Stud. Univ. Babeş-Bolyai Math. 65(2020), No. 4, 487–494
DOI: 10.24193/subbmath.2020.4.01

Differential sandwich theorem for certain class of analytic functions associated with an integral operator

Luminiţa-Ioana Cotîrlă and Adriana Cătaş

Abstract. In this paper we obtain some applications of first order differential subordination and superordination result involving an integral operator for certain normalized analytic function.

Mathematics Subject Classification (2010): 30C45.

Keywords: Integral operator, subordination and superordination, analytic functions, sandwich theorem.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0, \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$.

If f and g are analytic functions in U, we say that f is subordinate to g in U, written symbolically as $f \prec g$ or $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in U$. In particular, if the function g is univalent in U, the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [2], [3]).

For the function f given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

The set of all functions f that are analytic and injective on $\overline{U} - E(f)$, denote by Q where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$, (see [4]).

If $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h is univalent in U with $q \in Q$. In [3] Miller and Mocanu consider the problem of determining conditions on admissible functions ψ such that

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.2}$$

implies that $p(z) \prec q(z)$ for all functions $p \in \mathcal{H}[a,n]$ that satisfy the differential subordination (1.2).

Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ and $h \in \mathcal{H}$ with $q \in \mathcal{H}[a,n]$. In [4] and [5] is studied the dual problem and determined conditions on ϕ such that

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \tag{1.3}$$

implies $q(z) \prec p(z)$ for all functions $p \in Q$ that satisfy the above subordination. They also found conditions so that the functions q is the largest function with this property, called the best subordinant of the subordination (1.3) .

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disc.

For n a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + \ldots\}.$$

The integral operator I^m of a function f is defined in [6] by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt,$$

$$\ldots$$

$$I^m f(z) = I\left(I^{m-1}f(z)\right),\ z \in U.$$

Lemma 1.1. [3] *Let q be univalent in U, $\zeta \in \mathbb{C}^*$ and suppose that*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\left(\frac{1}{\zeta}\right)\right\}. \tag{1.4}$$

If p is analytic in U with $p(0) = q(0)$ and

$$p(z) + \zeta zp'(z) \prec q(z) + \zeta zq'(z) \tag{1.5}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2. [3] *Let the function q be univalent in the unit disk and let θ, φ be analytic in domain D containing $q(U)$ with $\varphi(w) \neq 0$, where $w \in q(U)$. Set*

$$Q(z) = zq'(z)\varphi(q(z)) \quad \textit{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

$$Q \textit{ is starlike univalent in } U;$$

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0,\ \textit{for } z \in U.$$

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) \tag{1.6}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.3. [1] *Let q be convex in the unit disc U, $q(0) = a$ and $\zeta \in \mathbb{C}$, $\mathrm{Re}\,(\zeta) > 0$. If $p \in \mathcal{H}[a, 1] \cap Q$ and $p(z) + \zeta zp'(z)$ is univalent in U then*

$$q(z) + \zeta zq'(z) \prec p(z) + \zeta zp'(z) \tag{1.7}$$

implies $q(z) \prec p(z)$ and q is the best subordinant.

Lemma 1.4. [2] *Let the function q be convex and univalent in the unit disc U and θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

1. $\mathrm{Re}\left\{\dfrac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in U$ and*

2. $Q(z) = zq'(z)\varphi(q(z))$ is starlike univalent in U.

If $p \in \mathcal{H}[q(0), 1] \cap Q$ with $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)) \tag{1.8}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

2. Main results

Theorem 2.1. *Let q be univalent in U, with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$, and let $\sigma \in \mathbb{C}^*$, $f \in \mathcal{A}$ and suppose that f and g satisfy the next conditions:*

$$\frac{I^{m+1}(f(z))}{z} \neq 0, z \in U \tag{2.1}$$

and

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \text{ for } z \in U. \tag{2.2}$$

If

$$\frac{I^m(f(z))}{I^{m+1}(f(z))} \prec 1 + \frac{zq'(z)}{\sigma q(z)}, \tag{2.3}$$

then

$$\left(\frac{I^{m+1}(f(z))}{z}\right)^\sigma \prec q(z)$$

and q is the best dominant of (2.3).

Proof. Let

$$p(z) = \left(\frac{I^{m+1}(f(z))}{z}\right)^\sigma, \ z \in U. \tag{2.4}$$

Because the integral operator I^m satisfies the identity $z\left[I^{m+1}(f(z))\right]' = I^m(f(z))$ and the function $p(z)$ is analytic in U, by differentiating (2.4) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)} = \sigma\left(\frac{I^m(f(z))}{I^{m+1}(f(z))} - 1\right). \tag{2.5}$$

In order to prove our result we will use Lemma 1.2. In this lemma we consider

$$\theta(w) = 1 \text{ and } \varphi(w) = \frac{1}{\sigma w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{zq'(z)}{\sigma q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \frac{zq'(z)}{\gamma\sigma q(z)}$$

from (2.2) we see that $Q(z)$ is a starlike function in U. We also have

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \text{ for } z \in U$$

and then, by using Lemma 1.2 we deduce that subordination (2.3) implies $p(z) \prec q(z)$ and the function q is the best dominant of (2.3). □

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.1, it easy to check that the assumption

$$p(z) + \frac{1}{\sigma}zp'(z) \prec q(z) + \frac{\alpha}{\sigma}zq'(z)$$

holds, hence we obtain the next result.

