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Existence of solutions for a p-Laplacian Kirchhoff type problem with nonlinear term of superlinear and subcritical growth

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Abstract. This paper is concerned by the study of the existence of nonnegative and nonpositive solutions for a nonlocal quasilinear Kirchhoff problem by using the Mountain Pass lemma technique.

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1. Introduction

Many research are interested to study the existence of nontrivial solutions of Kirchhoff type equations for its huge importance. The Kirchhoff equation was introduced for the first time in 1876, which describe the free transverse vibrations of a tight rope of length L and a constant density (assumed to be equal to 1). The rope is described by a variable x taking its values in the interval $[0, L]$. The system of equations describing this phenomena and which was given by Kirchhoff is

$$u_{tt} - \Big(g(\lambda)u_x\Big)_x = 0,\ 0 < x < L,\ t > 0, \tag{1.1}$$

$$v_{tt} - \Big(g(\lambda)(1 + v_x)\Big)_x = 0,\ 0 < x < L,\ t > 0, \tag{1.2}$$

$$u(0,t) = u(L,t) = v(0,t) = v(L,t),\ \ t \geq 0, \tag{1.3}$$

where λ is the deformation of the cord given by $\lambda(x,t) = \Big(|1+v_x|^2+|u_x|^2\Big)^{\frac{1}{2}} - 1$, and $g(\lambda) = \frac{\sigma(\lambda)}{1+\lambda}$ with $\sigma(\lambda)$ represents the rope (cord) constraint corresponding to λ; finally and most important, the unknowns $u(x,t)$ and $v(x,t)$ represent the transversal and longitudinal displacements of the material point x at the time t. In order to separate the unknowns u and v and under some hypotheses, one can obtain

$$\begin{aligned} &u_{tt} - \Big(T_0 + \frac{E}{2L}\int_0^L |u_x|^2\,dx\Big)u_{xx} = 0, \quad 0 < x < L, t > 0, \\ &u(0,t) = u(L,t) = 0, \qquad\qquad t \geq 0, \end{aligned}$$

which is named the Kirchhoff equation. T_0 and $\frac{E}{2L}$ are two physical constants.
Since then, many researchers are interested in the Kirchhoff equation for its importance and it has been the subject of many studies; we cite here, in particular [4], which treats the following Kirchhoff type problem

$$\begin{cases} -\Big(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with 4-superlinear growth as $|u| \to +\infty$, and using minimax methods, it gives two interesting results, the existence of nontrivial solutions, and the existence of sign-changing solutions and multiple solutions. We cite also [3] which treat the existence and multiplicity of solutions for the semilinear elliptic problem given by

$$\begin{cases} -\Delta u + \ell(x)u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

by using the Mountain Pass technique. Note that, our work is practically based on the papers [3] and [4].
Let us consider the following nonlocal[1] Kirchhoff problem

$$\begin{cases} -\Big(a + b\displaystyle\int_\Omega |\nabla u|^p\,dx\Big)^{p-1}\Delta_p u + \ell(x)|u|^{p-2}u = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Δ_p is the p Laplacian operator: $-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$, a, b two strictly positive real numbers, $\ell \in L^{\frac{N}{p}}(\Omega) \cap L^\infty(\Omega)$ and f is a real continuous function defined on $\overline{\Omega} \times \mathbb{R}$. The induced norm in $W_0^{1,p}(\Omega)$ is given by

$$||u|| := \Big(\int_\Omega |\nabla u|^p\,dx\Big)^{\frac{1}{p}}, \quad \forall u \in W_0^{1,p}(\Omega).$$

2. Statement of the main result

The operator L defined by $Lu = -(a + b||u||^p)^{p-1}\Delta_p u + \ell|u|^{p-2}u$ possesses an unbounded eigenvalues sequence

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \to +\infty \text{ as } n \to +\infty,$$

[1] It is called nonlocal because of the term $M(||u||^p) = a + b||u||^p$ which implies that the equation is no more a punctual identity [1].

where λ_1 is simple and is characterized by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{(a+b||u||^p)^{p-1}||u||^p + \displaystyle\int_\Omega \ell(x)|u|^p\,dx}{\displaystyle\int_\Omega |u|^p\,dx}.$$

Remark 2.1. Our purpose in this remark is to study the following eigenvalue problem

$$L\phi = -(a+b||\phi||^p)^{p-1}\Delta_p\phi + \ell(x)|\phi|^{p-2}\phi = \lambda|\phi|^{p-2}\phi. \tag{2.1}$$

Let λ_k and $\widetilde{\phi}_k$ respectively eigenvalues and eigenfunctions of the operator

$$-\Delta_p + g|\phi|^{p-2} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

with $g \in L^\infty(\Omega)$ (see [5], [6]), which means that

$$-\Delta_p\widetilde{\phi}_k + g(x)|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k = \lambda_k|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k; \tag{2.2}$$

especially for $g(x) = \dfrac{\ell(x)}{(a+b||\widetilde{\phi}_k||^p)^{p-1}}$, i.e.,

$$-\Delta_p\widetilde{\phi}_k + \frac{\ell(x)}{(a+b||\widetilde{\phi}_k||^p)^{p-1}}|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k = \lambda_k|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k, \tag{2.3}$$

multiplying by $(a+b||\widetilde{\phi}_k||^p)^{p-1}$, we obtain

$$-(a+b||\widetilde{\phi}_k||^p)^{p-1}\Delta_p\widetilde{\phi}_k + \ell(x)|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k = \lambda_k(a+b||\widetilde{\phi}_k||^p)^{p-1}|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k, \tag{2.4}$$

so the sequence $(\widehat{\lambda}_k)$ defined by

$$\widehat{\lambda}_k = \lambda_k(a+b||\widetilde{\phi}_k||^p)^{p-1}$$

consist of eigenvalues of the operator L associated to the eigenfunctions $\widetilde{\phi}_k$. Since λ_1 is simple and strictly positive (see[5]) , it follows that $\widehat{\lambda}_1$, the first eigenvalue of (2.1), is also simple and strictly positive.

Proposition 2.1. *If λ is an eigenvalue of the operator L, then, there exist λ_k and $\widetilde{\phi}_k$ such that*

$$\lambda = \lambda_k(a+b||\widetilde{\phi}_k||^p)^{p-1}.$$

Proof. As λ is an eigenvalue of the operator L, one has that there exists $\phi \in W_0^{1,p}(\Omega)$ with $\phi \not\equiv 0$ which satisfies

$$-(a+b||\phi||^p)^{p-1}\Delta_p\phi + \ell(x)|\phi|^{p-2}\phi = \lambda|\phi|^{p-2}\phi \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega,$$

and this implies that

$$(a+b||\phi||^p)^{p-1}\int_\Omega |\nabla\phi|^p\,dx + \int_\Omega \ell(x)|\phi|^p\,dx = \lambda\int_\Omega |\phi|^p\,dx,$$

as a result

$$\lambda = \frac{(a+b||\phi||^p)^{p-1}||\phi||^p + \int_\Omega \ell(x)|\phi|^p\,dx}{\int_\Omega |\phi|^p\,dx},$$

and that

$$-\Delta_p \phi + \frac{\ell(x)}{(a+b||\phi||^p)^{p-1}}|\phi|^{p-2}\phi = \frac{\lambda}{(a+b||\phi||^p)^{p-1}}|\phi|^{p-2}\phi,$$

consequently, there exists $k \in \mathbb{N}^*$ such that $\lambda_k = \frac{\lambda}{(a+b||\widetilde{\phi}_k||^p)^{p-1}}$ and $\phi = \widetilde{\phi}_k$ for some eigenfunction associated to λ_k. $\lambda_k = \frac{\lambda}{(a+b||\widetilde{\phi}_k||^p)^{p-1}}$ implies that $\lambda = \lambda_k(a+b||\widetilde{\phi}_k||^p)^{p-1}$ and this conclude the proof of the proposition. □

For $p < N$ and concerning the embedding mapping $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, it is continuous for $r \in [1, p^*]$ and compact for $r \in [1, p^*)$ with $p^* = \frac{pN}{N-p}$, so we have that $S_r|u|_r \leq ||u||$ for all $u \in W_0^{1,p}(\Omega)$, where $|\cdot|_r$ denotes the norm in $L^r(\Omega)$ and S_r is the best constant corresponding to the embedding mapping (see [2]).

