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Exponential decay of the viscoelastic wave equation of Kirchhoff type with a nonlocal dissipation

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Abstract. The following viscoelastic wave equation of Kirchhoff type with nonlinear and nonlocal damping

$$u_{tt} - \psi\left(\|\nabla u\|_2^2\right)\Delta u - \alpha\Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + M\left(\|\nabla u\|_2^2\right)u_t = f(u),$$

where $M(r)$ is a $C^1([0,\infty))$ -function satisfying $M(r) \geq m_1 > 0$ for $r \geq 0$, is considered in a bounded domain Ω of $\mathbb{R}^N$. The existence of global solutions and decay rates of the energy are proved.

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1. Introduction

In this paper, we shall consider the initial boundary value problem for the following integro-differential problem

$$\begin{cases} u_{tt} - \psi\left(\|\nabla u\|_2^2\right)\Delta u - \alpha\Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)d\tau \\ \qquad +M\left(\|\nabla u\|_2^2\right)u_t = f(u), \quad \text{in} \quad \Omega\times(0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in\Omega, \\ u(x,t) = 0, \quad x\in\partial\Omega,\ t>0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. $\psi(r)$ is a positive locally Lipschitz function satisfying $\psi(r) \geq m_0 > 0$, for $r \geq 0$ like $\psi(r) = m_0 + br^\gamma$, $b \geq 0$ and $\gamma \geq 1$. $M(r)$ is a $C^1[0,\infty)$ -function satisfying $M(r) \geq m_1 > 0$ for $r \geq 0$, the scalar function $g(s)$ (so-called relaxation kernel) is assumed to satisfy (2.1) and f is a non-linear function as similar to $|u|^{p-2}u$, $p > 2$. Here $\alpha \geq 0$. The motivation for this problem comes from the following original equation

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = \left\{ p_0 + \frac{Eh}{L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f, \tag{1.2}$$

where $0 \leq x \leq L$, $t \geq 0$ and $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the cross-section area, L the length, E the Youngs modulus, p_0 the initial axial tension, δ the resistance modulus and f the external force. When $\delta = f = 0$, the equation (1.2) was first introduced by Kirchhoff [2].

In the absence of the term $M\left(\left\|\nabla u\right\|_2^2\right)u_t$. Wu and Tsai [7] studied (1.1) with $\alpha = 1$. The authors established the global existence and energy decay under the assumption $g'(t) \leq -rg(t)$, $\forall t \geq 0$ for some $r > 0$. Recently, this decay estimate of the energy function was improved by Wu in [6] under a weaker condition on g i.e. $g'(t) \leq 0$, $\forall t \geq 0$.

If we consider (1.1) with $[\psi \equiv 1, f = \alpha = 0]$ and the bi-harmonic instead of Laplace operator one we get the model

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau) d\tau + M\left(\left\|\nabla u\right\|_2^2\right) u_t = 0. \tag{1.3}$$

Cavalcanti et al. [1] investigated the global existence, uniqueness and stabilization of energy. By taking a bounded or unbounded open set Ω, the authors showed that the energy goes to zero exponentially provided that g goes to zero at the same form.

The main interest of the present paper is to examine whether there exists a global solution u to (1.3) under the presence of the nonlinear and nonlocal dissipation represented by $M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right) u_t$ and the real-value function $M : [0, +\infty) \to [m_1, +\infty)$, where $m_1 > 0$ will be considered of class C^1.

This kind of damping effect was firstly introduced by H. Lange and G. Perla Menzala [3] for the beam equation where the following model was considered

$$u_{tt} + \Delta^2 u + M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right) u_t = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+. \tag{1.4}$$

The nonlocal nonlinearity $M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right) u_t$ is indeed a damping term. It models a frictional mechanism acting on the body that depends on the average of u itself. Moreover, if such u does exist, we intend to investigate its asymptotic behavior as $t \to \infty$.

In this paper we show that under some conditions the solution is global in time and the energy decays exponentially. We first use Faedo-Galerkin method to study the existence of the simpler problem (3.1). Then, we obtain the local existence Theorem 3.2 by using contraction mapping principle. We obtain global existence of the solutions of (1.1) given in Theorem 4.4. Our technique of proof is similar to the one in [7] with some necessary modifications due to the nature of the problem treated here. Moreover, the asymptotic behavior of global solutions is investigated under some assumptions on the initial data.

2. Preliminaries

In this section we present some assumptions, notations and Lemmas. We first make the following hypotheses.
(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying

$$\int_0^\infty g(\tau)d\tau = l_1 > 0, \quad g(0) - K_1 \int_0^\infty g(\tau)d\tau = l_2 > 0, \tag{2.1}$$
$$-K_1 g(t) \le g'(t) \le -K_2 g(t),$$

here K_1 and K_2 are positive constants.
(A2) $f(0) = 0$ and there is a positive constant K_3 such that

$$\big|f(u) - f(v)\big| \le K_3 \big|u - v\big| \Big(\big|u\big|^{p-2} + \big|v\big|^{p-2}\Big) \quad \text{for} \quad u, v \in \mathbb{R}, \tag{2.2}$$

and

$$2 < p < \infty \quad \text{if} \quad N = 1, 2 \quad \text{and} \quad 2 < p \le \frac{2(N-1)}{N-2} \quad \text{if} \quad N \ge 3. \tag{2.3}$$

(A3) The function $M(r)$ for $r \ge 0$ belongs to the class $C^1[0, \infty)$ and satisfies

$$M(r) \ge m_1 > 0 \quad \text{for} \quad r \ge 0. \tag{2.4}$$

For functions $u(x,t)$, $v(x,t)$ defined on Ω, we introduce

$$(u, v) = \int_\Omega uv dx, \qquad \|u\|_2 = \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}, \qquad \|u\|_\infty = ess \sup_{x \in \Omega} |u(x)|,$$

$$\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}, \qquad \|u\|_{H^m} = \left(\sum_{|\beta| \le m} \|D^\beta\|_2^2\right)^{\frac{1}{2}}.$$

Lemma 2.1. (*Sobolev-Poincaré inequality* [5]) *If* $2 \le p \le \frac{2N}{N-2}$, *then*

$$\|u\|_p \le B_1 \|\nabla u\|_2, \tag{2.5}$$

for $u \in H_0^1(\Omega)$ *holds with some constant* B_1.

3. Local existence of solution

In this section, we shall discuss the local existence of solutions for (1.1) by using contraction mapping principle. An important step in the proof of local existence Theorem 3.2 below is the study of the following simpler problem:

$$\begin{cases} u_{tt} - \mu(t)\Delta u - \alpha \Delta u_t + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau) d\tau \\ \quad +\chi(t) u_t = f_1(x,t), \quad \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t \ge 0. \end{cases} \tag{3.1}$$

Here, $T>0$, $\alpha\geq 1$, f_1 is a fixed forcing term in $\Omega\times(0,T)$, $\mu(t)$ is a positive locally Lipschitz function on $[0,\infty)$ with $\mu(t)\geq m_0>0$ for $t\geq 0$ and $\chi(t)$ is C^1-function on $[0,\infty)$ such that $\chi(t)\geq 0$ for $t\geq 0$.

Lemma 3.1. *Suppose that* $(A1)$ *holds, and that* $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1\in H_0^1(\Omega)$ *and* $f_1\in L^2\big([0,T];L^2(\Omega)\big)$ *be given. Then the problem* (3.1) *admits a unique solution* u *such that*

$$u\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)),\quad u_t\in C([0,T];L^2(\Omega))\cap L^2([0,T];H_0^1(\Omega)),$$

$$u_{tt}\in L^2([0,T];L^2(\Omega)).$$

Proof. Let $(\omega_n)_{n\in\mathbb{N}}$ be a basis in $H^2(\Omega)\cap H_0^1(\Omega)$ and V^n be the space generated by $\omega_1,....,\omega_n$, $n=1,2,\cdots$. Let us consider

$$u^n(t)=\sum_{k=1}^{n}d_k^n(t)w_k,$$

be the weak solution of the following approximate problem corresponding to (3.1)

$$\begin{aligned}&\int_\Omega u_{tt}^n(t)\omega dx+\mu(t)\int_\Omega\nabla u^n(t)\cdot\nabla\omega dx\\&-\int_0^t g(t-\tau)\int_\Omega\nabla u^n(\tau)\cdot\nabla\omega dxd\tau+\alpha\int_\Omega\nabla u_t^n(t)\cdot\nabla\omega dx\\&+\chi(t)\int_\Omega u_t^n(t)\omega dx=\int_\Omega f_1(x,t)wdx\quad\text{ for }\ \omega\in V^n,\end{aligned}\tag{3.2}$$

with initial conditions

$$u^n(0)=u_0^n=\sum_{k=1}^{n}\int_\Omega u_0w_kdxw_k\longrightarrow u_0\ \text{ in }\ \ H^2(\Omega)\cap H_0^1(\Omega),\tag{3.3}$$

$$u_t^n(0)=u_1^n=\sum_{k=1}^{n}\int_\Omega u_1w_kdxw_k\longrightarrow u_1\ \text{ in }\ \ H_0^1(\Omega).\tag{3.4}$$

By standard methods in differential equations, we prove the existence of solutions to $(3.2)-(3.4)$ on some interval $[0,t_n)$, $0<t_n<T$. In order to extend the solution of $(3.2)-(3.4)$ to the whole interval $[0,T]$, we need the following a priori estimate.

