+\frac{\omega_2\theta}{2}\right]B-\frac{ps}{2(1-s)}<0 & \text{as } \varepsilon\to 1^-,\end{cases}$$

then, there exists $\varepsilon_* \in (0,1)$, such that

$$a(\varepsilon_*)=0 \text{ and } a(\varepsilon)>0, \quad \forall\varepsilon\in(0,\varepsilon_*).$$

Hence, we have

$$\gamma_1(\varepsilon)-\gamma_2(\varepsilon)\to\begin{cases}+\infty & \text{as } \varepsilon\to 0^+\\ -\infty & \text{as } \varepsilon\to\varepsilon_*^-.\end{cases}$$

Therefore, there exists $\varepsilon_0\in(0,\varepsilon_*)$, such that $\gamma_1(\varepsilon_0)=\gamma_2(\varepsilon_0)>0$. So, by setting

$$\begin{aligned} L(t) &= F'(t)-\gamma_1(\varepsilon_0)E(t),\\ M &= \gamma_1(\varepsilon_0),\end{aligned} \tag{3.7}$$

and by using (2.3), we obtain

$$\begin{aligned} L(0) &= (u_t(0),u(0))-\gamma_1(\varepsilon_0)E(0)\\ &> (u_t(0),u(0))-ME(0)>0.\end{aligned}$$

Moreover, with this choice of ε_0, (3.6) becomes

$$\frac{d}{dt}L(t)\geq a(\varepsilon_0)L(t),$$

which gives

$$L(t) \geq L(0)e^{a(\varepsilon_0)t}, \qquad \forall t \geq 0,$$

and hence

$$F'(t) \geq L(0)e^{a(\varepsilon_0)t}, \qquad \forall t \geq 0.$$

By integrating this last inequality over $(0, t)$, we get

$$\|u(t)\|_2^2 = 2F(t) \geq 2F(0) + 2\frac{L(0)}{a(\varepsilon_0)}\left[e^{a(\varepsilon_0)t} - 1\right], \quad \forall t \geq 0. \tag{3.8}$$

On the other hand, by using Hölder's inequality and (2.2), we have

$$\begin{aligned}
\|u(t)\|_2 &\leq \|u(0)\|_2 + \int_0^t \|u_\tau(\tau)\|_2 d\tau \\
&\leq \|u(0)\|_2 + C\int_0^t \|u_\tau(\tau)\|_m d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \int_0^t \|u_\tau(\tau)\|_m^m d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \int_0^t \frac{-1}{\mu}\frac{dE(\tau)}{d\tau} d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \left[\frac{E(0) - E(t)}{\mu}\right]^{1/m} \\
&\leq \|u(0)\|_2 + C\left[\frac{E(0)}{\mu}\right]^{1/m} t^{\frac{m-1}{m}},
\end{aligned}$$

which clearly contradicts (3.8). □

4. Lower bound for the blow-up time

In this section, we give a lower bound for the blow-up time $T_{\max}$. To this end, we define

$$G(t) := \frac{1}{p}\|u(t)\|_p^p.$$

Theorem 4.1. *Let u be the solution of (1.1), and assume that*

$$\begin{cases} 2 < p \leq \frac{\alpha(n-2)+2n}{2(n-\alpha)}, & \text{for } n > \alpha, \\ 2 < p < \infty, & \text{for } n = \alpha. \end{cases}$$

Then

$$T_{\max} \geq \int_{G(0)}^{+\infty} \left\{\tau + A_1\tau^{\frac{2}{\alpha}(p-1)} + A_2\right\}^{-1} d\tau,$$

where A_1 and A_2 are positive constants to be determined later in the proof.

Proof. By using inequality (2.1), we have

$$\frac{1}{2}\|u_t\|_2^2 + \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha = E(t) + \frac{1}{p}\|u(t)\|_p^p \leq E(0) + G(t). \tag{4.1}$$

Next, using the Schwarz inequality, the Sobolev-type inequality

$$\|u\|_q \leq C_q\|\nabla u\|_\alpha, \quad \forall q \in [1, \alpha^*], \quad \forall u \in W_0^{1,\alpha}(\Omega), \tag{4.2}$$

inequality (4.1) yields

$$\begin{aligned} G'(t) &= (|u|^{p-2}u, u_t) \\ &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_{2(p-1)}^{2(p-1)} \\ &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{C_{2(p-1)}^{2(p-1)}}{2}\|\nabla u\|_\alpha^{2(p-1)} \\ &\leq E(0) + G(t) + \frac{C_{2(p-1)}^{2(p-1)}}{2}\left[\alpha E(0) + \alpha G(t)\right]^{\frac{2}{\alpha}(p-1)}. \end{aligned} \tag{4.3}$$

From (4.3) and Jensen's inequality, we obtain the differential inequality

$$G'(t) \leq G(t) + A_1\left[G(t)\right]^{\frac{2}{\alpha}(p-1)} + A_2, \tag{4.4}$$

with

$$A_1 = C_*^{2(p-1)}2^{\frac{2}{\alpha}(p-1)-2}\alpha^{\frac{2}{\alpha}(p-1)} \quad \text{and} \quad A_2 = E(0) + A_1\left[E(0)\right]^{\frac{2}{\alpha}(p-1)}.$$

Hence, we get

$$T_{\max} \geq \int_0^{T_{\max}} \left\{G(s) + A_1\left[G(s)\right]^{\frac{2}{\alpha}(p-1)} + A_2\right\}^{-1} G'(s)ds.$$