In this paper, we assume that f is a continuous function on $\overline{\Omega} \times \mathbb{R}$ and satisfies:

(H1) for every $M > 0$, there exists a constant $L_M > 0$ such that $|f(x,s)| \leq L_M$ for $|s| \leq M$ and a.e. $x \in \Omega$,

(H2) $\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{p^*-2}s} = 0$, uniformly in a.e. $x \in \Omega$,

(H3) there exist a function $m \in L^{\frac{N}{p}}(\Omega)$, and a subset $\Omega' \subset \Omega$ with $|\Omega'| > 0$ such that

$$\limsup_{s\to 0} \frac{pF(x,s)}{|s|^p} \leq m(x) \leq \lambda_1,$$

uniformly in a.e. $x \in \Omega$, and $m < \lambda_1$ in Ω', where $F(x,s) = \int_0^s f(x,t)\,dt$ and $|\cdot|$ is the Lebesgue measure,

(H4) $\lim_{|s|\to\infty} \frac{F(x,s)}{s^{p^2}} = +\infty$ uniformly in a.e. $x \in \Omega$,

(H5) let $\overline{F}(x,u) = \frac{1}{p^2} f(x,u)u - F(x,u)$, then $\overline{F}(x,u) \to +\infty$ as $|u| \to +\infty$ uniformly in $x \in \Omega$, and there exists $\sigma > \max\{1, \frac{N}{p}\}$ such that $|f(x,u)|^\sigma \leq C\overline{F}(x,u)(|u|^{p-1})^\sigma$ for $|u|$ large.

Furthermore, we suppose that one of the two conditions is satisfied $\Big(\ell(x) - m(x) \geq 0\Big)$ or $\Big(\ell(x) \geq 0$ and $a^{p-1} \geq \frac{|m|_{L^\infty}}{S_p^p}$ when $p \geq 2\Big)$.

Example: consider the function

$$f(x,s) = \begin{cases} s^3\ln(1+s) + \frac{s^4}{4(1+s)} - \frac{1}{4}[s^3+s^2+s],\ s \geq 0, \\ s^3\ln(1-s) - \frac{s^4}{4(1-s)} - \frac{1}{4}[s^3-s^2+s],\ s < 0, \end{cases}$$

then f satisfies all the above hypotheses for $p = 2$ and $N = 3$.

Our main result is the following theorem

Theorem 2.1. *Assume that hypotheses (H1)-(H5) hold, and that $sf(x,s) \geq 0$ for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Then problem (1.4) has at least a nonnegative solution and a nonpositive solution.*

3. Preliminaries

Let $E = W_0^{1,p}(\Omega)$ and define the functional

$$\Phi(u) = \frac{1}{p}\widehat{M}(||u||^p) + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \int_\Omega F(x,u)\,dx, \quad u \in E,$$

where $\widehat{M}(t) = \int_0^t [M(s)]^{p-1}\,ds$ and $M(s) = a + bs$, in other words,

$$\Phi(u) = \frac{1}{bp^2}\Big[(a+b||u||^p)^p - a^p\Big] + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \int_\Omega F(x,u)\,dx, \quad u \in E.$$

The variational formulation associated to the problem is

$$\Big[M(||u||^p)\Big]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega \ell(x)|u|^{p-2}uv\,dx = \int_\Omega f(x,u)v\,dx,\ \forall v \in E,$$

and by $(H1)$ and $(H2)$, one can verify that $\Phi \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle \Phi'(u), v\rangle = \Big[M(||u||^p)\Big]^{p-1}&\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega \ell(x)|u|^{p-2}uv\,dx\\ &- \int_\Omega f(x,u)v\,dx,\ \forall u,v \in E;\end{aligned}$$

the weak solutions of the variational formulation are the critical points of Φ in E. Following the paper [3] and in order to obtain nonnegative and nonpositive solutions, we let $\widetilde{f}(x,s) = f(x,s) - m(x)|s|^{p-2}s$ and truncate $\widetilde{f}$ above or below $s = 0$, i.e., let

$$\widetilde{f}_+(x,s) = \begin{cases} \widetilde{f}(x,s), & s \geq 0,\\ 0, & s < 0,\end{cases} \qquad \text{and} \qquad \widetilde{f}_-(x,s) = \begin{cases} \widetilde{f}(x,s), & s \leq 0,\\ 0, & s > 0,\end{cases}$$

and $\widetilde{F}_+(x,s) = \int_0^s \widetilde{f}_+(x,t)\,dt$, $\widetilde{F}_-(x,s) = \int_0^s \widetilde{f}_-(x,t)\,dt$. Under $(H1)$ and $(H2)$, the functionals $\widetilde{\Phi}_+$ and $\widetilde{\Phi}_-$ defined as follows

$$\begin{aligned}\widetilde{\Phi}_+(u) &= \frac{1}{p}\widehat{M}(||u||^p) + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx - \int_\Omega \widetilde{F}_+(x,u)\,dx,\\ &= \frac{1}{bp^2}\Big[(a+b||u||^p)^p - a^p\Big] + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx\\ &\quad - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx - \int_\Omega \widetilde{F}_+(x,u)\,dx,\\ \widetilde{\Phi}_-(u) &= \frac{1}{p}\widehat{M}(||u||^p) + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx - \int_\Omega \widetilde{F}_-(x,u)\,dx,\\ &= \frac{1}{bp^2}\Big[(a+b||u||^p)^p - a^p\Big] + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx\\ &\quad - \int_\Omega \widetilde{F}_-(x,u)\,dx,\end{aligned}$$

belong to $C^1(E,\mathbb{R})$ and

$$\langle \widetilde{\Phi}'_+(u), v\rangle = (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx + \int_\Omega \ell(x)|u|^{p-2}uv\,dx - \int_\Omega m(x)|u|^{p-2}uv\,dx - \int_\Omega \widetilde{f}_+(x,u)v\,dx,$$

$$\langle \widetilde{\Phi}'_-(u), v\rangle = (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx + \int_\Omega \ell(x)|u|^{p-2}uv\,dx - \int_\Omega m(x)|u|^{p-2}uv\,dx - \int_\Omega \widetilde{f}_-(x,u)v\,dx,$$

for all $u, v \in E$.

4. Proof of main results

We recall one critical point theorem which is the Mountain Pass lemma.

