Stud. Univ. Babeş-Bolyai Math. 67(2022), No. 4, 717–730
DOI: 10.24193/subbmath.2022.4.04

Blow-up results for damped wave equation with fractional Laplacian and non linear memory

Tayeb Hadj Kaddour and Ali Hakem

Abstract. The goal of this paper is to study the nonexistence of nontrivial solutions of the following Cauchy problem

$$\begin{cases} u_{tt} + (-\Delta)^{\beta/2}u + u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p \, d\tau, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where $p > 1$, $0 < \gamma < 1$, $\beta \in (0,2)$ and $(-\Delta)^{\beta/2}$ is the fractional Laplacian operator of order $\frac{\beta}{2}$. Our approach is based on the test function method.

Mathematics Subject Classification (2010): 26A33, 35K55, 74G25, 74H35.

Keywords: Damped wave equation, blow-up, Fujita's exponent, fractional derivative.

1. Introduction

The main goal of this paper is to discuss the critical exponent to the following Cauchy problem

$$\begin{cases} u_{tt} + (-\Delta)^{\beta/2}u + u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p \, d\tau, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $(-\Delta)^s$, $s \in (0,1)$, is the fractional Laplacian operator defined by

$$(-\Delta)^s f(x) = C_{n,s} \; P.V \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n, \tag{1.2}$$

Received 02 February 2020; Accepted 10 May 2020.

as long as the right-hand side exists, where $P.V$ stands for the Cauchy's principal value and

$$C_{n,s} = \frac{4^s\Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}\Gamma(-s)}$$

is the normalization constant and Γ denotes the Gamma function. Indeed, the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$ is a pseudo-differential operator of symbol $p(x,\xi) = |\xi|^{2s}$, $\xi \in \mathbb{R}^n$, defined by

$$(-\Delta)^s v = \mathcal{F}^{-1}\big(|\xi|^{2s}\mathcal{F}v(\xi)\big), \quad \text{for all } v \in \mathcal{S}'(\mathbb{R}^n), \tag{1.3}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are, respectively, the Fourier transform and its inverse. In fact $(-\Delta)^s$ is a particular case of Levy operator $\mathcal{L}$ defined by

$$\mathcal{L}v(x) = \mathcal{F}^{-1}\big(a(\xi)\mathcal{F}v(\xi)\big)(x), \quad \text{for all } v \in \mathcal{S}'(\mathbb{R}^n),\ x \in \mathbb{R}^n. \tag{1.4}$$

For more details about these notions, we refer to ([1], [8], [13], [9], [3], [14]) and the references therein.

Before we present our results, let us mention below some motivations for studying the problem of the type (1.1). In [2], Cazenave and al. considered the corresponding equation

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-\tau)^{-\gamma}|u(\tau,\cdot)|^{p-1}u(\tau,\cdot)d\tau, \\ 0 \le \gamma < 1, \quad u_0 \in C_0(\mathbb{R}^n). \end{cases} \tag{1.5}$$

It was shown that, if

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(n-2+2\gamma)_+} \quad \text{and } p^* = \max\left(p_\gamma, \gamma^{-1}\right),$$

where

$$(n-2+2\gamma)_+ = \max(n-2+2\gamma,\, 0).$$

Then

1. If $\gamma \neq 0$, $p \le p^*$ and $u_0 > 0$, then the solution u of (1.5) blows up in finite time.
2. If $\gamma \neq 0, p > p^*$ and $u_0 \in L_{q^*}(\mathbb{R}^n)$ (where $q^* = \frac{(p-1)n}{4-2\gamma}$) with $\|u_0\|_{L_{q^*}}$ small enough, then u exists globally. In particular, They proved that the critical exponent in Fujita's sense p^* is not the one predicted by scaling. This is not a surprising result since it is well known that scaling is efficient only for parabolic equations and not for pseudo-parabolic ones. To show this, it is sufficient to note that, formally, equation (1.5) is equivalent to

$$D^\alpha_{0|t}u_t - D^\alpha_{0|t}\Delta u = \Gamma(\alpha)\,|u|^{p-1}\,u,$$

where $\alpha = 1-\gamma$ and $D^\alpha_{0|t}$ is the fractional derivative operator of order α ($\alpha \in (0,1)$) in Riemann-Liouville sense defined by

$$D^\alpha_{0|t}u = \frac{d}{dt}J^{1-\alpha}_{0|t}u, \tag{1.6}$$

and $J^{1-\alpha}_{0|t}$ is the fractional integral of order $1-\alpha$ defined by the formula (2.2) bellow.