Since $\lim_{t\to T_{\max}^-} G(t) = +\infty$, so the previous inequality implies

$$T_{\max} \geq \int_{G(0)}^{+\infty} \left\{\tau + A_1\tau^{\frac{2}{\alpha}(p-1)} + A_2\right\}^{-1} d\tau.$$

□

In the next theorem, when $n > \alpha$, the upper bound for p is enlarged. We define

$$H(t) := \frac{1}{\sigma}\|u(t)\|_\sigma^\sigma,$$

where $\sigma = \frac{\alpha(n-2)+2n}{2(n-\alpha)}$. Then, we have

Theorem 4.2. *Let u be the solution of (1.1), and assume that*

$$\frac{\alpha(n-2)+2n}{2(n-\alpha)} < p \leq \frac{\alpha n(2n-\alpha+2)-2\alpha^2}{2n(n-\alpha)}. \tag{4.5}$$

Then

$$T_{\max} \geq \int_{H(0)}^{+\infty} \left\{B_1\tau^{b_1} + B_2\tau^{b_2} + B_3\right\}^{-1} d\tau,$$

where B_1, B_2, B_3, b_1 and b_2 are positive constants to be determined later in the proof.

Proof. By using inequality (2.1), we have

$$\frac{1}{2}\|u_t\|_2^2+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha=E(t)+\frac{1}{p}\|u(t)\|_p^p\le E(0)+\frac{1}{p}\|u(t)\|_p^p. \tag{4.6}$$

Using the Schwarz inequality, the Sobolev-type inequality (4.2), with $q=\alpha^*$, and inequality (4.6) we get

$$\begin{aligned}
H'(t)&=(|u|^{\sigma-2}u,u_t)\\
&\le\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|u\|_{2(\sigma-1)}^{2(\sigma-1)}\\
&\le\frac{1}{2}\|u_t\|_2^2+\frac{C_*^{\alpha^*}}{2}\|\nabla u\|_\alpha^{\alpha^*}\\
&\le E(0)+\frac{1}{p}\|u\|_p^p+\frac{C_*^{\alpha^*}}{2}\left[\alpha E(0)+\frac{\alpha}{p}\|u\|_p^p\right]^{\frac{n}{n-\alpha}}.
\end{aligned}\tag{4.7}$$

Next, the interpolation inequality, the Sobolev inequality and Young's inequality give

$$\begin{aligned}
\|u\|_p^p&\le\|u\|_{\alpha^*}^{\theta p}.\|u\|_\sigma^{(1-\theta)p},\qquad \theta=\frac{\alpha^*(p-\sigma)}{p(\alpha^*-\sigma)},\\
&\le C_*^{\theta p}\|\nabla u\|_\alpha^{\theta p}.\|u\|_\sigma^{(1-\theta)p},\\
&\le\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+B\|u\|_\sigma^r,
\end{aligned}\tag{4.8}$$

where

$$B=C_*\left(1-\frac{\theta p}{\alpha}\right)(p\theta C_*)^{\frac{p\theta}{\alpha-p\theta}}\quad\text{and}\quad r=\frac{\alpha p(1-\theta)}{\alpha-\theta p}.$$

Note that in virtue of (4.5), we have $\alpha>\theta p$. Hence, by (2.1) we have

$$\|u\|_p^p\le E(0)+\frac{1}{p}\|u\|_p^p+B\|u\|_\sigma^r, \tag{4.9}$$

which gives

$$\frac{1}{p}\|u\|_p^p\le\frac{1}{p-1}\left(E(0)+B\|u\|_\sigma^r\right).$$

Inserting this last inequality in (4.7), and using Jensen's inequality, we obtain

$$\begin{aligned}
H'(t)&\le\frac{pE(0)}{p-1}+\frac{B}{p-1}\|u\|_\sigma^r+\frac{C_*^{\alpha^*}}{2}\left[\frac{\alpha pE(0)}{p-1}+\frac{\alpha B}{p-1}\|u\|_\sigma^r\right]^{\frac{n}{n-\alpha}}\\
&=B_1\left(H(t)\right)^{b_1}+B_2\left(H(t)\right)^{b_2}+B_3,
\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}
B_1&=\frac{B\sigma^r}{p-1},\quad B_2=\frac{C_*^{\alpha^*}}{2}2^{\frac{\alpha}{n-\alpha}}\left[\frac{\alpha B\sigma^r}{p-1}\right]^{\frac{n}{n-\alpha}},\\
B_3&=\frac{pE(0)}{p-1}+\frac{C_*^{\alpha^*}}{2}2^{\frac{\alpha}{n-\alpha}}\left[\frac{\alpha pE(0)}{p-1}\right]^{\frac{n}{n-\alpha}},\\
b_1&=\frac{r}{\sigma},\quad b_2=\frac{rn}{\sigma(n-\alpha)}.
\end{aligned}$$

Finally, integrating inequality (4.10) over $(0, T_{\max})$ we get

$$T_{\max} \geq \int_0^{T_{\max}} \left\{ B_1 \left(H(s)\right)^{b_1} + B_2 \left(H(s)\right)^{b_2} + B_3 \right\}^{-1} H'(s)ds,$$

and so

$$T_{\max} \geq \int_{H(0)}^{+\infty} \left\{ B_1 \tau^{b_1} + B_2 \tau^{b_2} + B_3 \right\}^{-1} d\tau.$$

□

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