Theorem 4.1. *Let $(X, ||\cdot||_X)$ be a Banach space, suppose that $\Phi \in C^1(X,\mathbb{R})$ satisfies $\Phi(0) = 0$ and*

(i) *(the first geometric condition) there exist positive constants R_0 and α_0 such that*

$$\Phi(u) \geq \alpha_0 \text{ for all } u \in X \text{ with } ||u||_X = R_0,$$

(ii) *(the second geometric condition) there exists $e \in X$ with $||e||_X > R_0$ such that $\Phi(e) < 0$,*

(iii) *(the Palais-Smale condition) Φ satisfies the (C_c) condition, that is, for $c \in \mathbb{R}$, every sequence $(u_n) \subset X$ such that*

$$\Phi(u_n) \to c, \quad ||\Phi'(u_n)||\,(1+||u_n||) \to 0$$

has a convergent subsequence. Then $c := \inf_{\gamma\in\Gamma}\sup_{s\in[0,1]} \Phi(\gamma(s))$ is a critical value of Φ, where

$$\Gamma := \{\gamma \in C([0,1],X); \gamma(0) = 0, \gamma(1) = e\}.$$

We need also the following lemmas.

Lemma 4.1. *Assume that $N \geq 3$ and $v \in L^{\frac{N}{p}}(\Omega)$, then the functional*

$$\psi(u) := \int_\Omega v(x)|u|^p\,dx, \quad u \in W_0^{1,p}(\Omega)$$

is weakly continuous.

Proof. As in [8], the functional ψ is well defined by the Sobolev and Hölder inequalities. Assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and consider an arbitrary subsequence (w_n) of (u_n). Since

$$w_n \to u \text{ in } L^p_{loc}(\Omega), \quad w_n^+ \to u^+ \text{ in } L^p_{loc}(\Omega) \quad \text{and} \quad w_n^- \to u^- \text{ in } L^p_{loc}(\Omega)$$

going if necessary to a subsequence, we can assume that

$$w_n \to u \text{ a.e. on } \Omega, \quad w_n^+ \to u^+ \text{ a.e. on } \Omega \quad \text{and} \quad w_n^- \to u^- \text{ a.e. on } \Omega.$$

Since both (w_n^+) and (w_n^-) are bounded in $L^{p^*}(\Omega)$, $((w_n^+)^p)$ and $((w_n^-)^p)$ are bounded in $L^{\frac{N}{N-p}}(\Omega)$. Hence $(w_n^+)^p \rightharpoonup (u^+)^p$ and $(w_n^-)^p \rightharpoonup (u^-)^p$ in $L^{\frac{N}{N-p}}(\Omega)$, and so

$$\int_\Omega v(x)|w_n|^p\,dx \to \int_\Omega v(x)|u|^p.$$

As a result, ψ is weakly continuous. □

Lemma 4.2. *Assume that* $m \in L^{\frac{N}{p}}(\Omega)$, *and there exists* $\Omega' \subset \Omega$ *with* $|\Omega'| > 0$ *such that*

$$m \le \lambda_1 \text{ in } \Omega \text{ and } m < \lambda_1 \text{ in } \Omega'$$

then

$$d := \inf_{u\in W_0^{1,p}(\Omega), u\neq 0} \frac{\Big(a+b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx - \int_\Omega m(x)|u|^p\,dx}{||u||^p}$$

is strictly positive $(d > 0)$.

Proof. Since $\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega), u\neq 0} \frac{\Big(a+b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx}{|u|_p^p}$ and from the assumption that $m \le \lambda_1$ in Ω, we have that $d \ge 0$, because $m \le \lambda_1$ implies that

$$-\int_\Omega m|u|^p\,dx \ge -\int_\Omega \lambda_1|u|^p\,dx,$$

and consequently, we have

$$\begin{aligned}
&\frac{\Big(a+b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx - \int_\Omega m(x)|u|^p\,dx}{||u||^p}\\
\ge\ &\frac{\Big(a+b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx - \int_\Omega \lambda_1|u|^p\,dx}{||u||^p}\\
=\ &\frac{\Big(a+b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx}{||u||^p} - \lambda_1\frac{\int_\Omega |u|^p\,dx}{||u||^p}\\
\ge\ &0,
\end{aligned}$$

by definition of λ_1. It remains to prove that $d \neq 0$; for that, we let

$$\begin{aligned}
J(u) &:= \int_\Omega \ell(x)|u|^p\,dx, u \in W_0^{1,p}(\Omega),\\
K(u) &:= \int_\Omega m(x)|u|^p\,dx, u \in W_0^{1,p}(\Omega),\\
L(u) &:= \Big(a+b||u||^p\Big)^{p-1}||u||^p + J(u) - K(u), u \in W_0^{1,p}(\Omega).
\end{aligned}$$

Supposing by contradiction that $d = 0$, it follows that there exists a sequence $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that

$$||u_n|| = 1 \quad \text{and} \quad \lim_{n\to+\infty} L(u_n) = 0,$$

by the boundedness of $(u_n)_n$ in $W_0^{1,p}(\Omega)$, we can extract a subsequence such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Since J and K are weakly continuous, we have

$$\lim_{n\to+\infty} J(u_n) = J(u), \lim_{n\to+\infty} K(u_n) = K(u), \tag{4.1}$$

and from the weak lower semicontinuity of L, we obtain

$$0 \leq L(u) \leq \liminf_n L(u_n) = \lim_n L(u_n) = 0,$$

then we have

$$L(u) = \Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) - K(u) = \lim_n L(u_n) = 0, \tag{4.2}$$

which implies that

$$\Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) = K(u) = \int_\Omega m(x)|u|^p\, dx \leq \lambda_1 \int_\Omega |u|^p\, dx,$$

so, we have

$$\Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) \leq \lambda_1 \int_\Omega |u|^p\, dx \leq \Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u),$$

consequently

$$\Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) = \lambda_1 \int_\Omega |u|^p\, dx. \tag{4.3}$$

If $u = 0$, from (4.1), (4.2), we have that

$$\lim_n L(u_n) = \lim_n \Big(\Big(a + b||u_n||^p\Big)^{p-1} ||u_n||^p\Big) + J(0) - K(0) = 0,$$

which implies that $\lim_{n\to+\infty} ||u_n|| = 0$, and this is a contradiction with $||u_n|| = 1$. So $u \neq 0$, then u is an eigenfunction corresponding to λ_1; since $m \leq \lambda_1$ in Ω and $m < \lambda_1$ in Ω' with $|\Omega'| > 0$, it follows that,

$$\begin{aligned} \Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) = K(u) &= \int_\Omega m(x)|u|^p\, dx \\ &= \int_{\Omega'} m(x)|u|^p\, dx + \int_{\Omega\setminus\Omega'} m(x)|u|^p \\ &< \lambda_1 \int_{\Omega'} |u|^p\, dx + \int_{\Omega\setminus\Omega'} \lambda_1|u|^p\, dx \\ &= \int_\Omega \lambda_1 |u|^p\, dx, \end{aligned}$$

which is in contradiction with (4.3). Consequently, $d > 0$. □

Lemma 4.3. *Assume $(H1)$, $(H2)$ and $(H3)$ hold, then $\widetilde{\Phi}_+$ satisfies the first geometric condition.*