In the special case $\gamma = 0$, Souplet [15] proved that the nonzero positive solution of (1.5) blows -up in finite time. Note that the classical damped wave equation with nonlinear memory, namely

$$u_{tt} - \Delta u + u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau, \tag{1.7}$$

was investigated by Fino [4]. He studied the global existence and blow-up of solutions. He used as the main tool the weighted energy method with a weight similar to the one introduced by G. Todorova an B. Yardanov [16], while he employed the test function method to derive nonexistence results. In particular, he found the same p_γ and so the same critical exponent p^* founded by Cazenave and al in [2]. More recently, the Authors of [6] generalized the results of [2] and [4] by establishing nonexistence results for the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + D^{\sigma}_{0|t} u_t = \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau, \quad t > 0. \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.8}$$

Remark 1.1. Throughout, C denotes a positive constant, whose value may change from line to line.

2. Blow up solutions

This section is devoted to prove blow-up results of problem (1.1). The method which we will use for our task is the test function method considered by Mitidieri and Pohozaev ([10], [11]), Pohozaev and Tesei [12], Fino [4], Hadj-Kaddour and Hakem ([5], [6]); it was also used by Zhang [17].
Before that, one can show that the problem (1.1) can be written in the following form:

$$\begin{cases} u_{tt} + (-\Delta)^{\beta/2} u + u_t = \Gamma(\alpha) J^{\alpha}_{0|t}(|u|^p), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \text{for all} \quad x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $\alpha = 1 - \gamma$ and $J^{\alpha}_{0|t}$ is the fractional integral of order α ($\alpha \in (0,1)$) defined for all $v \in L^1_{loc}(\mathbb{R})$, by

$$J^{\alpha}_{0|t} v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \tag{2.2}$$

where $(-\Delta)^{\beta/2}$ is the fractional Laplacian operator of order $\beta/2$, $\beta \in (0,2)$.
First, let us introduce what we mean by a weak solution for problem (2.1).

Definition 2.1. Let $T > 0$, $\gamma \in (0,1)$ and $\beta \in (0,2)$. A weak solution for the Cauchy problem (2.1) in $[0,T) \times \mathbb{R}^n$ with initial data $(u_0, u_1) \in L^1_{loc}(\mathbb{R}^n) \times L^1_{loc}(\mathbb{R}^n)$ is a locally integrable function $u \in L^p\big((0,T), L^p_{loc}(\mathbb{R}^n)\big)$ that satisfies

$$\begin{aligned}\Gamma(\alpha)&\int_0^T\int_{\mathbb{R}^n} J^\alpha_{0|t}(|u|^p)\varphi(t,x)dtdx + \int_{\mathbb{R}^n}(u_0(x)+u_1(x))\varphi(0,x)dx\\ &-\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx\\ &-\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_t(t,x)dtdx - \int_0^T\int_{\mathbb{R}^n} u(t,x)(-\Delta)^{\beta/2}\varphi(t,x)dtdx, \end{aligned}\tag{2.3}$$

for all non-negative test function $\varphi \in \mathcal{C}^2([0,T]\times\mathbb{R}^n)$ such that $\varphi(T,\cdot) = \varphi_t(T,\cdot) = 0$ and $\alpha = 1-\gamma$. If $T = \infty$, we call u a global in time weak solution to (2.1).

Now, we are ready to state the main results of this paper. For all $\gamma \in (0,1)$, $\beta \in (0,2)$ and $n \in \mathbb{N}$, we put

$$p_\gamma(\beta) = 1 + \frac{\beta(2-\gamma)}{(n-\beta(1-\gamma))_+} \quad \text{and} \quad p^* = \max\{p_\gamma(\beta), \gamma^{-1}\}. \tag{2.4}$$

Theorem 2.2. *Let $0<\gamma<1$, $p \in (1,\infty)$ for $n = 1,2$ and $1 < p < \frac{n}{n-2}$ for $n \geq 3$. We assume that $(u_0, u_1) \in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ satisfying the following relation:*

$$\int_{\mathbb{R}^n} u_i(x)dx > 0, \quad i = 0,1. \tag{2.5}$$

Moreover, we suppose the condition

$$p \leq p^*.$$

Then, the problem (2.1) admits no global weak solution.

The proof of our main result is given in the next section.

3. Proofs

In this section, we give the proof of Theorem 2.2. For this task, we choose a test function for some $T>0$, as follows:

$$\varphi(t,x) = D^\alpha_{t|T}\psi(t,x) = \varphi_1^\ell(x) D^\alpha_{t|T}\varphi_2(t), \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \tag{3.1}$$

where $\ell > 1$ and $D^\alpha_{t|T}$ is the right fractional derivative operator of order α in the sense of Riemann-Liouville defined by

$$D^\alpha_{t|T}v(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{v(s)}{(s-t)^\alpha}ds, \tag{3.2}$$

and the functions φ_1 and φ_2 are given by

$$\varphi_1(x) = \phi\Big(\frac{x^2}{K}\Big), \quad \varphi_2(t) = \Big(1-\frac{t}{T}\Big)_+^\sigma, \tag{3.3}$$

with $K>0$, $\sigma > 1$ and ϕ is a smooth non-increasing function such that

$$\phi(s) = \begin{cases} 1 & \text{if } \ 0 \leq s \leq 1,\\ 0 & \text{if } \ s \geq 2,\end{cases} \quad 0 \leq \phi \leq 1 \text{ everywhere and } |\phi'(s)| \leq \frac{C}{s}. \tag{3.4}$$

