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Approximation by a generalization of Szász-Mirakjan type operators

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Abstract. In the present paper we propose a new generalization of Szász-Mirakjan-type operators. We discuss their weighted convergence and rate of convergence via weighted modulus of continuity. We also give an asymptotic estimate through Voronovskaja type result for these operators.

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1. Introduction

In [7] Rempulska et al. introduced the following operators of Szász-Mirakjan type

$$L_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{2k}{n}\right), \tag{1.1}$$

with

$$p_{n,k}(x) = \frac{1}{\cosh nx} \frac{(nx)^{2k}}{(2k)!}, \ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{1.2}$$

where $f \in C_B$ and C_B is the space of real-valued functions uniformly continuous and bounded on $\mathbb{R}^+ = [0, \ \infty)$ and the norm in C_B is given as

$$\|f\| = \sup_{x \in \mathbb{R}^+} |f(x)|.$$

In [8, 9] a Voronovskaja-type theorem was proved for these operators.

In 2014, Aral et al. [1] introduced a very interesting generalization of the Szász-Mirakjan operators [10] using a function ρ as

$$\begin{aligned} S_n^{\rho}(f;x) =& e^{-n\rho(x)} \sum_{k=0}^{\infty} \left(f \circ \rho^{-1}\right) \left(\frac{k}{n}\right) \frac{(n\rho(x))^k}{k!} \\ =& (S_n(f \circ \rho^{-1}) \circ \rho)(x) \\ =& e^{-n\rho(x)} \sum_{k=0}^{\infty} f\left(\rho^{-1}\left(\frac{k}{n}\right)\right) \frac{(n\rho(x))^k}{k!}, \end{aligned} \tag{1.3}$$

where the function ρ satisfies following properties:
(ρ_1) ρ is continuously differentiable on $\mathbb{R}^+$,
(ρ_2) $\rho(0) = 0$, $\inf_{x \in \mathbb{R}^+} \rho'(x) \geq 1$.

We propose a similar generalization of the operators (1.1) as follows

$$L_n^{\rho}(f;x) = \frac{1}{\cosh(n\rho(x))} \sum_{k=0}^{\infty} (f \circ \rho^{-1}) \left(\frac{2k}{n}\right) \frac{(n\rho(x))^{2k}}{(2k)!}, \tag{1.4}$$

where $x \in \mathbb{R}^+$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and function ρ satisfies conditions (ρ_1) and (ρ_2).

We see that these new operators are positive linear operators. For $\rho(x) = x$, these operators (1.4) reduce to the operators (1.1). Also from conditions (ρ_1) and (ρ_2) we can draw out that
(i) $\lim_{x \in \mathbb{R}^+} \rho(x) = \infty$,
(ii) $|t - x| \leq |\rho(t) - \rho(x)|$ for all x, $t \in \mathbb{R}^+$.

In this paper we study some approximation properties of these new operators. Firstly we prove a theorem for the weighted convergence of $L_n^{\rho} f$ to f with the help of a weighted Korovkin-type theorem [4], [3]. Then we determine an estimate of the rate of the weighted convergence using weighted modulus of continuity defined in [5]. At the end we prove a Voronovskaja type result for these new operators.

2. Weighted convergence of $L_n^{\rho}(f;x)$

From the definition of the operators L_n^{ρ} one can easily derive the following results.

Lemma 2.1. *For the operators defined in* (1.4) *we have*

$$L_n^{\rho}(1;x) = 1, \tag{2.1}$$

$$L_n^{\rho}(\rho;x) = \rho(x) \tanh(n\rho(x)), \tag{2.2}$$

$$L_n^{\rho}(\rho^2;x) = \rho^2(x) + \frac{\rho(x)}{n} \tanh(n\rho(x)), \tag{2.3}$$

$$L_n^{\rho}(\rho^3;x) = \rho^3(x) \tanh(n\rho(x)) + \frac{3\rho^2(x)}{n} + \frac{\rho(x)}{n} \tanh(n\rho(x)), \tag{2.4}$$

$$L_n^{\rho}(\rho^4;x) = \rho^4(x) + \frac{6\rho^3(x)}{n} \tanh(n\rho(x)) + 7\frac{\rho^2(x)}{n^2} + \frac{\rho(x)}{n^3} \tanh(n\rho(x)). \tag{2.5}$$