Proof. In the same way as in the paper [3], by $(H3)$ and for $\varepsilon \in \Big(0, \frac{dS_p^p}{2}\Big)$, there exists a positive constant $M_1 < 1$ such that

$$F_+(x, s) = F(x, s^+) \leq \frac{1}{p}(m(x) + \varepsilon)(s^+)^p, \text{ for } |s| \leq M_1 \text{ and a.e. } x \in \Omega \tag{4.4}$$

with $s^+ = \max(s, 0)$; for the chosen ε and from $(H1)$ and $(H2)$, we have

$$\exists M_2 > 1, \exists L_{M_2} : |f_+(x,s)| = |f(x,s^+)| \le \varepsilon (s^+)^{p^*-1} + L_{M_2} \tag{4.5}$$

and

$$F_+(x,s) \le \frac{1}{p}(m(x)+\varepsilon)(s^+)^p + \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)(s^+)^{p^*}, \tag{4.6}$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. From (4.6), we have

$$\begin{aligned}
\widetilde{\Phi}_+(u) &= \frac{1}{bp^2}\Big[(a+b||u||^p)^p - a^p\Big] + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx \\
&- \frac{1}{p}\int_\Omega m(x)|u|^p\,dx - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&= \frac{1}{bp^2}(a+b||u||^p)^p + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx \\
&- \frac{a^p}{bp^2} - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&= \frac{1}{bp^2}(a+b||u||^p)^{p-1}(a+b||u||^p) + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx \\
&- \frac{a^p}{bp^2} - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&\ge \frac{1}{bp^2}(a+b||u||^p)^{p-1}b||u||^p + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx \\
&- \frac{a^p}{bp^2} - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&= \frac{1}{p^2}(a+b||u||^p)^{p-1}||u||^p + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx \\
&- \frac{a^p}{bp^2} - \int_\Omega \widetilde{F}_+(x,u)\,dx, \\
&= \frac{1}{p}\Big[(a+b||u||^p)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx - \int_\Omega m(x)|u|^p\,dx\Big] \\
&- \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&\ge \frac{d}{p}||u||^p - \int_\Omega \widetilde{F}_+(x,u)\,dx - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p \\
&= \frac{d}{p}||u||^p - \int_\Omega F_+(x,u)\,dx + \frac{1}{p}\int_\Omega m(x)|u^+|^p\,dx \\
&- \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p \\
&\ge \frac{d}{p}||u||^p - \frac{1}{p}\int_\Omega (m(x)+\varepsilon)(u^+)^p\,dx - \int_\Omega \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)(u^+)^{p^*}\,dx \\
&+ \frac{1}{p}\int_\Omega m(x)|u^+|^p\,dx - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p
\end{aligned}$$

$$= \frac{d}{p}||u||^p - \frac{1}{p}\int_\Omega \varepsilon (u^+)^p\,dx - \int_\Omega \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)(u^+)^{p^*}\,dx$$
$$- \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p$$
$$\geq \frac{d}{p}||u||^p - \frac{\varepsilon}{pS_p^p}||u||^p - \Big(\frac{L_{M_2}M_2}{M_1^{2^*}} + \frac{\varepsilon}{2^*}\Big)\Big(\frac{1}{S_{p^*}}\Big)||u||^{p^*}$$
$$- \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p$$
$$= \frac{d}{2p}||u||^p - \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)\Big(\frac{1}{S_{p^*}}\Big)||u||^{p^*} - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p,$$
$$\forall u \in W_0^{1,p}(\Omega).$$

Let $C_1 = \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)\Big(\frac{1}{S_{p^*}}\Big)$, we have that

$$\widetilde{\Phi}_+(u) \geq \frac{d}{2p}||u||^p - C_1||u||^{p^*} - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p, \quad \forall u \in W_0^{1,p}(\Omega).$$

For R_0 sufficiently small, with $\|u\| = R_0$, one can have $||u||^{p^*} < ||u||^p$ and

$$(a+b||u||^p)^{p-1}||u||^p < ||u||^p$$

and

$$\widetilde{\Phi}_+(u) \geq \frac{d}{2p}||u||^p - C_1||u||^{p^*} - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p \geq \alpha_0 > 0,\ \forall u \in W_0^{1,p}(\Omega).$$

Consequently, the first geometric condition is satisfied. □

Lemma 4.4. *Assume that* $(H1)$ *and* $(H4)$ *hold, then* $\widetilde{\Phi}_+$ *satisfies the second geometric condition.*

Proof. Note that, using the following standard inequality, for $\alpha, \beta \geq 0$ and $p \geq 1$, we have $(\alpha+\beta)^p \leq 2^{p-1}(\alpha^p + \beta^p)$, then,

$$\begin{aligned}\widehat{M}(||u||^p) &= \frac{1}{pb}\Big[(a+b||u||^p)^p - a^p\Big]\\ &\leq \frac{1}{pb}\Big[2^{p-1}\Big(a^p + b^p||u||^{p^2}\Big) - a^p\Big]\\ &\leq \frac{1}{pb}\Big[(2^{p-1}-1)a^p + 2^{p-1}b^p||u||^{p^2}\Big];\end{aligned}$$

let $c_1 = 2^{p-1}-1$ and $c_2 = 2^{p-1}b^p$, we have

$$\widehat{M}(||u||^p) \leq \frac{1}{pb}\Big[c_1a^p + c_2||u||^{p^2}\Big]. \tag{4.7}$$

From $(H1)$ and $(H4)$, we have

$$\forall \Lambda > 0,\ \exists M_3 > 0,\ F_+(x,s) \geq \Lambda(s^+)^{p^2} - L_{M_3}M_3,$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then for $t > 0$, $\phi_1 > 0$, the first eigenfunction and using (4.7), we have

$$\begin{aligned}
\widetilde{\Phi}_+(t\phi_1) &= \frac{1}{p}\widehat{M}(||t\phi_1||^p) + \frac{1}{p}\int_\Omega \ell(x)|t\phi_1|^p\,dx - \frac{1}{p}\int_\Omega m(x)|t\phi_1|^p\,dx - \int_\Omega \widetilde{F}_+(x,t\phi_1)\,dx \\
&\le \frac{1}{p^2b}\left[c_1a^p + c_2||t\phi_1||^{p^2}\right] + \frac{1}{p}\int_\Omega \ell(x)|t\phi_1|^p\,dx - \frac{1}{p}\int_\Omega m(x)|t\phi_1|^p\,dx \\
&- \int_\Omega \widetilde{F}_+(x,t\phi_1)\,dx \\
&= \frac{1}{p^2b}\left[c_1a^p + c_2||t\phi_1||^{p^2}\right] + \frac{1}{p}\int_\Omega \ell(x)|t\phi_1|^p\,dx - \frac{1}{p}\int_\Omega m(x)|t\phi_1|^p\,dx \\
&+ \frac{1}{p}\int_\Omega m(x)((t\phi_1)^+)^p\,dx - \int_\Omega F(x,(t\phi_1)^+)\,dx \\
&\le \frac{1}{p^2b}\left[c_1a^p + c_2||t\phi_1||^{p^2}\right] + \frac{1}{p}\int_\Omega \ell(x)|t\phi_1|^p\,dx - \Lambda\int_\Omega t^{p^2}\phi_1^{p^2}\,dx + L_{M_3}M_3|\Omega| \\
&= t^{p^2}\left[\frac{c_2}{p^2b}||\phi_1||^{p^2} - \Lambda\int_\Omega \phi_1^{p^2}\,dx\right] + \frac{t^p}{p}\int_\Omega \ell(x)\phi_1^p\,dx + \frac{c_1a^p}{p^2b} + L_{M_3}M_3|\Omega| \\
&= At^{p^2} + Bt^p + C = P(t),
\end{aligned}$$

where

$$A = \frac{c_2}{p^2b}||\phi_1||^{p^2} - \Lambda\int_\Omega \phi_1^{p^2}\,dx,\ B = \frac{1}{p}\int_\Omega \ell(x)\phi_1^p\,dx,\ \text{ and } C = \frac{c_1a^p}{p^2b} + L_{M_3}M_3|\Omega| > 0;$$

by choosing

$$\Lambda > \frac{c_2||\phi_1||^{p^2}}{p^2b\int_\Omega \phi_1^{p^2}\,dx},$$

we then have $A < 0$. For t_0 sufficiently large, we have that $P(t_0) < 0$ and then by taking

$$e = t_0\phi_1 \in W_0^{1,p}(\Omega),$$

we conclude that $\widetilde{\Phi}_+$ satisfies the second geometric condition. □