We also denote by Ω_K for the support of φ_1, that is

$$\Omega_K = supp\varphi_1 = \{x \in \mathbb{R}^n,\ |x|^2 \le 2K\}, \tag{3.5}$$

and by Δ_K for the set containing the support of $\Delta\varphi_1$ which is defined as follows:

$$\Delta_K = \{x \in \mathbb{R}^n,\ K \le |x|^2 \le 2K\}. \tag{3.6}$$

Furthermore, for every $f, g \in \mathcal{C}([0,T])$ such that $D^\alpha_{0|t} f(t)$ and $D^\alpha_{t|T} g(t)$ exist and are continuous, for all $t \in [0,T],\ 0 < \alpha < 1$ we have the formula of integration by parts([14])

$$\int_0^t f(t) D^\alpha_{t|T} g(t)dt = \int_0^t \big(D^\alpha_{0|t} f(t)\big) g(t)dt, \tag{3.7}$$

Note also that, for all $u \in \mathcal{C}^n[0,T]$ and all integers $n \ge 0$, we have

$$(-1)^n \partial_t^n D^\alpha_{t|T} u(t) = D^{\alpha+n}_{t|T} u(t), \tag{3.8}$$

where ∂_t^n is the $n-$times ordinary derivative with respect to t. Moreover, for all $1 \le q \le \infty$, the following formula

$$\big(D^\alpha_{0|t} \circ I^\alpha_{0|t}\big)(u) = u \text{ for all } u \in L^q\left([0,T]\right), \tag{3.9}$$

holds almost everywhere on [0,T].

The following Lemmas are crucial in the proof of Theorem 2.2.

Lemma 3.1. *Let $\sigma > 1$ and φ_2 be the function defined by*

$$\varphi_2(t) = \Big(1 - \frac{t}{T}\Big)_+^\beta.$$

Then, for all $\alpha \in (0,1)$ we have

$$D^\alpha_{t|T}\varphi_2(t) = C_1 T^{-\beta}(T-t)_+^{\beta-\alpha} = CT^{-\alpha}\Big(1-\frac{t}{T}\Big)_+^{\beta-\alpha},$$

$$D^{\alpha+1}_{t|T}\varphi_2(t) = C_2 T^{-\beta}(T-t)_+^{\beta-\alpha-1} = CT^{-\alpha-1}\Big(1-\frac{t}{T}\Big)_+^{\beta-\alpha-1},$$

and

$$D^{\alpha+2}_{t|T}\varphi_2(t) = C_3 T^{-\beta}(T-t)_+^{\beta-\alpha-2} = CT^{-\alpha-2}\Big(1-\frac{t}{T}\Big)_+^{\beta-\alpha-2}.$$

In particular, for all $\alpha \in (0,1)$, one has

$$D^{\alpha+j}_{t|T}\varphi_2(0) = C_j T^{-\alpha-2}, \quad \textit{for all}\ \ j = 0,1,2, \tag{3.10}$$

and

$$C_j = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1-j)}, \quad j = 0,1,2. \tag{3.11}$$

Proof. The proof of Lemma 3.1 is straight-forward. For all $\alpha \in (0,1)$, we have by definition (3.2)

$$D^\alpha_{t|T}\varphi_2(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{\varphi_2(s)}{(s-t)^\alpha}ds.$$

By using the Euler's change of variable

$$s \mapsto y = \frac{s-t}{T-t}, \tag{3.12}$$

we get,

$$\begin{aligned} D^{\alpha}_{t|T}\varphi_2(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{(1-\frac{s}{T})^{\beta}}{(s-t)^{\alpha}}ds \\ &= \frac{T^{-\beta}}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\left((T-t)^{\beta-\alpha+1}\int_0^1 y^{-\alpha}(1-y)^{\beta}dy\right) \\ &= \frac{(\beta-\alpha+1)\mathcal{B}(1-\alpha,\beta+1)}{\Gamma(1-\alpha)}T^{-\beta}(T-t)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}\left(1-\frac{t}{T}\right)^{\beta-\alpha}, \end{aligned}$$

where $\mathcal{B}$ is the *Beta function* defined by

$$\mathcal{B}(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt, \quad \mathcal{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{3.13}$$

For the second and the third, we apply directly formula (3.8) to show that

$$\forall t \in [0,T] : D^{\alpha+i}_{t|T}\varphi_2(t) = (-1)^i \partial_t D^{\alpha}_{t|T}\varphi_2(t), \quad \text{for all } i = 1, 2.$$