Step 1. (The first priori estimate) Replacing w by $2u_t^n(t)$ in (3.2), we have

$$\begin{aligned}&\tfrac{d}{dt}\left[\|u_t^n(t)\|_2^2+\mu(t)\|\nabla u^n(t)\|_2^2\right]+2\alpha\|\nabla u_t^n(t)\|_2^2+2\chi(t)\|u_t^n(t)\|_2^2\\&\quad=\mu'(t)\|\nabla u^n(t)\|_2^2+2\int_0^t g(t-\tau)\int_\Omega\nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\\&\quad+2\int_\Omega f_1(x,t)u_t^n(t)dx\leq\mu'(t)\|\nabla u^n(t)\|_2^2+\|\nabla u_t^n(t)\|_2^2\\&\quad+\|g\|_{L^1}\int_0^t g(t-\tau)\|\nabla u^n(\tau)\|_2^2d\tau+\|f_1\|_2^2+\|u_t^n(t)\|_2^2.\end{aligned}\tag{3.5}$$

Then, integrating (3.5) from 0 to t, we get

$$\begin{aligned}&\|u_t^n(t)\|_2^2+\mu(t)\|\nabla u^n(t)\|_2^2+(2\alpha-1)\int_0^t\|\nabla u_\tau^n(\tau)\|_2^2d\tau\leq c_1\\&+\int_0^t\left[1+\frac{1}{\mu(\tau)}\left(|\mu'(\tau)|+\|g\|_{L^1}^2\right)\right]\left[\|u_\tau^n(\tau)\|_2^2+\mu(\tau)\|\nabla u^n(\tau)\|_2^2\right]d\tau,\end{aligned}\tag{3.6}$$

where

$$c_1=\|u_1^n\|_2^2+\mu(0)\|\nabla u_0^n\|_2^2+\int_0^t\|f_1\|_2^2dt.$$

Taking into account (3.3) and (3.4), we obtain from Gronwall's Lemma the first priori estimate

$$\|u_t^n(t)\|_2^2+\mu(t)\|\nabla u^n(t)\|_2^2+\int_0^t\|\nabla u_t^n(\tau)\|_2^2d\tau\leq L_1,\tag{3.7}$$

for all $t\in[0,T]$. Here L_1 is a positive constant independent of $n\in\mathbb{N}$ and $t\in[0,T]$.
Step 2. (The second priori estimate) Replacing ω by $u_{tt}^n(t)$ in (3.2), we have

$$\begin{aligned}&\tfrac{d}{dt}\left[\mu(t)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx+\frac{\alpha}{2}\|\nabla u_t^n(t)\|_2^2+\frac{\chi(t)}{2}\|u_t^n(t)\|_2^2\right]\\&\quad+\|u_{tt}^n(t)\|_2^2=\mu'(t)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx+\mu(t)\|\nabla u_t^n(t)\|_2^2\\&\quad+\tfrac{\chi'(t)}{2}\|u_t^n(t)\|_2^2+\tfrac{d}{dt}\left(\int_0^tg(t-\tau)\int_\Omega\nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\right)\\&\quad-g(0)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx+\int_\Omega f_1(x,t)u_{tt}^n(t))dx\\&\quad-\int_0^tg'(t-\tau)\int_\Omega\nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau.\end{aligned}\tag{3.8}$$

By $(A1)$, Hölder's inequality and Young's inequality, one has than we have

$$\begin{aligned}-\int_0^tg'(t-\tau)\int_\Omega\nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau&\leq\frac{1}{2}\|\nabla u_t^n(t)\|_2^2\\&+\tfrac{\xi_1^2\|g\|_{L^1}}{2}\int_0^tg(t-\tau)\|\nabla u^n(\tau)\|_2^2d\tau.\end{aligned}\tag{3.9}$$

Since $\mu(t)\geq m_0$ and from (3.7) we obtain

$$\begin{aligned}-g(0)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx&\leq\frac{1}{2}\|\nabla u_t^n(t)\|_2^2+\frac{g(0)^2}{2}\|\nabla u^n(t)\|_2^2\\&\leq\tfrac{1}{2}\|\nabla u_t^n(t)\|_2^2+\tfrac{g(0)^2L_1}{2m_0}.\end{aligned}\tag{3.10}$$

Since $\chi(t)$ is C^1-function on $[0,\infty)$ and using (3.7) we infer that

$$\frac{\chi'(t)}{2}\|u_t^n(t)\|_2^2\leq\frac{A_1}{2}\|u_t^n(t)\|_2^2\leq\frac{A_1}{2}L_1.\tag{3.11}$$

Moreover,

$$\begin{aligned}\left|\mu'(t)\int_\Omega\nabla u^n(t)\cdot\nabla u_t^n(t)dx\right|&\leq\tfrac{1}{2}\|\nabla u_t^n(t)\|_2^2+\tfrac{M_1^2}{2}\|\nabla u^n(t)\|_2^2\\&\leq\tfrac{1}{2}\|\nabla u_t^n(t)\|_2^2+\tfrac{M_1^2L_1}{2m_0},\end{aligned}\tag{3.12}$$

where $M_1 = \sup_{0\leq t\leq T} \{|\mu'(t)|\}$ and $A_1 = \max_{0\leq t\leq T} \{|\chi'(t)|\}$. Then, by using (3.9) – (3.12), we obtain from (3.8)

$$\begin{aligned}
&\tfrac{d}{dt}\left[\mu(t)\int_\Omega \nabla u^n(t)\cdot\nabla u^n_t(t)dx + \frac{\alpha}{2}\|\nabla u^n_t(t)\|_2^2 + \frac{\chi(t)}{2}\|u^n_t(t)\|_2^2\right]\\
&\quad +\tfrac{1}{2}\|u^n_{tt}(t)\|_2^2 \leq c_2 + \tfrac{\xi_1^2\|g\|_{L^1}}{2}\int_0^t g(t-\tau)\|\nabla u^n(\tau)\|_2^2 d\tau\\
&\quad +\tfrac{d}{dt}\left(\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u^n_t(t)dxd\tau\right)\\
&\quad +\left(\tfrac{3}{2}+M_2\right)\|\nabla u^n_t(t)\|_2^2,
\end{aligned} \tag{3.13}$$

where $c_2 = \left(\frac{g(0)^2+M_1^2+A_1 m_0}{2m_0}\right)L_1 + \frac{1}{2}\|f_1\|_2^2$ and $M_2 = \sup_{0\leq t\leq T}\{|\mu(t)|\}$. Thus, integrating (3.13) over $(0,t)$, we obtain

$$\begin{aligned}
&\tfrac{\alpha}{2}\|\nabla u^n_t(t)\|_2^2 + \tfrac{1}{2}\int_0^t \|u^n_{\tau\tau}(\tau)\|_2^2 dt + \frac{\chi(t)}{2}\|u^n_t(t)\|_2^2\\
&\leq c_3 + \mu(t)\left|\int_\Omega \nabla u^n(t)\cdot\nabla u^n_t(t)dx\right| + \mu(0)\left|\int_\Omega \nabla u^n_0\cdot\nabla u^n_1 dx\right|\\
&+\left(M_2+\tfrac{3}{2}\right)\int_0^t \|\nabla u^n_\tau(\tau)\|_2^2 d\tau + \int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u^n_t(t)dxd\tau,
\end{aligned} \tag{3.14}$$

where $\left(c_3 = c_2 + \xi_1^2\|g\|_{L^1}^2 L_1\right)T + \frac{\alpha}{2}\|\nabla u^n_1\|_2^2 + \frac{\chi(0)}{2}\|u^n_1\|_2^2$. We note that using the inequality $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$, where $\eta > 0$ is arbitrary, it follows that