Lemma 4.5. *Assume that $(H1)$, $(H2)$ and $(H5)$ hold, then $\widetilde{\Phi}_+$ satisfies the Palais-Smale condition.*

Proof. Claim 1: Under the same hypotheses in the above lemma, any (C_c) sequence is bounded.
Indeed, for $c \in \mathbb{R}$, and $(u_n)_n \subset W_0^{1,p}(\Omega)$, such that

$$\widetilde{\Phi}_+(u_n) \to c \text{ and } (1 + ||u_n||)\widetilde{\Phi}'_+(u_n) \to 0, \tag{4.8}$$

we have for n large, that

$$
\begin{aligned}
C_0 &\geq \widetilde{\Phi}_+(u_n) - \frac{1}{p^2}\widetilde{\Phi}'_+(u_n)u_n \\
&= \frac{1}{p}\widehat{M}(||u_n||^p) + \frac{1}{p}\int_\Omega \ell(x)|u_n|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u_n|^p\,dx - \int_\Omega \widetilde{F}_+(x,u_n)\,dx \\
&- \frac{1}{p^2}\Big[\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx - \int_\Omega m(x)|u_n|^p\,dx \\
&- \int_\Omega \widetilde{f}_+(x,u_n)u_n\,dx\Big] \\
&= \frac{1}{p}\widehat{M}(||u_n||^p) + \frac{1}{p}\int_\Omega \ell(x)|u_n|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u_n|^p\,dx + \frac{1}{p}\int_\Omega m(x)(u_n^+)^p\,dx \\
&- \int_\Omega F_+(x,u_n)\,dx - \frac{1}{p^2}\Big[\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx \\
&- \int_\Omega m(x)|u_n|^p\,dx + \int_\Omega m(x)(u_n^+)^p\,dx - \int_\Omega f_+(x,u_n)u_n\,dx\Big]
\end{aligned}
$$

because,

$$\widetilde{F}_+(x,s) = F(x,s^+) - m(x)\frac{(s^+)^p}{p}$$

and

$$\widetilde{f}_+(x,s) = f(x,s^+)s^+ - m(x)(s^+)^p = f(x,s^+)s - m(x)(s^+)^p,$$

then

$$
\begin{aligned}
C_0 \quad &\geq \quad \frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega \ell(x)|u_n|^p\,dx \\
&\qquad - \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega m(x)|u_n^-|^p\,dx - \int_\Omega F_+(x,u_n)\,dx + \frac{1}{p^2}\int_\Omega f_+(x,u_n)u_n\,dx \\
&= \quad \frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega \ell(x)|u_n|^p\,dx \\
&\qquad - \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega m(x)|u_n^-|^p\,dx + \int_\Omega \overline{F}(x,u_n^+)\,dx;
\end{aligned}
$$

note that the quantity $\frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p$ is positive, so we obtain

$$
\begin{aligned}
C_0 \quad &\geq \quad \widetilde{\Phi}_+(u_n) - \frac{1}{p^2}\widetilde{\Phi}'_+(u_n)u_n \\
&\geq \quad \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega \ell(x)|u_n|^p\,dx - \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega m(x)|u_n^-|^p\,dx + \int_\Omega \overline{F}(x,u_n^+)\,dx.
\end{aligned}
$$

If $\ell(x) - m(x) \geq 0$ and since $(u^-)^p \leq |u|^p$, we have

$$\begin{aligned} C_0 &\geq \Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega \ell(x)|u_n|^p \, dx - \Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega m(x)|u_n|^p \, dx + \int_\Omega \overline{F}(x, u_n^+) \, dx \\ &= \Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega (\ell(x) - m(x))|u_n|^p \, dx + \int_\Omega \overline{F}(x, u_n^+) \, dx \\ &\geq \int_\Omega \overline{F}(x, u_n^+) \, dx. \end{aligned}$$

If $\ell(x) \geq 0$ and $a^{p-1} \geq \frac{|m|_{L^\infty}}{S_p^p}$ where $p \geq 2$, and using the fact that

$$\begin{aligned} \Big|\int_\Omega m(x) u_n^p \, dx\Big| &\leq |m|_{L^\infty} |u_n|_p^p \\ &\leq |m|_{L^\infty} \frac{1}{S_p^p} ||u_n||^p, \end{aligned}$$

we have

$$\begin{aligned} C_0 &\geq \frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1} ||u_n||^p + \Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega \ell(x)|u_n|^p \, dx \\ &\quad -\Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega m(x)|u_n|^p \, dx + \int_\Omega \overline{F}(x, u_n^+) \, dx \\ &\geq \frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1} ||u_n||^p - \Big(\frac{1}{p} - \frac{1}{p^2}\Big)|m|_{L^\infty} \frac{1}{S_p^p} ||u_n||^p \\ &\quad +\Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega \ell(x)|u_n|^p \, dx + \int_\Omega \overline{F}(x, u_n^+) \, dx \\ &\geq \int_\Omega \overline{F}(x, u_n^+) \, dx. \end{aligned}$$

So, in both cases, one can obtain

$$\begin{aligned} C_0 &\geq \widetilde{\Phi}_+(u_n) - \frac{1}{p^2}\widetilde{\Phi}'_+(u_n)u_n \\ &\geq \int_\Omega \overline{F}(x, u_n^+) \, dx; \end{aligned} \tag{4.9}$$

let suppose by contradiction that $||u_n|| \to +\infty$, and set

$$v_n = \frac{u_n}{||u_n||}.$$

Then $||v_n|| = 1$, and

$$|v_n|_s \leq \frac{1}{S_s} ||v_n|| = \frac{1}{S_s},$$

for $s \in [1, p^*]$.