Hence the result is conclude. □

Lemma 3.2 (Ju Cordoba). ([7]) *Let* $0 \le \beta \le 2$, $\ell \ge 1$ *and* $(-\Delta)^{\beta/2}$ *be the operator defined by (*1.3*). Then for all* $\Psi \in D((-\Delta)^{\beta/2})$, *the following inequality holds*

$$(-\Delta)^{\beta/2}\Psi^{\ell} \le \ell\Psi^{\ell-1}(-\Delta)^{\beta/2}\Psi.$$

Proof. (Theorem 2.2) The proof is by contradiction. Suppose that u is a global weak solution to (2.1). Introducing the test function defined by (3.1), using the formula of integration by parts (3.7) and the identity (3.9) we get easily

$$\begin{aligned} \int_0^T\int_{\mathbb{R}^n} J^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx &= \int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)D^{\alpha}_{t|T}\psi(t,x)dtdx \\ &= \int_0^T\int_{\mathbb{R}^n} D^{\alpha}_{0|T}\big(J^{\alpha}_{0|T}(|u|^p)\big)\psi(t,x)dtdx \\ &= \int_0^T\int_{\mathbb{R}^n} |u|^p\psi(t,x)dtdx. \end{aligned} \tag{3.14}$$

For the second term of the left-hand side of equality (2.3), thanks to Lemma 3.1, we have

$$\begin{aligned} \int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi(0,x)dx &= \int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi_1^{\ell}(x)D^{\alpha}_{t|T}\varphi_2(t)_{|t=0}\,dx \\ &= CT^{-\alpha}\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi_1^{\ell}(x)dx. \end{aligned} \tag{3.15}$$

Analogously, we obtain for the third term of the left hand-side of the weak formulation (2.3)

$$\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx = -CT^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^\ell(x)dx. \tag{3.16}$$

Therefore, using formula (3.8) with $n=1$ and $n=2$, we get respectively

$$\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_t(t,x)dtdx = -\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^\ell(x)D_{t|T}^{\alpha+1}\varphi_2(t)dtdx, \tag{3.17}$$

and

$$\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^\ell(x)D_{t|T}^{\alpha+2}\varphi_2(t)dtdx. \tag{3.18}$$

Finally for the third term of the right-hand side of the weak formulation (2.3), we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n} u(t,x)(-\Delta)^{-\beta/2}\varphi(t,x)dtdx\\ &\leq \ell\times\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^{\ell-1}(-\Delta)^{-\beta/2}\varphi_1(x)D_{t|T}^{\alpha}\varphi_2(t)dtdx,\end{aligned} \tag{3.19}$$

where we have used Lemma 3.2 with $\Psi=\varphi_1$.
Inserting all the formulas (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19) in the weak formulation (2.3) we arrive at

$$\begin{aligned}&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dtdx + CT^{-\alpha}\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi_1^\ell(x)dx\\ &+CT^{-\alpha-1}\int_{\mathbb{R}^n}u_0(x)\varphi_1^\ell(x)dx \leq C\Big(\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx\\ &+\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|dtdx\\ &\quad+\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^{\ell-1}(-\Delta)^{-\beta/2}\varphi_1(x)|D_{t|T}^{\alpha}\varphi_2(t)|dtdx\Big),\end{aligned} \tag{3.20}$$

where $C>0$ independent of T. Next, using the fact that (2.5) imply

$$\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi_1^\ell(x)dx>0\,\text{and}\,\int_{\mathbb{R}^n}u_0(x)\varphi_1^\ell(x)dx>0, \tag{3.21}$$

we deduce easily from (3.20) the inequality

$$\int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dtdx \leq C(J_1+J_2+J_3), \tag{3.22}$$

where

$$J_1 = \int_0^T \int_{\mathbb{R}^n} |u(t,x)| \varphi_1^\ell(x) |D_{t|T}^{\alpha+2} \varphi_2(t)| dtdx, \tag{3.23}$$

$$J_2 = \int_0^T \int_{\mathbb{R}^n} |u(t,x)| \varphi_1^\ell(x) |D_{t|T}^{\alpha+1} \varphi_2(t)| dtdx, \tag{3.24}$$

$$J_3 = \int_0^T \int_{\mathbb{R}^n} |u(t,x)| \varphi_1^{\ell-1} (-\Delta)^{-\beta/2} \varphi_1(x) |D_{t|T}^{\alpha} \varphi_2(t)| dtdx. \tag{3.25}$$

Now, the main goal is to estimate the integrals J_1, J_2 and J_3. To do so, we apply the following $\varepsilon-$ *Young inequality*

$$AB \leq \varepsilon A^p + C(\varepsilon) B^q, \; pq = p + q, \quad C(\varepsilon) = (\varepsilon p)^{-q/p} q^{-1}.$$

It is quite easy to check that

$$\begin{aligned} J_1 &= \int_0^T \int_{\mathbb{R}^n} |u(t,x)| \psi^{\frac{1}{p}} \psi^{-\frac{1}{p}} \varphi_1^\ell(x) |D_{t|T}^{\alpha+2} \varphi_2(t)| dtdx \\ &\leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \psi dtdx + C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^\ell \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dtdx. \end{aligned} \tag{3.26}$$