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u^n_t(t)dxd\tau \leq \eta\|\nabla u^n_t(t)\|_2^2\\
&\quad +\tfrac{1}{4\eta}\|g\|_{L^1(0,\infty)}\|g\|_{L^\infty(0,\infty)}\int_0^t \|\nabla u^n(\tau)\|_2^2 d\tau \leq \eta\|\nabla u^n_t(t)\|_2^2\\
&\quad +\frac{\|g\|_{L^1(0,\infty)}\|g\|_{L^\infty(0,\infty)}}{4\eta m_0}L_1 T,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\mu(t)\left|\int_\Omega \nabla u^n(t)\cdot\nabla u^n_t(t)dx\right| &\leq \eta\|\nabla u^n_t(t)\|_2^2 + \frac{M_2^2}{4\eta}\|\nabla u^n(t)\|_2^2\\
&\leq \eta\|\nabla u^n_t(t)\|_2^2 + \frac{M_2^2}{4\eta m_0}L_1.
\end{aligned} \tag{3.16}$$

By plugging (3.15) and (3.16) into (3.14) with $0 < \eta \leq \frac{\alpha}{4}$, we obtain from $\chi(t) \geq 0$ that

$$\begin{aligned}
\left(\tfrac{\alpha}{2}-2\eta\right)\|\nabla u^n_t(t)\|_2^2 \quad &+\tfrac{1}{2}\int_0^t \|u^n_{\tau\tau}(\tau)\|_2^2 dt \leq c_4\\
&+\left(M_2+\tfrac{3}{2}\right)\int_0^t \|\nabla u^n_\tau(\tau)\|_2^2 d\tau,
\end{aligned} \tag{3.17}$$

where

$$c_4 = c_3 + \mu(0)\|\nabla u^n_0\|_2\|\nabla u^n_1\|_2 + \frac{M_2^2}{4\eta m_0}L_1 + \frac{\|g\|_{L^1}\|g\|_{L^\infty}}{4\eta m_0}L_1 T. \tag{3.18}$$

Taking into account (3.3) – (3.4), we obtain from Gronwall's Lemma the second priori estimate

$$\|\nabla u_t^n(t)\|_2^2 + \int_0^t \|u_{\tau\tau}^n(\tau)\|_2^2 d\tau \leq L_2, \tag{3.19}$$

for all $t \in [0,T]$. Here L_2 is a positive constant independent of $n \in \mathbb{N}$ and $t \in [0,T]$.
Step 3. (The third priori estimate) Replacing ω by $-\Delta u^n(t)$ in (3.2), we have

$$\begin{aligned}
&\tfrac{d}{dt}\left[-\int_\Omega u_t^n(t)\Delta u^n(t)dx + \frac{\alpha}{2}\|\Delta u^n(t)\|_2^2 + \frac{\chi(t)}{2}\|\nabla u^n(t)\|_2^2\right] \\
&\quad -\|\nabla u_t^n(t)\|_2^2 + \mu(t)\|\Delta u^n(t)\|_2^2 \\
&\quad = \tfrac{\chi'(t)}{2}\|\nabla u^n(t)\|_2^2 + \int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau)\cdot\Delta u^n(t)dxd\tau \\
&\quad + \int_\Omega f_1(x,t)(-\Delta u^n(t))dx \leq \frac{A_1}{2}\|\nabla u^n(t)\|_2^2 + 2\eta\|\Delta u^n(t)\|_2^2 \\
&\quad + \tfrac{\|g\|_{L^1}}{4\eta}\int_0^t g(t-\tau)\|\Delta u^n(\tau)\|_2^2 d\tau + \frac{1}{4\eta}\|f_1\|_2^2,
\end{aligned} \tag{3.20}$$

where $0 < \eta \leq \frac{m_0}{2}$ is some positive constant. From $\mu(t) \geq m_0 > 0$, we deduce by integration

$$\begin{aligned}
&\tfrac{\alpha}{2}\|\Delta u^n(t)\|_2^2 + (m_0 - 2\eta)\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau + \frac{\chi(t)}{2}\|\nabla u^n(t)\|_2^2 \\
&\leq \int_0^t \|\nabla u_\tau^n(\tau)\|_2^2 dt + \frac{A_1}{2}\int_0^t \|\nabla u^n(\tau)\|_2^2 d\tau + \left|\int_\Omega u_t^n(t)\Delta u^n(t)dx\right| \\
&\quad + \left|\int_\Omega u_t^n(0)\Delta u^n(0)dx\right| + \tfrac{1}{4\eta}\int_0^t \|f_1\|_2^2 dt \\
&\quad + \tfrac{\alpha}{2}\|\Delta u_0^n\|_2^2 + \tfrac{\chi(0)}{2}\|\nabla u_0^n\|_2^2 + \tfrac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \\
&\quad \leq c_5 + \left|\int_\Omega u_t^n(t)\Delta u^n(t)dx\right| + \tfrac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau,
\end{aligned} \tag{3.21}$$

where

$$c_5 = \|u_1^n\|_2\|\Delta u_0^n\|_2 + \frac{\alpha}{2}\|\Delta u_0^n\|_2^2 + \frac{1}{4\eta}\int_0^t \|f_1\|_2^2 d\tau + \frac{\chi(0)}{2}\|\nabla u_0^n\|_2^2 + \left(\frac{A_1}{m_0}L_1 + L_2\right)T.$$

We note that using the inequality $ab \leq \frac{1}{4}a^2 + b^2$, it follows that

$$\int_\Omega u_t^n(t)\Delta u^n(t)dx \leq \frac{1}{4}\|\Delta u^n(t)\|_2^2 + \|u_t^n(t)\|_2^2. \tag{3.22}$$

Plugging (3.22) into (3.21), we obtain from $\chi(t) \geq m_1 > 0$ that

$$\begin{aligned}
&\left(\tfrac{\alpha}{2} - \tfrac{1}{4}\right)\|\Delta u^n(t)\|_2^2 + (m_0 - 2\eta)\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \\
&\qquad + \tfrac{\chi(t)}{2}\|\nabla u^n(t)\|_2^2 \leq c_6 + \tfrac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u_m(\tau)\|_2^2 d\tau,
\end{aligned} \tag{3.23}$$

where

$$c_6 = c_5 + L_1.$$

Taking into account (3.3) – (3.4), we obtain from Gronwall's Lemma the third priori estimate,

$$\|\Delta u^n(t)\|_2^2 + \int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \leq L_3, \tag{3.24}$$

for all $t \in [0,T]$ and L_3 is a positive constant independent of $n \in \mathbb{N}$ and $t \in [0,T]$.
Step 4. Let $p \geq n$ be two natural numbers, and consider $z^n = u^p - u^n$. Then, applying the same way as in the estimate step 1 and step 3 and observing that $\{u_0^n\}$ and $\{u_1^n\}$ are Cauchy sequence in $H_0^1(\Omega) \cap H^2(\Omega)$ and $H_0^1(\Omega)$, respectively, we deduce for all $t \in [0,T]$

$$\|z_t^n(t)\|_2^2 + \mu(t)\|\nabla z^n(t)\|_2^2 + \int_0^t \|\nabla z_\tau^n(\tau)\|_2^2 d\tau \to 0, \tag{3.25}$$

and

$$\|\Delta z^n(t)\|_2^2 + \int_0^t \|\Delta z^n(\tau)\|_2^2 d\tau \to 0, \quad \text{as} \quad n \to \infty. \tag{3.26}$$

Therefore, (3.7), (3.19), (3.24), (3.25) and (3.26), we see that

$$u^n \to u \ \text{ strongly in } \ C(0,T; H_0^1(\Omega)), \tag{3.27}$$

$$u_t^n \to u_t \ \text{ strongly in } \ C(0,T; L^2(\Omega)). \tag{3.28}$$

$$u_t^n \to u_t \ \text{ strongly in } \ L^2(0,T; H_0^1(\Omega)), \tag{3.29}$$

$$u_{tt}^n \to u_{tt} \ \text{ weakly in } \ L^2(0,T; L^2(\Omega)). \tag{3.30}$$

Then (3.27) – (3.30) are sufficient to pass the limit in (3.2) to obtain in $L^2(0,T; H^{-1}(\Omega))$

$$u_{tt} - \mu(t)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \alpha\Delta u_t + \chi(t)u_t = f_1(x,t). \tag{3.31}$$

Next, we want to show the uniqueness of (3.1). Let $u^{(1)}$ and $u^{(2)}$ be two solutions of (3.1). Then $y = u^{(1)} - u^{(2)}$ satisfies for $\omega \in H_0^1(\Omega)$

$$\begin{aligned} &\mu(t)\int_\Omega \nabla y(t)\cdot\nabla\omega dx - \int_0^t g(t-\tau)\int_\Omega \nabla y(\tau)\cdot\nabla\omega dx d\tau \\ &+ \int_\Omega y_{tt}(t)\omega dx + \alpha\int_\Omega \nabla y_t(t)\cdot\nabla\omega dx + \chi(t)\int_\Omega y_t(t)\omega dx = 0, \\ &y(x,0) = 0, \quad y_t(x,0) = 0, \quad x \in \Omega, \\ &y(x,t) = 0, \quad x \in \partial\Omega, \ t \geq 0. \end{aligned} \tag{3.32}$$

Setting $w = 2y_t(t)$ in (3.32), then as in deriving (3.7), we see that

$$\begin{aligned} &\|y_t(t)\|_2^2 + \mu(t)\|\nabla y(t)\|_2^2 + (2\alpha - 1)\int_0^t \|\nabla y_\tau(\tau)\|_2^2 d\tau \\ &\leq \int_0^t \left[1 + \frac{1}{\mu(\tau)}\left(|\mu'(\tau)| + \|g\|_{L^1}^2\right)\right]\left[\|y_\tau(\tau)\|_2^2 + \mu(\tau)\|\nabla y(\tau)\|_2^2\right] d\tau. \end{aligned} \tag{3.33}$$

Thus employing Gronwall's Lemma, we conclude that

$$\|y_t(t)\|_2 = \|\nabla y(t)\|_2 = 0 \quad \text{for all} \quad t \in [0,T]. \tag{3.34}$$

Therefore, we have the uniqueness. This finishes the proof of Lemma 3.1. □

Now, let us prove the local existence of the problem (1.1).