Lemma 2.2. *For the operators defined in* (1.4) *we have*

$$\begin{aligned} L_n^\rho(\rho(t)-\rho(x);x) =& \rho(x)(\tanh(n\rho(x))-1),\\ L_n^\rho((\rho(t)-\rho(x))^2;x) =& \left(2\rho^2(x)-\frac{\rho(x)}{n}\right)(1-\tanh(n\rho(x)))+\frac{\rho(x)}{n},\\ L_n^\rho((\rho(t)-\rho(x))^4;x) =& \left(8\rho^4(x)-\frac{12\rho^3(x)}{n}+\frac{4\rho^2(x)}{n^2}-\frac{\rho(x)}{n^3}\right)\\ &\times(1-\tanh(n\rho(x)))+\frac{3\rho^2(x)}{n^2}+\frac{\rho(x)}{n^3}. \end{aligned}$$

Now we give a very useful lemma.

Lemma 2.3. *For the operators defined in* (1.4) *we have*

$$\begin{aligned} &\lim_{n\to\infty} nL_n^\rho(\rho(t)-\rho(x);x)=0,\\ &\lim_{n\to\infty} nL_n^\rho((\rho(t)-\rho(x))^2;x)=\rho(x). \end{aligned}$$

Proof. From Lemma 2.2

$$\begin{aligned} nL_n^\rho(\rho(t)-\rho(x);x) =& n\rho(x)(\tanh(n\rho(x))-1)\\ =& \frac{-2n\rho(x)}{e^{2n\rho(x)}+1}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} nL_n^\rho(\rho(t)-\rho(x);x) = 0.$$

Again from Lemma 2.2

$$\begin{aligned} nL_n^\rho((\rho(t)-\rho(x))^2;x) &= \left(2\rho(x)-\frac{1}{n}\right)n\rho(x)(1-\tanh(n\rho(x)))+\rho(x)\\ &= \left(2\rho(x)-\frac{1}{n}\right)\left(\frac{2n\rho(x)}{e^{2n\rho(x)}+1}\right)+\rho(x). \end{aligned}$$

Thus we have

$$\lim_{n\to\infty} nL_n^\rho((\rho(t)-\rho(x))^2;x)=\rho(x).$$ □

We prove the convergence theorem using weighted Korovkin type theorem. Korovkin's theorem [6] was extended to unbounded intervals and a weighted Korovkin type theorem in a subspace of continuous functions on the real axis $\mathbb{R}$ was proved in [4], [3]. It was shown that the test functions 1, x, x^2 of original Korovkin's theorem can be replaced by 1, ρ, ρ^2 under certain additional conditions on the function ρ. We recall some notations and results given in [1], [4], [3].

Let $\varphi(x) = 1+\rho^2(x)$, where ρ satisfies conditions (ρ_1) and (ρ_2). Thus we see that ρ is continuous and strictly increasing function on positive real axis. We will consider following weighted space:

$$B_\varphi(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R} : \ |f(x)| \leq M_f \varphi(x), \ x \in \mathbb{R}^+\},$$

where M_f is positive constant depending only on f. $B_\varphi(\mathbb{R}^+)$ is a normed space with the norm

$$||f||_\varphi = \sup_{x\in\mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}.$$

We denote the subspace of all continuous function in $B_\varphi(\mathbb{R}^+)$ by $C_\varphi(\mathbb{R}^+)$. $C^k_\varphi(\mathbb{R}^+)$ denotes the subspace of all functions $f \in C_\varphi(\mathbb{R}^+)$ with the property

$$\lim_{x\to\infty} \frac{|f(x)|}{\varphi(x)} = k_f,$$

where k_f is a constant depending on f. $U_\varphi(\mathbb{R}^+)$ be the subspace of all functions f in $C_\varphi(\mathbb{R}^+)$ such that $\frac{f(x)}{\varphi(x)}$ is uniformly continuous. Then obviously

$$C^k_\varphi(\mathbb{R}^+) \subset U_\varphi(\mathbb{R}^+) \subset C_\varphi(\mathbb{R}^+) \subset B_\varphi(\mathbb{R}^+).$$

Lemma 2.4 ([4, 3]). *The linear positive operators L_n, $n \geq 1$, act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$ if and only if*

$$|L_n(\varphi; x)| \leq K\varphi(x),$$

where $x \in \mathbb{R}^+$, $\varphi(x)$ is the weight function and K is a positive constant.