Corollary 2.2. *Let $\sigma \in \mathbb{C}^*$ and $f \in \mathcal{A}$. Suppose*

$$\frac{I^{m+1}(f(z))}{z} \neq 0, z \in U.$$

If

$$\frac{I^m(f(z))}{I^{m+1}(f(z))} \prec 1 + \frac{z(A-B)}{\sigma(1+Az)(1+Bz)},$$

then

$$\left(\frac{I^{m+1}(f(z))}{z}\right)^{\sigma} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem 2.1, it easy to check that the assumption

$$p(z) + \frac{1}{\sigma}zp'(z) \prec q(z) + \frac{\alpha}{\sigma}zq'(z)$$

holds, hence we obtain the next result.

Corollary 2.3. *Let $\sigma \in \mathbb{C}^*$ and $f \in \mathcal{A}$. Suppose*

$$\frac{I^{m+1}(f(z))}{z} \neq 0, z \in U.$$

If

$$\frac{I^m(f(z))}{I^{m+1}(f(z))} \prec 1 + \frac{2z}{\sigma(1-z)(1+z)},$$

then

$$\left(\frac{I^{m+1}(f(z))}{z}\right)^{\sigma} \prec \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ *is the best dominant.*

Theorem 2.4. *Let* q *be univalent in* U, *with* $q(0) = 1$. *Let* $\sigma \in \mathbb{C}^*$ *and* $t, \nu, \eta \in \mathbb{C}$ *with* $\nu + \eta \neq 0$. *Let* $f \in \mathcal{A}$ *and suppose that* f *and* g *satisfy the next conditions*

$$\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon+\eta) z} \neq 0, z \in U \tag{2.6}$$

and

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -\operatorname{Re} t\}, \ z \ \in U. \tag{2.7}$$

If

$$\begin{aligned}\psi(z) = t&\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon+\eta) z}\right]^{\sigma} \\ &+ \sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right]\end{aligned} \tag{2.8}$$

and

$$\psi(z) \prec tq(z) + \frac{zq'(z)}{q(z)} \tag{2.9}$$

then

$$\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon+\eta) z}\right]^{\sigma} \prec q(z)$$

and q *is the best dominant.*

Proof. Let

$$p(z) = \left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon+\eta) z}\right]^{\sigma}, z \in U. \tag{2.10}$$

According to (2.3) the function $p(z)$ is analytic in U and differentiating (2.10) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)} = \sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right] \tag{2.11}$$

and hence

$$zp'(z) = \sigma\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon+\eta) z}\right]^{\sigma} \cdot \left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right].$$

In order to prove our result we will use Lemma 1.2. In this lemma we consider

$$\theta(w) = tw \quad \text{and} \quad \varphi(w) = \frac{1}{w}$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right]$$

and

$$h(z) = \theta(q(z)) + Q(z)$$
$$= t\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon+\eta)z}\right]^{\sigma} + \sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right]$$

from (2.6) we see that $Q(z)$ is a starlike function in U. We also have

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{t + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \text{ for } z \in U$$

and then, by using Lemma 1.2 we deduce that the subordination (2.9) implies $p(z) \prec q(z)$. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 2.4 and according to

$$\frac{zp'(z)}{p(z)} = \sigma\left(\frac{I^{m+1}(f(z))}{I^{m+2}(f(z))} - 1\right)$$

the condition (2.7) becomes $\max\{0, -\operatorname{Re}(t)\} \le \frac{1-|B|}{1+|B|}$. Hence, for the special case $\upsilon = 1$ and $\eta = 0$ we obtain the following result.

Corollary 2.5. *Let $t \in \mathbb{C}$ with $\max\{0, -\operatorname{Re}(t)\} \le \frac{1-|B|}{1+|B|}$. Let $f \in \mathcal{A}$ and suppose that*

$$\frac{I^{m+1}(f(z))}{z} \neq 0, z \in U.$$

If

$$t\left[\frac{I^{m+1}(f(z))}{z}\right]^{\sigma} + \sigma\left[\frac{z\left(I^{m+1}(f(z))\right)'}{I^{m+1}(f(z))} - 1\right] \prec t\frac{1+Az}{1+Bz} + \frac{(1-B)z}{(1+Az)(1+Bz)}$$

then

$$\left(\frac{I^{m+1}(f(z))}{z}\right)^{\sigma} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

Taking $\upsilon = m = 1,\ \eta = 0$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 2.1, we obtain the next result.