Observe that

$$\begin{aligned}\widetilde{\Phi}'_+(u_n)u_n &= (a+b||u_n||^p)^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx - \int_\Omega m(x)|u_n|^p\,dx\\ &- \int_\Omega \widetilde{f}(x,u_n^+)u_n\,dx\\ &= (a+b||u_n||^p)^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx - \int_\Omega m(x)|u_n|^p\,dx\\ &- \int_\Omega f(x,u_n^+)u_n\,dx + \int_\Omega m(x)(u_n^+)^p\,dx\\ &= (a+b||u_n||^p)^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx - \int_\Omega m(x)(u_n^-)^p\,dx\\ &- \int_\Omega f(x,u_n^+)u_n\,dx\\ &= ||u_n||^{p^2}\Big(\frac{(a+b||u||^p)^{p-1}||u||^p}{||u_n||^{p^2}} + \frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} - \frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}}\\ &- \frac{\int_\Omega f(x,u_n^+)v_n\,dx}{||u_n||^{p^2-1}}\Big).\end{aligned}$$

From (4.8), $\widetilde{\Phi}'_+(u_n) \to 0$ as $n \to +\infty$, so we have

$$\lim_n\Big(\frac{(a+b||u||^p)^{p-1}||u||^p}{||u_n||^{p^2}} + \frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} - \frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} - \frac{\int_\Omega f(x,u_n^+)v_n\,dx}{||u_n||^{p^2-1}}\Big) = 0.$$

Let's show that

$$\lim_n \frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} = 0$$

and

$$\lim_n \frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} = 0.$$

We have

$$\frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} = \frac{\int_\Omega \ell(x)|v_n|^p\,dx}{||u_n||^p}$$

and

$$\frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} = \frac{\int_\Omega m(x)(v_n^-)^p\,dx}{||u_n||^p};$$

since $v_n \to v$ in L^r $(r \in [1, p^*))$ and from Lemma 4.1, we deduce that

$$\lim_n \int_\Omega \ell(x)|v_n|^p\,dx = \int_\Omega \ell(x)|v|^p\,dx \text{ and } \lim_n \int_\Omega m(x)(v_n^-)^p\,dx = \int_\Omega m(x)(v^-)^p\,dx,$$

and since $||u_n|| \to +\infty$, we conclude that

$$\lim_n \frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} = \lim_n \frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} = 0.$$

We have also that

$$\lim_{n\to+\infty}\frac{(a+b||u_n||^p)^{p-1}||u||^p}{||u_n||^{p^2}}=\lim_{n\to+\infty}\frac{b^{p-1}||u||^{p(p-1)+p}}{||u_n||^{p^2}}=b^{p-1}.$$

Then

$$\lim_{n}\int_\Omega\frac{f(x,u_n^+)v_n\,dx}{||u_n||^{p^2-1}}=b^{p-1}. \tag{4.10}$$

Set for $r\ge 0$,

$$g(r)=\inf\{\overline{F}(x,u^+):x\in\Omega\text{ and }u^+\in\mathbb{R}_+\text{ with }u^+\ge r\}.$$

$(H5)$ implies that $g(r)>0$ for all r large, and $g(r)\to+\infty$ as $r\to+\infty$. Set for $0\le\alpha<\beta\le+\infty$,

$$\Lambda_n(\alpha,\beta):=\{x\in\Omega:\ \alpha\le|u_n^+(x)|<\beta\}$$

and

$$\sigma_\alpha^\beta:=\inf\Big\{\frac{\overline{F}(x,u^+)}{|u^+(x)|^p}:x\in\Omega\text{ and }u^+\in\mathbb{R}_+\text{ with }\alpha\le u^+<\beta\Big\}.$$

For large α, we have $\overline{F}(x,u^+)>0$, $\sigma_\alpha^\beta>0$ and

$$\overline{F}(x,u_n^+)\ge\sigma_\alpha^\beta|u_n^+|^p,\qquad\text{for }x\in\Lambda_n(\alpha,\beta).$$

It follows from (4.9) that

$$\begin{aligned}C_0&\ge\int_{\Lambda_n(0,\alpha)}\overline{F}(x,u_n^+)+\int_{\Lambda_n(\alpha,\beta)}\overline{F}(x,u_n^+)+\int_{\Lambda_n(\beta,+\infty)}\overline{F}(x,u_n^+)\\&\ge\int_{\Lambda_n(0,\alpha)}\overline{F}(x,u_n^+)+\sigma_\alpha^\beta\int_{\Lambda_n(\alpha,\beta)}|u_n^+|^p+g(\beta)\,|\Lambda_n(\beta,+\infty)|.\end{aligned}$$

Since $g(r)\to+\infty$ as $r\to+\infty$,

$$|\Lambda_n(\beta,+\infty)|\to 0,\qquad\text{as }\beta\to+\infty,\ \text{uniformly in }n,$$

which implies that, by the Hölder inequality, that for any $s\in[1,p^*)$,

$$\begin{aligned}\int_{\Lambda_n(\beta,+\infty)}|v_n|^s&\le\Big(\int_{\Lambda_n(\beta,+\infty)}(|v_n|^s)^{\frac{p^*}{s}}\Big)^{\frac{s}{p^*}}|\Lambda_n(\beta,+\infty)|^{\frac{p^*-s}{p^*}}\\&\le\frac{1}{S_{p^*}^s}|\Lambda_n(\beta,+\infty)|^{\frac{p^*-s}{p^*}}\\&\to 0\end{aligned} \tag{4.11}$$

as $\beta\to+\infty$ uniformly in n. Furthermore, for any fixed $0<\alpha<\beta$,

$$\begin{aligned}\int_{\Lambda_n(\alpha,\beta)}|v_n^+|^p&=\frac{1}{||u_n||^p}\int_{\Lambda_n(\alpha,\beta)}|u_n^+|^p=\frac{1}{||u_n||^p}\int_{\Lambda_n(\alpha,\beta)}\frac{\sigma_\alpha^\beta|u_n^+|^p}{\sigma_\alpha^\beta}\\&\le\frac{1}{||u_n||^p\sigma_\alpha^\beta}\int_{\Lambda_n(\alpha,\beta)}\overline{F}(x,u_n^+)\\&\le\frac{C_0}{||u_n||^p\sigma_\alpha^\beta}\\&\to 0.\end{aligned} \tag{4.12}$$

Set $0 < \eta < \frac{b^{p-1}}{3}$. From (4.5) (from $(H1)$ and $(H2)$), we have

$$\begin{aligned}
\int_{\Lambda_n(0,\alpha)} \frac{f(x,u_n^+)u_n}{||u_n||^{p^2}} &\leq \int_{\Lambda_n(0,\alpha)} \frac{|f(x,u_n^+)u_n|}{||u_n||^{p^2}} = \int_{\Lambda_n(0,\alpha)} \frac{|f(x,u_n^+)|\,(u_n^+ + u_n^-)}{||u_n||^{p^2}} \\
&= \int_{\Lambda_n(0,\alpha)} \frac{|f(x,u_n^+)|\,u_n^+}{||u_n||^{p^2}} + \int_{\Lambda_n(0,\alpha)} \frac{|f(x,u_n^+)|\,u_n^-}{||u_n||^{p^2}} \\
&\leq \int_{\Lambda_n(0,\alpha)} \frac{(\varepsilon (u_n^+)^{p^*} + L_{M_2} u_n^+)\,dx}{||u_n||^{p^2}} \\
&+ \int_{\Lambda_n(0,\alpha)} \frac{(\varepsilon (u_n^+)^{p^*-1} + L_{M_2})\,u_n^-\,dx}{||u_n||^{p^2}} \\
&\leq \int_{\Lambda_n(0,\alpha)} \frac{(\varepsilon \alpha^{p^*} + L_{M_2}\alpha)\,dx}{||u_n||^{p^2}} + \int_{\Lambda_n(0,\alpha)} \frac{L_{M_2}\,u_n^-\,dx}{||u_n||^{p^2}} \to 0,
\end{aligned}$$

because

$$\begin{aligned}
\int_{\Lambda_n(0,\alpha)} \frac{L_{M_2}\,u_n^-\,dx}{||u_n||^{p^2}} &= \frac{L_{M_2}}{||u_n||^{p^2-1}} \int_{\Lambda_n(0,\alpha)} v_n^-\,dx \\
&\leq \frac{L_{M_2}}{||u_n||^{p^2-1}} |\Omega|^{\frac{1}{p'}} |v_n^-|_{L^p} \\
&\leq \frac{L_{M_2}|\Omega|^{\frac{1}{p'}}}{||u_n||^{p^2-1}} |v_n|_{L^p} \\
&\leq \frac{L_{M_2}|\Omega|^{\frac{1}{p'}}}{S_p||u_n||^{p^2-1}} \to 0,
\end{aligned}$$

so there exists n_1, such that for $n > n_1$,

$$\int_{\Lambda_n(0,\alpha)} \frac{f(x,u_n^+)u_n}{||u_n||^{p^2}} < \eta. \tag{4.13}$$