Theorem 3.2. *Assume that (A1), (A2) and (A3) are fulfilled. Suppose that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ *be given. Then there exists a unique solution* u *of* (1.1) *satisfying* $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ *and* $u_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, *and at least one of the following statements is valid:*

$$\begin{array}{l} (i) \;\; T = \infty, \\ (ii) \;\; e\big(u(t)\big) \equiv \|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \to \infty \quad as \quad t \to T^-. \end{array} \tag{3.35}$$

Proof. Define the following two-parameter space:

$$X_{T,R_0} = \left\{ \begin{array}{l} v \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)), \\ v_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) : \\ e\big(v(t)\big) \le R_0^2, \; t \in [0,T], \quad \text{with} \quad v(0) = u_0, \quad v_t(0) = u_1. \end{array} \right\},$$

for $T > 0$, $R_0 > 0$. Then X_{T,R_0} is a complete metric space with the distance

$$d(y,z) = \sup_{0 \le t \le T} e\left(y(t) - z(t)\right)^{\frac{1}{2}}, \tag{3.36}$$

where $y, z \in X_{T,R_0}$. Given $v \in X_{T,R_0}$, we consider the following problem

$$\left\{ \begin{array}{l} u_{tt} - \psi\left(\left\|\nabla v\right\|_2^2\right) \Delta u - \alpha \Delta u_t + \displaystyle\int_0^t g(t-\tau) \Delta u(\tau) d\tau \\ \qquad +M\left(\left\|\nabla v\right\|_2^2\right) u_t = f(v), \;\; \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \; t \ge 0. \end{array} \right. \tag{3.37}$$

By $(A2)$, we see that $f(v) \in L^2\big(0,T; L^2(\Omega)\big)$. Thus, by Lemma 3.1, we derive that problem (3.37) admits a unique solution $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ and $u_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$. Then, we define the nonlinear mapping $Sv = u$, and we would like to show that there exist $T > 0$ and $R_0 > 0$ such that S is a contraction mapping from X_{T,R_0} into itself. For this, we multiply the first equation of (3.37) by $2u_t$ and integrate it over Ω to get

$$\begin{array}{l} \frac{d}{dt}\left[\left(\psi\left(\|\nabla v\|_2^2\right) - \displaystyle\int_0^t g(\tau) d\tau\right) \|\nabla u(t)\|_2^2 + \big(g \circ \nabla u\big)(t)\right] \\ \quad + \frac{d}{dt}\left[\|u_t(t)\|_2^2\right] + 2\alpha \|\nabla u_t(t)\|_2^2 + 2M\left(\left\|\nabla v\right\|_2^2\right) \|u_t(t)\|_2^2 \\ \quad -\big(g' \circ \nabla u\big)(t) + g(t) \|\nabla u(t)\|_2^2 \\ \quad = \left(\frac{d}{dt}\psi\left(\left\|\nabla v\right\|_2^2\right)\right) \|\nabla u(t)\|_2^2 + 2\displaystyle\int_\Omega f(v) u_t dx. \end{array} \tag{3.38}$$

The equality in (3.38) is obtained, because

$$\begin{array}{l} -2\displaystyle\int_0^t \int_\Omega g(t-\tau) \nabla u(\tau) \cdot \nabla u_t(t) dx d\tau = -\big(g' \circ \nabla u\big)(t) \\ +g(t) \|\nabla u(t)\|_2^2 + \frac{d}{dt}\left[\big(g \circ \nabla u\big)(t) - \displaystyle\int_0^t g(\tau) \|\nabla u(t)\|_2^2 d\tau\right], \end{array} \tag{3.39}$$

where

$$\big(g \circ \nabla u\big)(t) = \int_0^t g(t-\tau) \int_\Omega |\nabla u(\tau) - \nabla u(t)|^2 dx d\tau.$$

Next, multiplying the first equation of (3.37) by $-2\Delta u$, and integrating it over Ω, we have

$$\begin{aligned}&\tfrac{d}{dt}\left[\alpha\|\Delta u(t)\|_2^2-2\int_\Omega u_t\Delta u dx+M\Big(\big\|\nabla v\big\|_2^2\Big)\|\nabla u(t)\|_2^2\right]\\&\quad+2\psi\Big(\big\|\nabla v\big\|_2^2\Big)\|\Delta u(t)\|_2^2-2\|\nabla u_t(t)\|_2^2\\&\quad=\Big(\tfrac{d}{dt}M\Big(\big\|\nabla v\big\|_2^2\Big)\Big)\|\nabla u(t)\|_2^2-2\int_\Omega f(v)\Delta u dx\\&\quad+2\int_0^t g(t-\tau)\Delta u(\tau)\cdot\Delta u(t)dxd\tau.\end{aligned}\tag{3.40}$$

Multiplying (3.40) by ϵ, $0\le\epsilon\le 1$, adding (3.38) together and taking into account $(A1)$ and $(A3)$, we obtain

$$\frac{d}{dt}e^*\big(u(t)\big)+2(\alpha-\epsilon)\|\nabla u_t(t)\|_2^2+2\epsilon\psi\Big(\big\|\nabla v\big\|_2^2\Big)\|\Delta u(t)\|_2^2\le I_1+I_2+I_3,\tag{3.41}$$

where

$$\begin{aligned}e^*\big(u(t)\big)&=\|u_t(t)\|_2^2+\left(\psi\Big(\|\nabla v\|_2^2\Big)-\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\&\quad+\big(g\circ\nabla u\big)(t)+\epsilon\alpha\|\Delta u(t)\|_2^2-2\epsilon\int_\Omega u_t\Delta u dx\\&\quad+\epsilon M\Big(\big\|\nabla v\big\|_2^2\Big)\|\nabla u(t)\|_2^2.\end{aligned}\tag{3.42}$$

$$I_1=2\int_\Omega f(v)\big(u_t-\epsilon\Delta u\big)dx,$$

$$I_2=\left(\frac{d}{dt}\psi\Big(\big\|\nabla v\big\|_2^2\Big)+\epsilon\frac{d}{dt}M\Big(\big\|\nabla v\big\|_2^2\Big)\right)\|\nabla u(t)\|_2^2,$$

and

$$I_3=2\epsilon\int_0^t g(t-\tau)\Delta u(\tau)\cdot\Delta u(t)dxd\tau.$$

Estimate for $I_1=2\int_\Omega f(v)\big(u_t-\epsilon\Delta u\big)dx$. From $(A2)$ and making use of Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}I_1&=2\int_\Omega f(v)\big(u_t-\epsilon\Delta u\big)dx\\&\le 2\int_\Omega\Big|f(v)u_t\Big|dx+2\epsilon\int_\Omega\Big|f(v)\Delta u\Big|dx\\&\le 2K_3\int_\Omega\big|v\big|^{p-1}\big|u_t\big|dx+2\epsilon K_3\int_\Omega\big|v\big|^{p-1}\big|\Delta u\big|dx\\&\le 2K_3B_1^{2(p-1)}\|\Delta v\|_2^{p-1}\|u_t\|_2+2\epsilon K_3B_1^{2(p-1)}\|\Delta v\|_2^{p-1}\|\Delta u\|_2\\&\le 2K_3B_1^{2(p-1)}R_0^{p-1}e\big(u(t)\big)^{\frac12}+2\epsilon K_3B_1^{2(p-1)}R_0^{p-1}e\big(u(t)\big)^{\frac12}\\&=2K_3(1+\epsilon)B_1^{2(p-1)}R_0^{p-1}e\big(u(t)\big)^{\frac12}.\end{aligned}\tag{3.43}$$