Theorem 2.5 ([4, 3]). *Let $(L_n)_{n\geq1}$ be the sequence of positive linear operators which act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$ satisfying the conditions*

$$\lim_{n\to\infty} ||L_n(\rho^i) - \rho^i||_\varphi = 0, \ i = 0, 1, 2.$$

then for any function $f \in C^k_\varphi(\mathbb{R}^+)$

$$\lim_{n\to\infty} ||L_n(f) - f||_\varphi = 0.$$

Lemma 2.6. *The linear positive operators L^ρ_n, $n \in \mathbb{N}$, act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$, where $\varphi(x) = 1 + \rho^2(x)$ is the weight function.*

Proof. In view of (2.1) and (2.3) we see that operators L^ρ_n, $n \in \mathbb{N}$ satisfy the condition of the Lemma 2.4. Thus the result follows. □

In [8] the following inequality was proved

$$0 \leq x^r(1 - tanh(nx)) \leq 2^{1-r}r!n^{-r}, \ n, r \in N \text{ and } x \geq 0.$$

Similarly for $\rho(x)$ satisfying ρ_1 and ρ_2 and $n, r \in N$ we can get the following inequality

$$0 \leq \rho^r(x)(1 - tanh(n\rho(x)) \leq 2^{1-r}r!n^{-r}. \tag{2.6}$$

Now we prove the convergence theorem for the operators (L^ρ_n).

Theorem 2.7. *Let $(L_n^\rho)_{n\in\mathbb{N}}$ be the sequence of linear positive operators defined by* (1.4)*. Then for any $f \in C_\varphi^k(\mathbb{R}^+)$ we have*

$$\lim_{n\to\infty} ||L_n^\rho(f) - f||_\varphi = 0.$$

Proof. Using Theorem 2.5 we see that in order to prove the theorem, it is sufficient to prove the following three conditions

$$\lim_{n\to\infty} ||L_n^\rho(\rho^v) - \rho^v||_\varphi = 0,\ v = 0, 1, 2.$$

Now from (2.1) we have

$$\lim_{n\to\infty} ||L_n^\rho(1) - 1||_\varphi = 0.$$

From (2.2) we get

$$||L_n^\rho(\rho) - \rho||_\varphi \le \sup_{x\in\mathbb{R}^+} \frac{\rho(x)}{1+\rho^2(x)}(1 - \tanh(n\rho(x)),$$

so using (2.6) for $r = 1$ we have

$$||L_n^\rho(\rho) - \rho||_\varphi \le \frac{1}{n}.$$

This leads to

$$\lim_{n\to\infty} ||L_n^\rho(\rho) - \rho||_\varphi = 0.$$

Again from (2.3)

$$\begin{aligned} L_n^\rho(\rho^2) - \rho^2 &= \frac{\rho(x)}{n} \tanh(n\rho(x)) \\ &= \frac{\rho(x)}{n} - \frac{\rho(x)}{n}(1 - \tanh(n\rho(x))), \end{aligned}$$

thus

$$||L_n^\rho(\rho^2) - \rho^2||_\varphi \le \sup_{x\in\mathbb{R}^+} \left[\frac{\rho(x)}{n(1+\rho^2(x))} + \frac{\rho(x)}{n(1+\rho^2(x))}(1 - \tanh(n\rho(x)))\right]$$

and using (2.6) we get

$$||L_n^\rho(\rho^2) - \rho^2||_\varphi \le \left[\frac{1}{n} + \frac{1}{n}\right] = \frac{2}{n}. \tag{2.7}$$

So we have

$$\lim_{n\to\infty} ||L_n^\rho(\rho^2) - \rho^2||_\varphi = 0.$$

This completes the proof. □

3. Rate of convergence via weighted modulus of continuity

In this section we compute the rate of convergence of the operators defined in (1.4) in terms of weighted modulus of continuity. In [5] Holhoş defined for all $f \in C_\varphi(\mathbb{R}^+)$ and for every $\delta \geq 0$, the weighted modulus of continuity as

$$\omega_\rho(f,\delta) = \sup_{x\in\mathbb{R}^+,\ |\rho(t)-\rho(x)|\leq\delta} \frac{|f(t)-f(x)|}{\varphi(t)+\varphi(x)}.$$

We see that $\omega_\rho(f,0) = 0$ for all $f \in C_\varphi(\mathbb{R}^+)$ and also that $\omega_\rho(f,\delta)$ is a nonnegative and nondecreasing function with respect to δ. The properties of weighted modulus of continuity were discussed in [5]. The following results were given by Holhos [5].