Similarly, for J_2 and J_3, we obtain

$$\begin{aligned} J_2 \leq \varepsilon & \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p \psi(t,x) dtdx \\ &+ C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^\ell(x) \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dtdx, \end{aligned} \tag{3.27}$$

$$\begin{aligned} J_3 \leq \varepsilon & \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p \psi(t,x) dtdx \\ &+ C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^{\ell-\frac{p}{p-1}} \big(-\Delta)^{\beta/2} \varphi_1 \big)^{\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha} \varphi_2|^{\frac{p}{p-1}} dtdx. \end{aligned} \tag{3.28}$$

Plugging the estimates (3.26), (3.27), (3.28) into (3.22) we find, for ε small enough, the estimate

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t,x) dtdx &\leq C \Big(\int_0^T \int_{\mathbb{R}^n} \varphi_1^\ell \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dtdx \\ &+ \int_0^T \int_{\mathbb{R}^n} \varphi_1^\ell(x) \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dtdx \\ &+ \int_0^T \int_{\mathbb{R}^n} \varphi_1^{\ell-\frac{p}{p-1}} \big(-\Delta)^{\beta/2} \varphi_1 \big)^{\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha} \varphi_2|^{\frac{p}{p-1}} dtdx \Big) \\ &\leq C(I_1 + I_2 + I_3), \end{aligned} \tag{3.29}$$

where $C > 0$ independent of T, and

$$I_1 = \int_0^T \int_{\mathbb{R}^n} \varphi_1^\ell \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}} dtdx, \tag{3.30}$$

$$I_2 = \int_0^T \int_{\mathbb{R}^n} \varphi_1^\ell(x) \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}} dtdx, \tag{3.31}$$

$$I_3 = \int_0^T \int_{\mathbb{R}^n} \varphi_1^{\ell-\frac{p}{p-1}} \big(-\Delta)^{\beta/2}\varphi_1\big)^{\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha}\varphi_2|^{\frac{p}{p-1}} dtdx. \tag{3.32}$$

The aim, now, is to estimate the integrals I_1, I_2 and I_3. We have to distinguish two cases:

Case of $p \le p_\gamma(\beta)$

At this stage, we introduce the scaled variables.

$$x = T^{\frac{1}{\beta}} y \quad \text{and} \quad t = T\tau. \tag{3.33}$$

Let $K = T^{1/\beta}$. Using Fubini's theorem, we get, for I_1

$$\begin{aligned} I_1 &= \Big(\int_{\Omega_T} \varphi_1^\ell(x)dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) \\ &= \Big(T^{\frac{n}{\beta}} \int_0^2 \phi^\ell(y^2)dy\Big)\Big(T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^1 (1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}} d\tau\Big) \\ &= CT^{1-(\alpha+2)\frac{p}{p-1}+\frac{n}{\beta}}, \end{aligned} \tag{3.34}$$

where we have used

$$\int_{\Omega_T} \varphi_1^\ell(x)dx = T^{\frac{n}{\beta}} \int_0^2 \phi^\ell(y^2)dy = CT^{\frac{n}{\beta}}, \tag{3.35}$$

and

$$\int_0^1 (1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}} d\tau = C. \tag{3.36}$$

Similarly, for I_2 and I_3, we obtain

$$\begin{aligned} I_2 &= \Big(\int_{\Omega_T} \varphi_1^\ell(x)dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) \\ &= CT^{1-(\alpha+1)\frac{p}{p-1}+\frac{n}{\beta}}, \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} I_3 &= \int_0^T \int_{\mathbb{R}^n} \varphi_1^{\ell-\frac{p}{p-1}}(x) \big(-\Delta)^{\beta/2}\varphi_1(x)\big)^{\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}}(t) |D_{t|T}^{\alpha}\varphi_2(t)|^{\frac{p}{p-1}} dtdx \\ &= \int_{\Omega_T} \varphi_1^{\ell-\frac{p}{p-1}}(x) \big(-\Delta)^{\beta/2}\varphi_1(x)\big)^{\frac{p}{p-1}} dx \int_0^T \varphi_2^{-\frac{1}{p-1}}(t) |D_{t|T}^{\alpha}\varphi_2(t)|^{\frac{p}{p-1}} dt \\ &= CT^{1-(\alpha+\frac{2}{\beta})\frac{p}{p-1}+\frac{n}{\beta}}. \end{aligned} \tag{3.38}$$