Estimate for $I_2=\left(\frac{d}{dt}\psi\Big(\big\|\nabla v\big\|_2^2\Big)+\epsilon\frac{d}{dt}M\Big(\big\|\nabla v\big\|_2^2\Big)\right)\|\nabla u(t)\|_2^2$. First of all, we observe that

$$\begin{aligned}\tfrac{d}{dt}\psi\Big(\big\|\nabla v\big\|_2^2\Big)&=2\psi'\Big(\big\|\nabla v\big\|_2^2\Big)\int_\Omega\nabla v\cdot\nabla v_t dx\\&\le 2M_3\|\Delta v\|_2\|v_t\|_2\le 2M_3R_0^2,\end{aligned}\tag{3.44}$$

where $M_3 = \sup\left\{|\psi'(s)|;\ 0 \le s \le B_1^2 R_0^2\right\}$, and

$$\begin{aligned} \epsilon \tfrac{d}{dt} M\left(\left\|\nabla v\right\|_2^2\right) &= 2\epsilon M'\left(\left\|\nabla v\right\|_2^2\right) \int_\Omega \nabla v \cdot \nabla v_t dx \\ &\le 2\epsilon A_2 \|\Delta v\|_2 \|v_t\|_2 \le 2\epsilon A_2 R_0^2, \end{aligned} \tag{3.45}$$

where $A_2 = \max\left\{|M'(s)|;\ 0 \le s \le B_1^2 R_0^2\right\}$. Then, from (3.44) ,(3.45) and using (3.35) we arrive at

$$I_2 \le 2B_1^2 R_0^2 (M_3 + \epsilon A_2) e\big(u(t)\big). \tag{3.46}$$

Estimate for $I_3 = 2\epsilon \int_0^t g(t-\tau)\Delta u(\tau)\cdot\Delta u(t) dx d\tau$. Using the inequality $ab \le \frac{1}{4\eta} a^2 + \eta b^2$, where $\eta > 0$ is arbitrary, we get

$$\begin{aligned} I_3 &= 2\epsilon \int_0^t g(t-\tau) \int_\Omega \Delta u(\tau) \cdot \Delta u(t) dx d\tau \\ &\le 2\epsilon\eta \|\Delta u(t)\|_2^2 + \epsilon \tfrac{\|g\|_{L^1}}{2\eta} \int_0^t g(t-\tau) \|\Delta u(\tau)\|_2^2 d\tau. \end{aligned} \tag{3.47}$$

Combining these inequalities with $0 < \eta < \frac{\|g\|_{L^1}}{2}$, we get

$$\begin{aligned} &\tfrac{d}{dt} e^*\big(u(t)\big) + 2(\alpha - \epsilon)\|\nabla u_t(t)\|_2^2 + 2\epsilon \left(\psi\left(\|\nabla v\|_2^2\right) - \eta\right) \|\Delta u(t)\|_2^2 \\ &\le 2B_1^2 R_0^2 (M_3 + \epsilon A_2) e\big(u(t)\big) + 2K_3 (1+\epsilon) B_1^{2(p-1)} R_0^{p-1} e\big(u(t)\big)^{\frac{1}{2}} \\ &+ \epsilon \tfrac{\|g\|_{L^1}}{2\eta} \int_0^t g(t-\tau) \|\Delta u(\tau)\|_2^2 d\tau. \end{aligned} \tag{3.48}$$

When we take $\epsilon = 0$ in (3.48), we see that

$$\begin{aligned} &\tfrac{d}{dt} \left[\left(\psi\left(\|\nabla v\|_2^2\right) - \int_0^t g(\tau) d\tau \right) \|\nabla u(t)\|_2^2 + \big(g \circ \nabla u\big)(t) \right] \\ &\quad + \tfrac{d}{dt} \left[\|u_t(t)\|_2^2\right] + 2\alpha \|\nabla u_t(t)\|_2^2 \\ &\quad \le 2B_1^2 R_0^2 M_3 e\big(u(t)\big) + 2K_3 B_1^{2(p-1)} R_0^{p-1} e\big(u(t)\big)^{\frac{1}{2}}. \end{aligned} \tag{3.49}$$

By Young's inequality, we get

$$2\epsilon \int_\Omega u_t \Delta u dx \le 2\epsilon \|u_t\|_2^2 + \frac{\epsilon}{2} \|\Delta u(t)\|_2^2.$$

Hence

$$\begin{aligned} &e^*\big(u(t)\big) \ge (1 - 2\epsilon)\|u_t\|_2^2 + \epsilon\left(\alpha - \tfrac{1}{2}\right)\|\Delta u(t)\|_2^2 + \big(g \circ \nabla u\big)(t) \\ &+ \epsilon M\left(\left\|\nabla v\right\|_2^2\right)\|\nabla u(t)\|_2^2 + \left(\psi\left(\|\nabla v\|_2^2\right) - \int_0^t g(\tau) d\tau\right)\|\nabla u(t)\|_2^2. \end{aligned} \tag{3.50}$$

Choosing $\epsilon = \frac{2}{5}$ and taking into account $(A1)$ and $(A3)$, we have

$$e^*\big(u(t)\big) \ge \frac{1}{5} e\big(u(t)\big), \tag{3.51}$$

and

$$\begin{aligned} &e^*\big(u_0\big) \le (1 + 2\epsilon)\|u_1\|_2^2 + \epsilon\left(\alpha + \tfrac{1}{2}\right)\|\Delta u_0\|_2^2 + \psi\left(\|\nabla u_0\|_2^2\right)\|\nabla u_0\|_2^2 \\ &+ \epsilon M\left(\left\|\nabla u_0\right\|_2^2\right)\|\nabla u_0\|_2^2 \le 2\|u_1\|_2^2 + \left(\alpha + \tfrac{1}{2}\right)\|\Delta u_0\|_2^2 \\ &+ \psi\left(\|\nabla u_0\|_2^2\right)\|\nabla u_0\|_2^2 + M\left(\left\|\nabla u_0\right\|_2^2\right)\|\nabla u_0\|_2^2 = c^*. \end{aligned} \tag{3.52}$$

Integrating (3.48) over $(0,t)$, we get

$$\begin{aligned} & e^*\big(u(t)\big) + \tfrac{4}{5}\left(m_0 - \eta - \tfrac{\|g\|_{L^1}^2}{4\eta}\right)\int_0^t \|\Delta u(\tau)\|_2^2 d\tau \\ & \le e^*\big(u_0\big) + \int_0^t \left[C_1 e^*\big(u(\tau)\big) + C_2 e^*\big(u(\tau)\big)^{\frac{1}{2}}\right] d\tau, \end{aligned} \tag{3.53}$$

where $C_1 = 10B_1^2R_0^2(M_3 + \frac{2}{5}A_2)$ and $C_2 = \frac{14\sqrt{5}}{5}K_1B_1^{2(p-1)}R_0^{p-1}$. Taking $\eta = \frac{\|g\|_{L^1}}{2\eta}$ in (3.53), then from $(A1)$, we deduce

$$\begin{aligned} e^*\big(u(t)\big) & \le e^*\big(u_0\big) + \int_0^t \left[C_1 e^*\big(u(\tau)\big) + C_2 e^*\big(u(\tau)\big)^{\frac{1}{2}}\right] d\tau \\ & \le c^* + \int_0^t \left[C_1 e^*\big(u(\tau)\big) + C_2 e^*\big(u(\tau)\big)^{\frac{1}{2}}\right] d\tau. \end{aligned} \tag{3.54}$$

Hence, by Gronwall's inequality, we have

$$e^*\big(u(t)\big) \le \left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1T}. \tag{3.55}$$

Then, by (3.51), we obtain

$$e\big(u(t)\big) \le 5\left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1T}. \tag{3.56}$$

for any $t \in (0,T]$. Therefore, we see that for the parameters T and R_0 satisfy

$$5\left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1T} \le R_0^2. \tag{3.57}$$