Set $\sigma' = \frac{\sigma}{\sigma-1}$. Since $\sigma > \max\{1, \frac{N}{p}\}$, one can see that $p\sigma' \in (p, p^*)$. By $||u_n|| \to +\infty$, we take $n_2 > n_1$ such that $||u_n|| \geq 1$, if $n \geq n_2$, and by (4.11), $(H5)$ and Hölder inequality, one can take β large such that

$$\begin{aligned}
\int_{\Lambda_n(\beta,+\infty)} \frac{f(x,u_n^+)v_n}{||u_n||^{p^2-1}} &\leq \int_{\Lambda_n(\beta,+\infty)} \frac{f(x,u_n^+)}{|u_n|^{p-1}} v_n^p \\
&\leq \int_{\Lambda_n(\beta,+\infty)} \frac{f(x,u_n^+)}{|u_n^+|^{p-1}} v_n^p \qquad (4.14) \\
&\leq \Big(\int_{\Lambda_n(\beta,+\infty)} \Big|\frac{f(x,u_n^+)}{|u_n^+|^{p-1}}\Big|^\sigma\Big)^{\frac{1}{\sigma}} \Big(\int_{\Lambda_n(\beta,+\infty)} v_n^{p\sigma'}\Big)^{\frac{1}{\sigma'}} \\
&\leq \Big(\int_{\Lambda_n(\beta,+\infty)} C\overline{F}(x,u_n^+)\Big)^{\frac{1}{\sigma}} \Big(\int_{\Lambda_n(\beta,+\infty)} v_n^{p\sigma'}\Big)^{\frac{1}{\sigma'}} \\
&< \eta. \qquad (4.15)
\end{aligned}$$

Note that there is $C = C(\alpha,\beta)$ independent of n such that (because of the continuity of $(x,s) \mapsto \frac{f(x,s)}{s}$ on the compact $\overline{\Omega} \times [\alpha,\beta]$, so it is bounded)

$$|f(x,u_n^+)| \le Cu_n^+ \le C|u_n|, \quad \text{for } x \in \Lambda_n(\alpha,\beta).$$

So by (4.12), there is $n_0 > n_2$ such that

$$\begin{aligned}
\int_{\Lambda_n(\alpha,\beta)} \frac{f(x,u_n^+)v_n}{||u_n||^{p^2-1}} &\le \int_{\Lambda_n(\alpha,\beta)} \frac{C|u_n^+|\,|v_n|}{||u_n||^{p^2-1}} \\
&= \frac{C}{||u_n||^{p^2-2}} \int_{\Lambda_n(\alpha,\beta)} v_n^+|v_n| \\
&= \frac{C}{||u_n||^{p^2-2}} \int_{\Lambda_n(\alpha,\beta)} v_n^+(v_n^+ + v_n^-) \\
&= \frac{C}{||u_n||^{p^2-2}} \int_{\Lambda_n(\alpha,\beta)} (v_n^+)^2 \\
&\le \frac{C}{||u_n||^{p^2-2}} \int_{\Lambda_n(\alpha,\beta)} (v_n)^2 \\
&\le \frac{C}{||u_n||^{p^2-2}} \frac{1}{S_2^2} ||v_n||^2 \\
&= \frac{C}{||u_n||^{p^2-2}} \frac{1}{S_2^2} \\
&< \eta, \qquad (4.16)
\end{aligned}$$

for all $n > n_0$. Now, combining (4.13), (4.14) and (4.16), we obtain that for $n > n_0$,

$$\int_\Omega \frac{f(x,u_n^+)u_n}{||u_n||^{p^2}} < 3\eta < b^{p-1},$$

which contradicts (4.10). As a result, $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.
Claim 2: Assume the same hypotheses as in the last lemma, then any (C_c) condition has a convergent subsequence.
Indeed, let (u_n) be the (C_c) sequence such that

$$\widetilde{\Phi}_+(u_n) \to c, \quad (1 + ||u_n||)\widetilde{\Phi}'_+(u_n) \to 0.$$

We have

$$\begin{aligned}
\widetilde{\Phi}'_+(u_n)(u-u_n) = (a + b||u_n||^p)^{p-1} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u-u_n)\,dx \\
- \int_\Omega f(x,u_n^+)(u-u_n)\,dx &+ \int_\Omega \ell(x)|u_n|^{p-2}u_n(u-u_n)\,dx \\
- \int_\Omega m(x)|u_n|^{p-2}u_n(u-u_n)\,dx &+ \int_\Omega m(x)|u_n^+|^{p-2}u_n^+(u-u_n)\,dx.
\end{aligned}$$

Since (u_n) is bounded, one can extract a subsequence, named in the same way (u_n), that satisfies

$$\begin{array}{rcl} u_n & \rightharpoonup & u \text{ in } W_0^{1,p}(\Omega), \\ u_n & \to & u \text{ in } L^s(\Omega), \text{ for } 1 \leq s < p^*, \\ u_n & \to & u \text{ a.e. in } \Omega, \\ u_n^- & \rightharpoonup & u^- \text{ in } W_0^{1,p}(\Omega), \\ u_n^- & \to & u^- \text{ in } L^s(\Omega), \text{ for } 1 \leq s < p^*, \\ u_n^- & \to & u^- \text{ a.e. in } \Omega. \end{array} \tag{4.17}$$

For ε in (4.5), and from (4.17), there exists a positive constant $N(\varepsilon)$ such that

$$|u - u_n|_1 \leq \varepsilon, \forall n > N(\varepsilon); \tag{4.18}$$

from this, (4.5), Hölder inequality and $|u_n^+| \leq |u_n|$, it follows that for $n > N(\varepsilon)$,

$$\begin{aligned} \Big| \int_\Omega f(x, u_n^+)(u - u_n)\, dx \Big| &= \Big| \int_\Omega f_+(x, u_n)(u - u_n)\, dx \Big| \\ &\leq \int_\Omega \Big(\varepsilon (u_n^+)^{p^*-1} + L_{M_2} \Big) |u - u_n|\, dx \\ &= \int_\Omega \varepsilon (u_n^+)^{p^*-1} |u - u_n|\, dx + L_{M_2} |u - u_n|_1 \\ &\leq \varepsilon |u_n|_{p^*}^{p^*-1} |u - u_n|_{p^*} + L_{M_2} \varepsilon, \end{aligned}$$

using the fact that $|u_n|_{p^*} \leq \frac{1}{S_{p^*}} ||u_n||$, $|u - u_n|_{p^*} \leq \frac{1}{S_{p^*}} ||u - u_n||$, also the boundedness of (u_n) in $W_0^{1,0}(\Omega)$ i.e. there exists $C_3 > 0$ such that $||u_n|| \leq C_3$ and the following inequality

$$\begin{aligned} ||u - u_n|| &\leq ||u|| + ||u_n|| \\ &\leq \liminf_n ||u_n|| + ||u_n|| \\ &\leq 2C_3, \end{aligned}$$

we obtain

$$\Big| \int f(x, u_n^+)(u - u_n) \Big| \leq 2\varepsilon \Big(\frac{C_3}{S_{p^*}} \Big)^{p^*} + L_{M_2} \varepsilon;$$

this implies that $\lim_n \int_\Omega f(x, u_n^+)(u - u_n)\, dx = 0$.