Lemma 3.1 ([5]). *For every $f \in U_\varphi(\mathbb{R}^+)$, $\lim_{\delta\to 0}\omega_\rho(f,\delta) = 0$.*

Theorem 3.2 ([5]). *Let $(L_n)_{n\geq 1}$ be a sequence of linear positive operators acting from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$ with*

$$\begin{aligned} ||L_n(\rho^0)-\rho^0||_{\varphi^0} &= a_n,\\ ||L_n(\rho)-\rho||_{\varphi^{\frac{1}{2}}} &= b_n,\\ ||L_n(\rho^2)-\rho^2||_{\varphi} &= c_n,\\ ||L_n(\rho^3)-\rho^3||_{\varphi^{\frac{3}{2}}} &= d_n, \end{aligned}$$

where a_n, b_n, c_n and d_n tend to zero as n goes to infinity. Then

$$||L_n(f)-f||_{\varphi^{\frac{3}{2}}} \leq (7+4a_n+2c_n)\omega_\rho(f,\delta_n) + ||f||_\varphi a_n$$

for all $f \in C_\varphi(\mathbb{R}^+)$, where

$$\delta_n = 2\sqrt{(a_n+2b_n+c_n)(1+a_n)} + a_n + 3b_n + 3c_n + d_n.$$

Theorem 3.3. *For all $f \in C_\varphi(\mathbb{R}^+)$, we have*

$$||L_n^\rho(f)-f||_{\varphi^{\frac{3}{2}}} \leq \left(7+\frac{4}{n}\right)\omega_\rho(f,\delta_n),$$

where

$$\delta_n = \frac{4}{\sqrt{n}} + \frac{15}{n}.$$

Proof. From (2.1) and (2.2) we see that

$$||L_n(\rho^0)-\rho^0||_{\varphi^0} = a_n = 0,$$

$$b_n = ||L_n(\rho)-\rho||_{\varphi^{\frac{1}{2}}} \leq \sup_{x\in\mathbb{R}^+} \frac{\rho(x)}{\sqrt{1+\rho^2(x)}}(1-\tanh(n\rho(x))$$

and using (2.6) we get

$$b_n = ||L_n(\rho)-\rho||_{\varphi^{\frac{1}{2}}} \leq \frac{1}{n}.$$

From (2.7) we have

$$c_n = ||L_n(\rho^2)-\rho^2||_\varphi \leq \frac{2}{n}.$$

Again from (2.4) we obtain

$$\begin{aligned} d_n =& ||L_n(\rho^3) - \rho^3||_{\varphi^{\frac{3}{2}}} \\ =& \sup_{x\in\mathbb{R}^+} \frac{1}{(1+\rho^2(x))^{\frac{3}{2}}} \Big|\rho^3(x)\tanh(n\rho(x)) - \rho^3(x) + \frac{3\rho^2(x)}{n} \\ &+ \frac{\rho(x)}{n^2}\tanh(n\rho(x))\Big| \\ \le& \sup_{x\in\mathbb{R}^+} \frac{1}{(1+\rho^2(x))^{\frac{3}{2}}} \Big|\rho^3(x)(1-\tanh(n\rho(x))) + \frac{\rho(x)}{n^2}(1-\tanh(n\rho(x))) \\ &+ \frac{3\rho^2(x)}{n} + \frac{\rho(x)}{n^2}\Big| \\ \le& \frac{1}{n} + \frac{1}{n^2} + \frac{3}{n} + \frac{1}{n^2}. \end{aligned}$$

Using (2.6) and by the fact that $\frac{1}{n^2} \le \frac{1}{n}$ we obtain

$$d_n = ||L_n(\rho^3) - \rho^3||_{\varphi^{\frac{3}{2}}} \le \frac{6}{n}.$$

Thus we see that a_n, b_n, c_n and d_n tend to zero as n goes to infinity. So on applying Theorem 3.2, we get

$$||L_n^\rho(f) - f||_{\varphi^{\frac{3}{2}}} \le \Big(7 + \frac{4}{n}\Big)\omega_\rho(f,\delta_n),$$

where

$$\delta_n = \frac{4}{\sqrt{n}} + \frac{15}{n}.$$

This completes the proof. □

Remark 3.4. We see from Theorem 3.3 that as $n \to \infty$, $\delta_n \to 0$. Thus, using Lemma 3.1, we have

$$\lim_{n\to\infty} ||L_n^\rho(f) - f||_{\varphi^{\frac{3}{2}}} = 0$$

for every $f \in U_\varphi(\mathbb{R}^+)$.