That means S maps X_{T,R_0} into itself. Moreover, by Lemma 3.1,

$$u \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

On the other hand, it follows from (3.49) and (3.56) that

$$u_t \in L^2(0,T; H_0^1(\Omega)).$$

Next, we shall verify that S is a contraction mapping with respect to the metric $d(\cdot,\cdot)$. We take $v_1, v_2 \in X_{T,R_0}$, and denote $u^{(1)} = Sv_1$ and $u^{(2)} = Sv_2$. Hereafter we suppose that (3.57) is valid, thus $u^{(1)}, u^{(2)} \in X_{T,R_0}$. Putting $w(t) = \big(u^{(1)} - u^{(2)}\big)(t)$, then w satisfies

$$\begin{cases} w_{tt} - \psi\left(\|\nabla v_1\|_2^2\right)\Delta w + \displaystyle\int_0^t g(t-\tau)\Delta w(\tau)d\tau - \alpha\Delta w_t \\ \qquad + M\left(\|\nabla v_1\|_2^2\right) w_t = f(v_1) - f(v_2) \\ \qquad + \left[\psi\left(\|\nabla v_1\|_2^2\right) - \psi\left(\|\nabla v_2\|_2^2\right)\right]\Delta u^{(2)} \\ \qquad + \left[M\left(\|\nabla v_2\|_2^2\right) - M\left(\|\nabla v_1\|_2^2\right)\right] u_t^{(2)}, \\ w(0) = 0, \quad w_t(0) = 0, \\ w(x,t) = 0, \quad x \in \partial\Omega,\ t \ge 0. \end{cases} \tag{3.58}$$

We multiply the first equation of (3.58) by $2w_t$ and integrate it over Ω to get

$$\begin{aligned}&\tfrac{d}{dt}\left[\left(\psi\Big(\|\nabla v_1\|_2^2\Big)-\int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2+(g\circ\nabla w)(t)\right]\\&\quad+\tfrac{d}{dt}\left[\|w_t(t)\|_2^2\right]+2\alpha\|\nabla w_t(t)\|_2^2\le I_4+I_5+I_6+I_7.\end{aligned}\tag{3.59}$$

We now estimate I_4-I_7 (defined as below), respectively.

$$I_4=\left(\frac{d}{dt}\psi\left(\|\nabla v_1\|_2^2\right)\right)\|\nabla w(t)\|_2^2\le 2M_3B_1^2R_0^2e\big(w(t)\big),\tag{3.60}$$

$$\begin{aligned}I_5&=2\int_\Omega\Big[f(v_1)-f(v_2)\Big]w_tdx\\&\le 2K_3\int_\Omega\Big(|v_1|^{p-2}+|v_2|^{p-2}\Big)|v_1-v_2|w_tdx\\&\le 2K_3\Big[\|v_1\|_{N(p-2)}^{p-2}+\|v_2\|_{N(p-2)}^{p-2}\Big]\|v_1-v_2\|_{\frac{2N}{N-2}}\|w_t\|_2\\&\le 4K_3B_1^{2(p-1)}R_0^{p-2}e\big(v_1-v_2\big)^{\frac12}e\big(w(t)\big)^{\frac12},\end{aligned}\tag{3.61}$$

$$\begin{aligned}I_6&=2\left[\psi\left(\|\nabla v_1\|_2^2\right)-\psi\left(\|\nabla v_2\|_2^2\right)\right]\int_\Omega\Delta u^{(2)}w_tdx\\&\le 2L\Big(\|\nabla v_1\|_2+\|\nabla v_2\|_2\Big)\|\nabla v_1-\nabla v_2\|_2\left\|\Delta u^{(2)}\right\|_2\|w_t\|_2\\&\le 4LB_1^2R_0^2e\big(v_1-v_2\big)^{\frac12}e\big(w(t)\big)^{\frac12},\end{aligned}\tag{3.62}$$

where $L=L(R)$ is the Lipschitz constant of $\psi(s)$ in $[0,R_0]$.
Estimate for $I_7=2\left[M\left(\|\nabla v_2\|_2^2\right)-M\left(\|\nabla v_1\|_2^2\right)\right]\int_\Omega u_t^{(2)}w_tdx$. Assumption $(A3)$ gives

$$\begin{aligned}\left|M\left(\|\nabla v_2\|_2^2\right)-M\left(\|\nabla v_1\|_2^2\right)\right|&=\left|\int_{\|\nabla v_1\|_2^2}^{\|\nabla v_2\|_2^2}M'(r)dr\right|\\&\le\int_{\|\nabla v_1\|_2^2}^{\|\nabla v_2\|_2^2}|M'(r)|\,dr\le C_*\left|\|\nabla v_2\|_2^2-\|\nabla v_1\|_2^2\right|\\&\le C_*\Big(\|\nabla v_1\|_2+\|\nabla v_1\|_2\Big)\|\nabla v_2-\nabla v_1\|_2,\end{aligned}\tag{3.63}$$

where C_* is a positive constant. From (3.63) and (3.35), we have

$$\begin{aligned}I_7&=2\left[M\left(\|\nabla v_2\|_2^2\right)-M\left(\|\nabla v_1\|_2^2\right)\right]\int_\Omega u_t^{(2)}w_tdx\\&\le 2C_*\Big(\|\nabla v_1\|_2+\|\nabla v_1\|_2\Big)\|\nabla(v_2-v_1)\|_2\left\|u_t^{(2)}\right\|_2\|w_t\|_2\\&\le 2C_*B_1^2R_0^2e\big(v_1-v_2\big)^{\frac12}e\big(w(t)\big)^{\frac12}.\end{aligned}\tag{3.64}$$

Inserting (3.60) – (3.64) in (3.59), we get

$$\begin{aligned}&\tfrac{d}{dt}\left[\left(\psi\Big(\|\nabla v_1\|_2^2\Big)-\int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2+(g\circ\nabla w)(t)\right]\\&\quad\tfrac{d}{dt}\left[\|w_t(t)\|_2^2\right]+2\alpha\|\nabla w_t(t)\|_2^2\\&\quad\le C_3e\big(w(t)\big)+C_4e\big(v_1-v_2\big)^{\frac12}e\big(w(t)\big)^{\frac12},\end{aligned}\tag{3.65}$$

where $C_3 = 2M_3B_1^2R_0^2$ and $C_4 = 4K_3B_1^{2(p-1)}R_0^{p-2} + 4LB_1^2R_0^2 + 2C_*B_1^2R_0^2$.
On the other hand, multiplying the first equation in (3.58) by $-2\Delta w$, and integrating it over Ω, we get

$$\begin{aligned}&\tfrac{d}{dt}\left\{\alpha\|\Delta w(t)\|_2^2 - 2\int_\Omega w_t\Delta w dx + M\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\nabla w(t)\|_2^2\right\}\\ &+2\psi\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\Delta w(t)\|_2^2 - 2\|\nabla w_t\|_2^2 = I_8 + I_9 + I_{10} + I_{11} + I_{12}.\end{aligned} \tag{3.66}$$

We now estimate I_8-I_{11} (defined as below), respectively.
Applying the similar arguments as in estimating I_i, $i = 2, 3, 5, 6, 7$, we observe that

$$I_8 = \left(\frac{d}{dt}M\Big(\big\|\nabla v_1\big\|_2^2\Big)\right)\|\nabla w(t)\|_2^2 \le 2A_2R_0^2B_1^2 e\big(w(t)\big), \tag{3.67}$$

$$\begin{aligned}I_9 &= -2\int_\Omega \Big[f(v_1) - f(v_2)\Big]\Delta w dx\\ &\le 4K_3B_1^{2(p-1)}R_0^{p-2}e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}},\end{aligned} \tag{3.68}$$

$$\begin{aligned}I_{10} &= 2\left[\psi\left(\|\nabla v_1\|_2^2\right) - \psi\left(\|\nabla v_2\|_2^2\right)\right]\int_\Omega \Delta u^{(2)}\Delta w dx\\ &\le 4LB_1^2R_0^2e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}},\end{aligned} \tag{3.69}$$

$$\begin{aligned}I_{11} &= 2\left[M\left(\|\nabla v_2\|_2^2\right) - M\left(\|\nabla v_1\|_2^2\right)\right]\int_\Omega \Delta u^{(2)}\Delta w dx\\ &\le 2C_*B_1^2R_0^2e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}},\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}I_{12} &= 2\int_0^t g(t-\tau)\int_\Omega \Delta w(\tau)\cdot\Delta w(t) dx d\tau\\ &\le 2\eta\|\Delta w(t)\|_2^2 + \tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2 d\tau,\end{aligned} \tag{3.71}$$

where $\eta > 0$ is arbitrary. Combining these inequalities with $0 < \eta \le \frac{\|g\|_{L^1}}{2}$, we get

$$\begin{aligned}&\tfrac{d}{dt}\left\{\alpha\|\Delta w(t)\|_2^2 - 2\int_\Omega w_t\Delta w dx + M\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\nabla w(t)\|_2^2\right\}\\ &\quad+2\left(\psi\Big(\big\|\nabla v_1\big\|_2^2\Big) - 2\eta\right)\|\Delta w(t)\|_2^2 \le C_4e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}}\\ &\quad+\tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2 d\tau + 2\|\nabla w_t\|_2^2 + C_5e\big(w(t)\big),\end{aligned} \tag{3.72}$$

where $C_5 = 2A_2B_1^2R^2$. Multiplying (3.72) by ϵ, $0 < \epsilon \le 1$, and adding (3.65) together, we obtain

$$\begin{aligned}&\tfrac{d}{dt}e^{**}\big(w(t)\big) + 2(\alpha - \epsilon)\|\nabla w_t\|_2^2 + 2\epsilon\left(\psi\Big(\big\|\nabla v_1\big\|_2^2\Big) - 2\eta\right)\|\Delta w(t)\|_2^2\\ &\quad\le (C_3 + \epsilon C_5)e\big(w(t)\big) + (1+\epsilon)C_4e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}}\\ &\quad+\epsilon\tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2 d\tau,\end{aligned} \tag{3.73}$$

where

$$\begin{aligned} e^{**}\big(w(t)\big) &= \|w_t(t)\|_2^2 + \left(\psi\Big(\|\nabla v_1\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 \\ &+(g\circ\nabla w)(t) + \epsilon\alpha\|\Delta w(t)\|_2^2 - 2\epsilon\int_\Omega w_t\Delta w dx \\ &+\epsilon M\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\nabla w(t)\|_2^2. \end{aligned} \tag{3.74}$$