Combining (3.38), (3.37) and (3.36), it holds from (3.29)

$$\int_0^T \int_{\Omega_T} |u(t,x)|^p \psi(t,x) dtdx \le CT^{-\delta}, \tag{3.39}$$

for some positive constant C independent of T and

$$\delta = 1 - (\alpha + 1)\frac{p}{p-1} + \frac{n}{\beta}. \tag{3.40}$$

Now we distinguish between two other subcases as follows:
Sub-case: $p < p_\gamma(\beta)$
Noting that

$$p < p_\gamma(\beta) \Longleftrightarrow \delta > 0. \tag{3.41}$$

Then, by passing to the limit in (3.39) as T goes to ∞ and invoking the fact that

$$\lim_{T \longrightarrow \infty} \psi(t, x) = 1, \tag{3.42}$$

we get after applying the dominate convergence theorem of Lebesgue that

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |u(t, x)|^p dt dx = 0. \tag{3.43}$$

This means that $u = 0$ and this is a contradiction.
The second case is:
Sub-case: $p = p_\gamma(\beta)$
First, we remark that the condition $p = p_\gamma(\beta)$ is equivalent to $\delta = 0$. Then, by taking the limit as $T \to \infty$ in (3.39) together with the consideration $\delta = 0$ we get

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |u|^p \, dt dx < +\infty, \tag{3.44}$$

from which we can deduce that

$$\lim_{T \to \infty} \int_0^{+\infty} \int_{\Delta_T} |u|^p \, \psi dt dx = 0, \tag{3.45}$$

where Δ_T is defined by (3.6). Fixing arbitrarily R in $]0, T[$ for some $T > 0$ such that when $T \to \infty$ we don't have $R \to \infty$ at the same time and taking $K = R^{-\frac{1}{\beta}} T^{\frac{1}{\beta}}$. First, we apply the following Hölder's inequality

$$\int_X uv d\mu \leq \Big(\int_X u^p d\mu\Big)^{\frac{1}{p}} \Big(\int_X v^q d\mu\Big)^{\frac{1}{q}}, \tag{3.46}$$

which happens for all $u \in L^p(X)$ and $v \in L^q(X)$ such that $p, q \in (1, +\infty)$ and $pq = p + q$ instead of $\varepsilon-$ *Young's* one to estimate the integral J_3 defined by (3.25) on the set

$$\Omega_{TR^{-1}} = \Big\{x \in \mathbb{R}^n : |x|^2 \leq 2R^{-\frac{1}{\beta}} T^{\frac{1}{\beta}}\Big\} = supp\varphi_1. \tag{3.47}$$

Taking into account the fact that $supp\Delta\varphi_1 \subset \Delta_{TR^{-1}} \subset \Omega_{TR^{-1}}$ where $\Delta_{TR^{-1}}$ is defined by

$$\Delta_{TR^{-1}} = \Big\{x \in \mathbb{R}^n : R^{-\frac{1}{\beta}} T^{\frac{1}{\beta}} \leq |x|^2 \leq 2R^{-\frac{1}{\beta}} T^{\frac{1}{\beta}}\Big\}, \tag{3.48}$$

we obtain the estimate

$$\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^{\ell-1}(-\Delta)^{-\beta/2}\varphi_1(x)|D^\alpha_{t|T}\varphi_2(t)|dtdx\leq\Big(\int_0^T\int_{\Delta_{TR^{-1}}}|u|^p\psi dtdx\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^T\int_{\Delta_{TR^{-1}}}\psi^{-\frac{q}{p}}\varphi_1^{(\ell-1)q}\big((-\Delta)^{\beta/2}\varphi_1\big)^q|D^\alpha_{t|T}\varphi_2|^q dtdx\Big)^{\frac{1}{q}},\tag{3.49}$$

while we estimate J_1 and J_2 by using $\varepsilon-$*Young* inequality as we did in the first case. Then we have to estimate the integrals $I_1,\ I_2$ and $\tilde{I}_3$ where I_1 and I_2 are given by (3.30) and (3.31) respectively and $\tilde{I}_3$ is defined by

$$\tilde{I}_3=\Big(\int_0^T\int_{\Delta_{TR^{-1}}}\psi^{-\frac{q}{p}}\varphi_1^{(\ell-1)q}\big((-\Delta)^{\beta/2}\varphi_1\big)^q|D^\alpha_{t|T}\varphi_2|^q dtdx\Big)^{\frac{1}{q}}.\tag{3.50}$$

For this task, we consider the scaled change of variables

$$x=R^{-\frac{1}{\beta}}T^{\frac{1}{\beta}}\quad\text{and } t=T^{\frac{1}{\beta}}\tau.\tag{3.51}$$

In this way, we find after using Fubini's theorem

$$I_1+I_2\leq C\big(T^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{\beta}+1}+T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{\beta}+1}\big)R^{-\frac{n}{\beta}}.\tag{3.52}$$