Corollary 2.6. *Let $f \in \mathcal{A}$ and suppose that $\frac{I^2(f(z))}{z} \neq 0, z \in U$, $\sigma \in \mathbb{C}^*$. If*

$$t\left[\frac{I^2(f(z))}{z}\right]^{\sigma} + \sigma\left[\frac{z\left(I^2(f(z))\right)'}{I^2(f(z))} - 1\right] \prec t\frac{1+z}{1-z} + \frac{2z}{(1+z)(1-z)}$$

then

$$\left[\frac{I^2(f(z))}{z}\right]^{\sigma} \prec \frac{1+z}{1-z}$$

and $q(z) = \dfrac{1+z}{1-z}$ is the best dominant.

Theorem 2.7. *Let q be convex in U, with $q(0) = 1$. Let $\sigma \in \mathbb{C}^*$ and $t, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$ and $\operatorname{Re} t > 0$. Let $f \in \mathcal{A}$ and suppose that f satisfies the next conditions:*

$$\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z} \neq 0, z \in U \tag{2.12}$$

and

$$\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right]^{\sigma} \in \mathcal{H}[q(0), 1] \cap Q. \tag{2.13}$$

If the function ψ given by (2.8) is univalent in U and

$$tq(z) + \frac{zq'(z)}{q(z)} \prec \psi(z), \tag{2.14}$$

then

$$q(z) \prec \left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right]^{\sigma}$$

and $q(z)$ is the best subordinant of (2.14).

Proof. Let

$$p(z) = \left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right]^{\sigma}, \ z \in U. \tag{2.15}$$

According to (2.12) the function $p(z)$ is analytic in U and differentiating (2.15) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)} = \sigma \left[\frac{\upsilon z \left(I^{m+1}(f(z))\right)' + z\eta \left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right]. \tag{2.16}$$

In order to prove our result we will use Lemma 1.4. In this lemma we consider

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma \left[\frac{\upsilon z \left(I^{m+1}(f(z))\right)' + z\eta \left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right]$$

and

$$h(z) = \theta(q(z)) + Q(z)$$

$$= t\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right]^{\sigma} + \sigma \left[\frac{\upsilon z \left(I^{m+1}(f(z))\right)' + z\eta \left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right]$$

from (2.12) we see that $Q(z)$ is a starlike function in U. We also have

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{t + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \text{ for } z \in U$$

and then, by using Lemma 1.4 we deduce that the subordination (2.14) implies $q(z) \prec p(z)$ and the proof is completed. □

Corollary 2.8. *Let q_1, q_2 are two convex functions in U, with $q_1(0) = q_2(0) = 1$, $\sigma \in \mathbb{C}^*$, $t, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$ and* $\operatorname{Re} t > 0$. *Let $f \in \mathcal{A}$ and suppose that f satisfies the next conditions:*

$$\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z} \neq 0, z \in U$$

and

$$\left(\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right)^{\sigma} \in \mathcal{H}[q(0), 1] \cap Q.$$

If the function $\psi(z)$ given by (2.8) *is univalent in U and*

$$tq_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec tq_2(z) + \frac{zq_2'(z)}{q_2(z)}$$

then

$$q_1(z) \prec \left(\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right)^{\sigma} \prec q_2(z) \tag{2.17}$$

and q_1, q_2 are respectively, the best subordinant and the best dominant of (2.17).

References

[1] Bulboacă, T., *Classes of first-order differential superordinations*, Demonstr. Math., **35**(2002), no. 2, 287-292.

[2] Bulboacă, T., *Differential Subordinations and Superordinations,* Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[3] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, in Series on Monographs and Textbooks in Pure and Applied Mathematics, No. 225, Marcel Dekker, New York, 2000.

[4] Miller, S.S., Mocanu, P.T., *Subordinants of differential superordinations*, Complex Variables, **48**(10)(2003), 815-826.

[5] Miller, S.S., Mocanu, P.T., *Briot-Bouquet differential superordinations and sandwich theorems,* J. Math., Anal. Appl., **329**(2007), no. 1, 327-335.

[6] Sălăgean, G. Şt., *Subclasses of univalent functions*, Complex Analysis, Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, 362-372.

Luminiţa-Ioana Cotîrlă
Technical University of Cluj-Napoca,
Department of Mathematics,
Cluj-Napoca, Romania
e-mail: `luminita.cotirla@yahoo.com, Luminita.Cotirla@math.utcluj.ro`

Adriana Cătaş
University of Oradea,
Department of Mathematics and Computer Science,
1, University Street, 410087 Oradea, Romania
e-mail: `acatas@gmail.com`