By using Young's inequality on the fifth term of right hand side of (3.74), we get

$$\begin{aligned} e^{**}\big(w(t)\big) &\geq (1-2\epsilon)\|w_t(t)\|_2^2 + \epsilon\left(\alpha - \tfrac{1}{2}\right)\|\Delta w(t)\|_2^2 \\ &+ \left(\psi\Big(\|\nabla v_1\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 \\ &+(g\circ\nabla w)(t) + \epsilon M\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\nabla w(t)\|_2^2. \end{aligned} \tag{3.75}$$

Choosing $\epsilon = \frac{2}{5}$ and by (2.1), (2.4), we have

$$e^{**}\big(w(t)\big) \geq \frac{1}{5}e\big(w(t)\big). \tag{3.76}$$

Then, applying the some way as in obtained (3.53) and taking $\eta = \frac{\|g\|_{L^1}}{2\eta}$, we deduce

$$\begin{aligned} e^{**}\big(w(t)\big) \leq \int_0^t &\left[5\left(C_3 + \frac{2}{5}C_5\right)e^{**}\big(w(t)\big)\right. \\ &\left.+\tfrac{7\sqrt{5}}{5}C_4 e\big(v_1 - v_2\big)^{\frac{1}{2}} e^{**}\big(w(t)\big)^{\frac{1}{2}}\right]d\tau + e^{**}\big(w(0)\big). \end{aligned} \tag{3.77}$$

Thus, applying Gronwall's Lemma and noting that $e^{**}\big(w(0)\big) = 0$, we have

$$e^{**}\big(w(t)\big) \leq \frac{49}{20}C_4^2T^2e^{5\left(C_3+\frac{2}{5}C_5\right)T}\sup_{0\leq t\leq T} e(v_1 - v_2). \tag{3.78}$$

By (3.36) and (3.76), we have

$$d(u^{(1)}, u^{(2)}) \leq C(T, R_0)^{\frac{1}{2}}d(v_1, v_2), \tag{3.79}$$

where

$$C(T, R_0)^{\frac{1}{2}} = \frac{49}{4}C_4^2T^2e^{5\left(C_3+\frac{2}{5}C_5\right)T}. \tag{3.80}$$

Hence, under inequality (3.57), S is a contraction mapping if $C(T, R_0) < 1$. Indeed, we choose R_0 sufficient large and T sufficient small so that (3.57) and (3.79) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

The second statement of the theorem is proved by a standard continuation argument. Indeed, let $[0, T)$ be a maximal existence interval on which the solution of (1.1) exists. Suppose that $T < \infty$ and $\lim_{t\to T^-}\left(\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2\right) < \infty$. Then, there are a sequence $\{t_n\}$ and a constant $K > 0$ such that $t_n \to T^-$ as $n \to \infty$ and $\|u_t(t_n)\|_2^2 + \|\Delta u(t_n)\|_2^2 \leq K$, $n = 1, 2, \ldots$ Since for all $n \in \mathbb{N}$, there exists a unique solution of (1.1) with initial data $(u(t_n), u_t(t_n))$ on $[t_n, t_{n+\rho}]$, $\rho > 0$ depending on K and independent of $n \in \mathbb{N}$. Thus, we can get $T < t_n + \rho$ for $n \in \mathbb{N}$ large enough. It contradicts to the maximality of T. The proof of Theorem 3.2 is now completed. □

4. Global existence and energy decay

In this section, we consider the global existence and energy decay of solutions for a kind of the problem (1.1):

$$\begin{cases} u_{tt} - \psi\left(\left\|\nabla u\right\|_2^2\right)\Delta u - \alpha\Delta u_t + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau \\ \quad +M\left(\left\|\nabla u\right\|_2^2\right)u_t = |u|^{p-2}u, \quad x\in\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \\ u(x,t)=0, \quad x\in\partial\Omega,\ t>0, \end{cases} \tag{4.1}$$

where $2 < p \leq \frac{2(N-1)}{N-2}$, $\alpha \geq 1$ and $\psi(r) = 1 + br^{\gamma}$, $b \geq 0$, $\gamma \geq 1$ and $r \geq 0$.
To obtain the results of this section, we now define some functionals as follows:

$$I_1(t) = I_1\big(u(t)\big) = \left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t) - \|u(t)\|_p^p, \tag{4.2}$$

$$I_2(t) = I_2\big(u(t)\big) = I_1(t) + b\|\nabla u(t)\|_2^{2(\gamma+1)}, \tag{4.3}$$

$$\begin{aligned} J(t) = J\big(u(t)\big) &= \tfrac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \tfrac{1}{2}(g\circ\nabla u)(t) \\ &\quad + \tfrac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \tfrac{1}{p}\|u(t)\|_p^p. \end{aligned} \tag{4.4}$$

We define the energy of the solution u of (4.1) by

$$\begin{aligned} E(t) = E\big(u(t)\big) &= \tfrac{1}{2}\|u_t(t)\|_2^2 + J\big(u(t)\big) = \tfrac{1}{2}\|u_t(t)\|_2^2 \\ &\quad + \tfrac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \tfrac{1}{2}(g\circ\nabla u)(t) \\ &\quad + \tfrac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \tfrac{1}{p}\|u(t)\|_p^p. \end{aligned} \tag{4.5}$$

Lemma 4.1. *$E(t)$ is a non-increasing function for $t \geq 0$, that is*

$$\begin{aligned} E'(t) \leq -\Big[& m_1\|u_t(t)\|_2^2 + \alpha\|\nabla u_t(t)\|_2^2 + \tfrac{K_2}{2}\big(g\circ\nabla u\big)(t) \\ & + \tfrac{1}{2}g(t)\|\nabla u(t)\|_2^2\Big] \leq 0, \quad \text{for all} \quad t > 0. \end{aligned} \tag{4.6}$$

Proof. Multiplying the differential equation in (4.1) by u_t, integrating by parts over Ω and using $(A3)$, we obtain

$$\begin{aligned} & \tfrac{d}{dt}\left[\tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\|\nabla u(t)\|_2^2 + \tfrac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \tfrac{1}{p}\|u(t)\|_p^p\right] \\ &\quad = -\alpha\|\nabla u_t(t)\|_2^2 - M\Big(\|\nabla u\|_2^2\Big)\|u_t(t)\|_2^2 \\ &\quad + \int_0^t\int_\Omega g(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau \\ & \leq -\alpha\|\nabla u_t(t)\|_2^2 - m_1\|u_t(t)\|_2^2 + \int_0^t\int_\Omega g(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau. \end{aligned}$$

Exploiting (3.39) on the third term on the right hand side of the above inequality and using $(A1)$, we have the result. □

Lemma 4.2. *Let u be the solution of* (4.1). *Assume the conditions of Theorem 3.2 hold. If* $I_1(0) > 0$ *and*

$$\sigma = \frac{B_1^p}{l_1}\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{4.7}$$

then $I_2(t) > 0$, *for all* $t \geq 0$.