4. Voronovskaja type theorem

Now we give a Voronovskaja-type result using the technique of Cárdenas-Morales et al. [2].

Theorem 4.1. *Let $f \in C_\varphi(\mathbb{R}^+)$, $x \in \mathbb{R}^+$ and suppose that the first and second derivatives of $f \circ \rho^{-1}$ exist at $\rho(x)$. If the second derivative of $f \circ \rho^{-1}$ is bounded on $\mathbb{R}^+$, then we have*

$$\lim_{n\to\infty} n[L_n^\rho(f;x) - f(x)] = \frac{\rho(x)}{2}(f\circ\rho^{-1})''(\rho(x)).$$

Proof. By the Taylor expansion of $f\circ\rho^{-1}$ at the point $\rho(x)\in\mathbb{R}^+$, there exists ξ lying between x and t such that

$$\begin{aligned} f(t) =&(f\circ\rho^{-1})(\rho(t)) = (f\circ\rho^{-1})(\rho(x)) + (f\circ\rho^{-1})'(\rho(x))(\rho(t)-\rho(x)) \\ &+\frac{1}{2}(f\circ\rho^{-1})''(\rho(x))(\rho(t)-\rho(x))^2 + h(t;x)(\rho(t)-\rho(x))^2, \end{aligned}$$

where

$$h(t;x) = \frac{(f\circ\rho^{-1})''(\rho(\xi)) - (f\circ\rho^{-1})''(\rho(x))}{2}. \tag{4.1}$$

On applying the operator (1.4)

$$\begin{aligned} &n[L_n^\rho(f;x) - f(x)] \\ &\quad=(f\circ\rho^{-1})'(\rho(x))nL_n^\rho(\rho(t)-\rho(x);x) + \frac{1}{2}(f\circ\rho^{-1})''(\rho(x)) \\ &\quad\quad\times nL_n^\rho((\rho(t)-\rho(x))^2;x) + nL_n^\rho(h(t;x)(\rho(t)-\rho(x))^2;x). \end{aligned} \tag{4.2}$$

Now using Lemma 2.3 in (4.2) we get

$$\begin{aligned} &\lim_{n\to\infty} n[L_n^\rho(f;x) - f(x)] \\ &\quad=\frac{\rho(x)}{2}(f\circ\rho^{-1})''(\rho(x)) + \lim_{n\to\infty} nL_n^\rho(h(t;x)(\rho(t)-\rho(x))^2;x). \end{aligned} \tag{4.3}$$

From the hypothesis of the theorem we have $|h(t;x)|\le M$ and

$$\lim_{t\to x} h(t;x) = 0.$$

Thus, for any $\varepsilon>0$ there exist a $\delta>0$ such that

$$|h(t;x)|<\varepsilon \ for \ |t-x|<\delta.$$

But from the condition (ρ_2) we have

$$|t-x|\le|\rho(t)-\rho(x)|.$$

Therefore, if $|\rho(t)-\rho(x)|<\delta$, then

$$|h(t;x)(\rho(t)-\rho(x))^2| < \varepsilon(\rho(t)-\rho(x))^2$$

and if

$$|\rho(t)-\rho(x)|\ge\delta,$$

then

$$|h(t;x)(\rho(t)-\rho(x))^2| < \frac{M}{\delta^2}(\rho(t)-\rho(x))^4.$$

Hence

$$\begin{aligned} &L_n^\rho(h(t;x)(\rho(t)-\rho(x))^2;x) \\ &\quad< \varepsilon L_n^\rho((\rho(t)-\rho(x))^2;x) + \frac{M}{\delta^2}L_n^\rho((\rho(t)-\rho(x))^4;x). \end{aligned}$$

From Lemma 2.2 we see that

$$L_n^\rho((\rho(t)-\rho(x))^4;x) = O\Big(\frac{1}{n^2}\Big).$$

Thus we get

$$\lim_{n\to\infty} nL_n^\rho(h(t;x)(\rho(t)-\rho(x))^2;x)=0.$$

On applying this to (4.3) we get the desired result. □

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