Moreover, taking into account the hypothesis $\delta=0$ we get from (3.52) the estimate

$$I_1+I_2\leq CR^{-\frac{n}{\beta}},\tag{3.53}$$

for $C>0$ independent of R and T. In the other hand, we may estimate $\tilde{I}_3$ by using the same change of variables (3.51) as follows

$$\tilde{I}_3\leq CR^{\frac{1}{\beta}-q\frac{n}{\beta}}.\tag{3.54}$$

Combining the estimates (3.54) and (3.53) together with (3.22), we obtain the inequality

$$\begin{aligned}\int_0^T\int_{\Omega_{TR^{-1}}}|u(t,x)|^p\psi(t,x)dtdx&\leq CR^{-\frac{n}{\beta}}\\&+CR^{\frac{1}{\beta}-q\frac{n}{\beta}}\Big(\int_0^T\int_{\Delta_{TR^{-1}}}|u(t,x)|^p\psi(t,x)dtdx\Big)^{\frac{1}{p}}.\end{aligned}\tag{3.55}$$

Using (3.45) and the fact that $\lim\limits_{T\to+\infty}\psi(t,x)=1$ we obtain from (3.55) as $T\to+\infty$.

$$\int_0^\infty\int_{\mathbb{R}^n}|u|^p dtdx\leq CR^{-\frac{n}{\beta}},$$

which means that necessarily $R\to+\infty$ and this is a contradiction.
Now we deal with the second main result in Theorem 2.2.

Case of $p\leq\frac{1}{\gamma}$

Even this case is divided into two subcases as follows:

2. i. Subcase of $p<\frac{1}{\gamma}$

In this case we take $K=R^{\frac{1}{\beta}}$, where R is a fixed positive number. Now let us turn to estimate the integrals $J_1,\ J_2$ and J_3 by using $\varepsilon-$*Young* inequality as we did in the first case, so we obtain the estimate (3.29). The aim, now, is to estimate the integrals I_1, I_2 and I_3 defined respectively by (3.30), (3.31) and (3.32), on the set

$$\Omega_R := \left\{x \in \mathbb{R}^n : |x| \leq 2R^{\frac{1}{\beta}}\right\} = supp\varphi_1, \tag{3.56}$$

since they are null outside Ω_R. For this reason, we consider the following scaled variables

$$x = R^{\frac{n}{\beta}} y \quad \text{and} \quad t = T\tau. \tag{3.57}$$

So, for I_1 we have

$$\begin{aligned} I_1 &= \Big(\int_{\Omega_R} \varphi_1^\ell(x)dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) \\ &= \Big(R^{\frac{n}{\beta}} \int_0^2 \phi^\ell(y^2)dy\Big)\Big(T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^1 (1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}} d\tau\Big) \\ &= CR^{\frac{n}{\beta}} T^{1-(\alpha+2)\frac{p}{p-1}}, \end{aligned} \tag{3.58}$$

for some constant $C > 0$ independent of R and T. In the same way, we obtain

$$I_2 = CR^{\frac{n}{\beta}} T^{1-(\alpha+1)\frac{p}{p-1}}, \tag{3.59}$$

where $C > 0$ is of R and T. Finally

$$I_3 = CR^{(\frac{n}{2}-\frac{p}{p-1})\frac{1}{\beta}} T^{1-\alpha\frac{p}{p-1}}. \tag{3.60}$$

Including the estimates (3.60), (3.59) and (3.58) into (3.29) we arrive at

$$\begin{aligned} \int_0^T \int_{\Omega_R} |u(t,x)|^p \psi(t,x)dtdx &= CR^{\frac{n}{\beta}} \left(T^{1-(\alpha+2)\frac{p}{p-1}} + T^{1-(\alpha+1)\frac{p}{p-1}}\right) \\ &\quad + CR^{\left(\frac{n}{2}-\frac{p}{p-1}\right)\frac{1}{\beta}} T^{1-\alpha\frac{p}{p-1}}. \end{aligned} \tag{3.61}$$

First, we note that $p < \frac{1}{\gamma}$ implies that

$$1 - \alpha\frac{p}{p-1} < 0.$$

Therefore, the fact that

$$\alpha\frac{p}{p-1} < (\alpha+1)\frac{p}{p-1} < (\alpha+2)\frac{p}{p-1}$$

together with

$$\lim_{T\to+\infty} \psi(t,x) = \varphi_1^\ell(x), \tag{3.62}$$

allow us after taking the limit as $T \to +\infty$ in (3.61) to obtain

$$\int_0^{+\infty} \int_{\Omega_R} |u(t,x)|^p \varphi_1^\ell(x)dtdx = 0. \tag{3.63}$$

Next, taking the limit as $R \to +\infty$ in (3.63). Using the fact that $\lim_{R\to+\infty} \varphi_1^\ell(x) = 1$, we get

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |u(t,x)|^p dtdx = 0.$$

This implies that $u = 0$ which is contradiction.