Also we have by Hölder inequality, that

$$\int_\Omega \ell(x) |u_n|^{p-2} u_n (u - u_n)\, dx \to 0,$$

$$\int_\Omega m(x) |u_n|^{p-2} u_n (u - u_n)\, dx \to 0$$

and

$$\int_\Omega m(x) |u_n^+|^{p-2} u_n^+ (u - u_n)\, dx \to 0.$$

In addition, let $A(\nabla u_n) = |\nabla u_n|^{p-2}\nabla u_n$, then

$$\begin{aligned}(a+b||u_n||^p)^{p-1}&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u-u_n)\,dx\\ &= (a+b||u_n||^p)^{p-1}\int_\Omega A(\nabla u_n)\nabla(u-u_n)\,dx.\end{aligned}\tag{4.19}$$

Taking account all the previous estimations and limits, we obtain that

$$\int_\Omega A(\nabla u_n)\nabla(u-u_n)\,dx \to 0, \quad n\to+\infty.$$

From the fact that

$$\begin{aligned}\int_\Omega A(\nabla u_n)\nabla(u_n-u)\,dx &= \int_\Omega \Big(A(\nabla u_n)-A(\nabla u)\Big)\nabla(u_n-u)\,dx\\ &\quad+\int_\Omega A(\nabla u)\nabla(u_n-u)\,dx\end{aligned}$$

and

$$\int_\Omega A(\nabla u)\nabla(u_n-u)\,dx \to 0, \quad n\to+\infty,$$

we deduce

$$\int_\Omega \Big(A(\nabla u_n)-A(\nabla u)\Big)\nabla(u_n-u)\,dx \to 0, \quad n\to+\infty.$$

From the following inequality

$$C_p\int_\Omega |\nabla(u-u_n)|^p\,dx \le \int_\Omega \Big(A(\nabla u_n)-A(\nabla u)\Big)\nabla(u_n-u)\,dx,$$

we deduce that $u_n\to u$ in $W_0^{1,p}(\Omega)$. □

Proof of Theorem 2.1. By Lemmas 4.3, 4.4 and 4.5 and by applying theorem 4.1, one can deduce that $\widetilde{\Phi}_+$ has a nontrivial critical point u, that is, for any v in E,

$$\begin{aligned}\langle\widetilde{\Phi}'_+(u),v\rangle &= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx+\int_\Omega \ell(x)|u|^{p-2}uv\,dx\\ &\quad-\int_\Omega m(x)|u|^{p-2}uv\,dx-\int_\Omega \widetilde{f}_+(x,u)v\,dx = 0.\end{aligned}$$

Taking as a test function $v=u^-$ in the precedent equation, we obtain

$$\begin{aligned}\langle\widetilde{\Phi}'_+(u),u^-\rangle &= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u^-\,dx+\int_\Omega \ell(x)|u|^{p-2}uu^-\,dx\\ &\quad-\int_\Omega m(x)|u|^{p-2}uu^-\,dx-\int_\Omega \widetilde{f}_+(x,u)u^-\,dx\\ &= 0;\end{aligned}$$

from the definition of $\widetilde{f}_+$, we have $\int_\Omega \widetilde{f}_+(x,u)u^-\,dx = 0$, so

$$\begin{aligned}\langle \widetilde{\Phi}'_+(u), u^-\rangle &= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u^-\,dx + \int_\Omega \ell(x)|u|^{p-2}uu^-\,dx\\ &- \int_\Omega m(x)|u|^{p-2}uu^-\,dx\\ &= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}(\nabla u^+ - \nabla u^-)\cdot\nabla u^-\,dx\\ &+ \int_\Omega \ell(x)|u|^{p-2}(u^+-u^-)u^-\,dx - \int_\Omega m(x)|u|^{p-2}(u^+-u^-)u^-\,dx\\ &= -(a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx - \int_\Omega \ell(x)|u|^{p-2}|u^-|^2\,dx\\ &+ \int_\Omega m(x)|u|^{p-2}|u^-|^2\,dx\\ &= 0.\end{aligned}$$

If $\ell(x) - m(x) \geq 0$, then, one can have that

$$(a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx + \int_\Omega \Big(\ell(x)-m(x)\Big)|u|^{p-2}|u^-|^2\,dx = 0,$$

consequently, each term in the last equation is equal to zero, especially

$$\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx = 0,$$

since

$$\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx = \int_\Omega |\nabla u^-|^p\,dx,$$

one can deduce that $||u^-|| = 0$. If $\ell(x) \geq 0$ and $a^{p-1} \geq \frac{|m|_{L^\infty}}{S_p^p}$, then

$$\begin{aligned}0 = -\langle \widetilde{\Phi}'_+(u), u^-\rangle &= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx + \int_\Omega \ell(x)|u|^{p-2}|u^-|^2\,dx\\ &- \int_\Omega m(x)|u|^{p-2}|u^-|^2\,dx\\ &= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u^-|^p\,dx + \int_\Omega \ell(x)|u^-|^p\,dx - \int_\Omega m(x)|u^-|^p\,dx\\ &\geq (a+b||u||^p)^{p-1}||u^-||^p - |m|_{L^\infty(\Omega)}\frac{1}{S_p^p}||u^-||^p + \int_\Omega \ell(x)|u^-|^p\,dx\\ &\geq 0,\end{aligned}$$

as a result each term is equal to zero, consequently $||u^-|| = 0$.

So one can say that $u = u^+ \geq 0$. Then u is also a critical point of Φ_+, which means that,

$$\begin{aligned}\langle \Phi'_+(u), v\rangle &= (a + b||u||^p)^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_\Omega \ell(x)|u|^{p-2}uv \, dx \\ &\quad - \int_\Omega m(x)|u|^{p-2}uv \, dx - \int_\Omega f_+(x,u)v \, dx \\ &= 0, \ \forall v \in E.\end{aligned}$$

In addition, from $(H1)$, $(H2)$ and $\ell \in L^\infty(\Omega)$, we obtain that there exists a positive constant C_ε such that

$$|-a(x)u + f(x,u)| \leq C_\varepsilon\Big(1 + |u|^{p^*-1}\Big), \quad \text{for } s \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Now, consider

$$b(x) := \frac{-\ell(x)|u(x)|^{p-2}u(x) + f(x,u(x))}{(a + b||u||^p)^{p-1}(1 + |u(x)|)},$$

then $b \in L^{\frac{N}{p}}(\Omega)$ and

$$-\Delta_p u = b(x)(1 + |u(x)|).$$

Remark 4.1. Following [7], we believe that one can obtain a positive and negative solutions for our problem. Note that, for the case $p = 2$ and using the same techniques as in [3], we have proved the existence of positive and negative solutions.
In a similar way, one can obtain a nonpositive solution for problem (1.4) by treating with $\widetilde{\Phi}_-$.

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