2. ii. Subcase of $p = \frac{1}{\gamma}$

In this case, the assumption

$$p < \frac{n}{n-2} \qquad \text{if } n \geq 3, \tag{3.64}$$

is needed. First, we observe that (3.64) implies

$$\frac{n}{2} - \frac{p}{p-1} < 0. \tag{3.65}$$

Under these assumptions, remind our selves that $\alpha = 1-\gamma$, then we verify easily that

$$1 - \alpha\frac{p}{p-1} = 0, \;\; 1-(\alpha+1)\frac{p}{p-1} = -\frac{1}{1-\gamma} < 0, \tag{3.66}$$

and also

$$1-(\alpha+2)\frac{p}{p-1} = -\frac{2p}{p-1} = -\frac{2}{1-\gamma} < 0.$$

Hence, taking the limit as $T \to \infty$ in (3.61) with the considerations (3.66) and (3.62) we obtain

$$\int_0^\infty \int_{\Omega_R} |u(t,x)|^p \varphi_1^\ell(x) dt dx = CR^{\left(\frac{n}{2}-\frac{p}{p-1}\right)\frac{1}{\beta}}. \tag{3.67}$$

Finally, one can remark that if $n = 1, 2$ then $\frac{n}{2} - \frac{p}{p-1} < 0$ for all $p > 1$ and then by taking the limit as $R \to \infty$ in (3.67), using the facts that $\beta \in (0, 2)$ and

$$\lim_{R\to+\infty} \varphi_1^\ell(x) = 1,$$

one has

$$\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p dt dx = 0. \tag{3.68}$$

This implies that $u = 0$ and this is a contradiction.
If $n \geq 3$ then $\frac{n}{2} - \frac{p}{p-1}$ is negative then it is not hard to get (3.68) by letting $R \to \infty$ in (3.67), if we assume furthermore that (3.64) or equivalently (3.65) is satisfied. This achieved the proof of *Theorem* 2.2. □

References

[1] Biler, P., Karch, G., Woyczynski, W.A., *Asymptotics for conservation laws involving Leivy diffusion generators*, Studia Mathematica, **148**(2001), no. 2, 171-192.

[2] Cazenave, T., Dickstein, F., Weissler, F.D., *An equation whose Fujita critical exponent is not given by scaling*, Nonlinear Anal., **68**(2008), 862-874.

[3] Duistermaat, J.J., Hörmander, L., *Fourier integral II*, Acta Math., **128**(1972), 183-269.

[4] Fino, A., *Critical exponent for damped wave equations with nonlinear memory*, Hal Arch. Ouv. Id: 00473941v2, (2010).

[5] Hadj Kaddour, T., Hakem, A., *Local existence and sufficient conditions of non-global solutions for weighted damped wave equations*, Facta Univ. Ser. Math. Inform., **32**(2017), no. 5, 629-657.

[6] Hadj Kaddour, T., Hakem, A., *Sufficient conditions of non-global solutions for fractional damped wave equations with non-linear memory*, Adv. Theory of Nonlinear Analysis Appl., **2**(2018), no. 4, 224-237.

[7] Ju, N., *The maximum principle and the global attractor for the dissipative 2-D quasi-geostrophic equations*, Comm. Pure Appl. Math., (2005), 161-181.

[8] Kilbas, A.A., Sarivastana, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, Elsevier 2006.

[9] Martinez, A., *An Introduction to Semiclassical and Microlocal Analysis Lecture Notes*, Università di Bologna, Italy, 2001.

[10] Mitidieri, E., Pohozaev, S.I., *Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on* $\mathbb{R}^n$, J. Evol. Equ., **1**(2001), 189-220.

[11] Mitidieri, E., Pohozaev, S.I., *A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities*, Proc. Steklov. Inst. Math., **234**(2001), 1-383.

[12] Pohozaev, S.I., Tesei, A., *Blow-up of nonnegative solutions to quasilinear parabolic inequalities*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Math. Natur. Rend. Lincei. 9 Math. App, **11**(2000), no. 2, 99-109.

[13] Pozrikidis, C., *The fractional Laplacian*, Taylor Francis Group, LLC /CRC Press, Boca Raton (USA), 2016.

[14] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives, Theory and Application*, Gordon and Breach Publishers, 1987.

[15] Souplet, P., *Monotonicity of solutions and blow-up for semilinear parabolic equations with nonlinear memory*, Z. Angew. Math. Phys, **55**(2004).

[16] Todorova, G., Yardanov, B., *Critical exponent for a non linear wave equation with damping*, J. Differential Equations, **174**(2001), 464-489.

[17] Zhang, Q.S., *A Blow Up Result for a Nonlinear Wave Equation with Damping*, C.R. Acad. Sci., Paris, 2001.

Tayeb Hadj Kaddour
Laboratory ACEDP, Djillali Liabes University,
22000 Sidi Bel Abbes, Algeria
e-mail: `hktn2000@yahoo.fr`

Ali Hakem
Laboratory ACEDP, Djillali Liabes University,
22000 Sidi Bel Abbes, Algeria,
e-mail: `hakemali@yahoo.com`