Proof. Since $I_1(0) > 0$, it follows from the continuity of $u(t)$ that

$$I_1(t) > 0, \tag{4.8}$$

for some interval near $t = 0$. Let $t_{max} > 0$ be a maximal time (possibly $t_{max} = T$), when (4.8) holds on $[0, t_{max})$. From (4.2) and (4.4), we have

$$\begin{aligned} J(t) &\geq \tfrac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 + \tfrac{1}{2}\big(g \circ \nabla u\big)(t) - \tfrac{1}{p}\|u\|_p^p \\ &\geq \tfrac{p-2}{2p}\left[\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right] + \tfrac{1}{p}I_1(t) \\ &\geq \tfrac{p-2}{p}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 \geq \left(\tfrac{p-2}{2p}\right) l_1\|\nabla u\|_2^2. \end{aligned} \tag{4.9}$$

Using (4.9), (4.5) and $E(t)$ is non-increasing by (4.6), we get

$$l_1\|\nabla u\|_2^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0). \tag{4.10}$$

Exploiting Lemma 2.1 and (4.7), we obtain from (4.10) on $[0, t_{max})$

$$\begin{aligned} \|u\|_p^p &\leq B_1^p\|\nabla u\|_2^p = B_1^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2 \\ &\leq \tfrac{B_1^p}{l_1}\left(\tfrac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}} l_1\|\nabla u\|_2^2 = \sigma l_1\|\nabla u\|_2^2 \\ &< \left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2. \end{aligned}$$

Thus on $[0, t_{max})$, we have

$$I_1(t) = \left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \big(g \circ \nabla u\big)(t) - \|u(t)\|_p^p > 0. \tag{4.11}$$

This implies that we can take $t_{max} = T$. But, from (4.2) and (4.3), we see that

$$I_2(t) \geq I_1(t) > 0, \quad t \in [0, T]. \tag{4.12}$$

Therefore, we have $I_2(t) > 0$, $t \in [0, T]$.
Next, we want to show that $T = \infty$. Multiplying the first equation in (4.1) by $-2\Delta u$, and integrating it over Ω, we get

$$\begin{aligned} &\tfrac{d}{dt}\left\{\alpha\|\Delta u\|_2^2 - 2\int_\Omega u_t\Delta u dx + M\Big(\big\|\nabla u\big\|_2^2\Big)\|\nabla u\|_2^2\right\} \\ &\quad + \Big(2\psi\Big(\big\|\nabla u\big\|_2^2\Big) - 2\eta\Big)\|\Delta u\|_2^2 \leq 2\|\nabla u_t\|_2^2 - 2\int_\Omega |u|^{p-2}u\Delta u dx \\ &\quad + \tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau + \left(\frac{d}{dt}M\Big(\big\|\nabla u\big\|_2^2\Big)\right)\|\nabla u\|_2^2, \end{aligned} \tag{4.13}$$

where $0 < \eta \leq \frac{\|g\|_{L^1}}{2}$. On the other hand, multiplying the first equation in (4.1) by $2u_t$, and integrating it over Ω, we get

$$\begin{aligned} \frac{d}{dt}\left(2E(t)\right) + 2\alpha\|\nabla u_t\|_2^2 &= \left(g' \circ \nabla u\right)(t) - g(t)\|\nabla u(t)\|_2^2 \\ &\quad -2M\left(\left\|\nabla u\right\|_2^2\right)\|u_t\|_2^2. \end{aligned} \tag{4.14}$$

Multiplying (4.13) by ϵ, $0 < \epsilon \leq 1$, and adding (4.14) together, we obtain

$$\begin{aligned} \frac{d}{dt}E^*(t) \quad & +2(\alpha-\epsilon)\|\nabla u_t\|_2^2 + 2\epsilon\left(\psi\left(\left\|\nabla u\right\|_2^2\right) - 2\eta\right)\|\Delta u\|_2^2 \\ & \leq -2\epsilon\int_\Omega |u|^{\alpha-2}u\Delta u dx + \epsilon\left(\frac{d}{dt}M\left(\left\|\nabla u\right\|_2^2\right)\right)\|\nabla u\|_2^2 \\ & +2\epsilon\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau, \end{aligned} \tag{4.15}$$

where

$$E^*(t) = 2E(t) - 2\epsilon\int_\Omega u_t\Delta u dx + \epsilon\alpha\|\Delta u\|_2^2 + \epsilon M\left(\left\|\nabla u\right\|_2^2\right)\|\nabla u\|_2^2. \tag{4.16}$$

By young's inequality, we get

$$\left|2\epsilon\int_\Omega u_t\Delta u dx\right| \leq 2\epsilon\|u_t\|_2^2 + \frac{\epsilon}{2}\|\Delta u\|_2^2. \tag{4.17}$$

Hence, choosing $\epsilon = \frac{2}{5}$ and by (4.11), we see that

$$E^*(t) \geq \frac{1}{5}\left(\|u_t\|_2^2 + \|\Delta u\|_2^2\right). \tag{4.18}$$

Let us estimate $I_{13} = \left(\frac{d}{dt}M\left(\left\|\nabla u\right\|_2^2\right)\right)\|\nabla u\|_2^2$. Since $M \in C^1\left([0,\infty\right)$, using (4.10) and (4.18) we infer that

$$\begin{aligned} I_{13} &= \left(\frac{d}{dt}M\left(\left\|\nabla u\right\|_2^2\right)\right)\|\nabla u\|_2^2 \\ &= 2M'\left(\left\|\nabla u\right\|_2^2\right)\left(\int_\Omega \nabla u\cdot\nabla u_t dx\right)\|\nabla u\|_2^2 \\ &\leq 2A_3\|\Delta u\|_2\|u_t\|_2\|\nabla u\|_2^2 \leq 10A_3\left(\frac{2p}{l_1(p-2)}\right)E(0)E^*(t) = c_7E^*(t), \end{aligned} \tag{4.19}$$

where $c_7 = 10A_3(\frac{2p}{l_1(p-2)})E(0)$ and $A_3 = \max\{M'(r),\ 0 \leq r \leq (\frac{2p}{l_1(p-2)})E(0)\}$. Moreover, we note that

$$\begin{aligned} 2\left|\int_\Omega |u|^{p-2}u\Delta u dx\right| &\leq 2(p-1)\int_\Omega |u|^{p-2}|\nabla u|^2 dx \\ &\leq 2(p-1)\|u\|_{(p-2)\theta_1}^{p-2}\|\nabla u\|_{2\theta_2}^2, \end{aligned} \tag{4.20}$$

where $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$, so that, we put $\theta_1 = 1$ and $\theta_2 = \infty$, if $N = 1$; $\theta_1 = 1 + \epsilon_1$ (for arbitrary small $\epsilon_1 > 0$), if $N = 2$; and $\theta_2 = \frac{N}{N-2}$, if $N \geq 3$. Then, by Lemma 2.1, (4.10) and (4.18), we have

$$2\left|\int_\Omega |u|^{p-2}u\Delta u dx\right| \leq 2B_1^p(p-1)\|\nabla u\|_2^{p-2}\|\Delta u\|_2^2 \leq c_8E^*(t), \tag{4.21}$$

where $c_8 = 10B_1^p(p-1)\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}}$. Inserting (4.19) and (4.21) into (4.15), and then integrating it over $(0,t)$, we obtain

$$\begin{aligned} E^*(t) \quad &+\tfrac{4}{5}\left(m_0 - \eta - \tfrac{\|g\|_{L^1}^2}{4\eta}\right)\int_0^t \|\Delta u(\tau)\|_2^2 d\tau \\ &\le E^*(0) + \int_0^t c_9 E^*(\tau)d\tau, \end{aligned} \tag{4.22}$$

where $c_9 = c_7 + c_8$. Taking $\eta = \frac{\|g\|_{L^1}}{2}$ in (4.22), and by Gronwall's Lemma, we deduce

$$E^*(t) \le E^*(0)e^{c_9 t}, \tag{4.23}$$

for any $t \ge 0$. Therefore by Theorem 3.2, we have $T = \infty$. □

Lemma 4.3. *If u satisfies the assumptions of Lemma 4.2, then there exists $B > 0$ such that*

$$\|u\|_p^p \le BE(t). \tag{4.24}$$

Proof. Using Lemma 2.1 and (4.10), we have

$$\begin{aligned} \|u\|_p^p &\le B_1^p\|\nabla u\|_2^p = B_1^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2 \\ &\le \frac{B_1^p}{l_1}\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}} l_1\|\nabla u\|_2^2 = \sigma l_1\|\nabla u\|_2^2 \\ &\le \sigma\left(\frac{2p}{p-2}\right)E(t). \end{aligned}$$

Let $B = \sigma\left(\frac{2p}{p-2}\right)$, then we have (4.24). □

Theorem 4.4. *(Global existence and Energy decay) Suppose that (A1) and (A3) hold. Assume $I_1(u_0) > 0$ and (4.7) holds, then the problem (4.1) admits a global solution u if $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Moreover, we have the following decay estimates*

$$E(t) \le ce^{-\kappa\epsilon t}, \quad \forall t \ge 0 \quad \text{and} \quad \epsilon \in (0, \epsilon_1],$$

where c, κ and ϵ_1 are positive constants.

Proof. Defining the perturbed energy by

$$E_\epsilon(t) = E(t) + \epsilon\varphi(t), \tag{4.25}$$

where

$$\varphi(t) = \int_\Omega u(t)u_t(t)dx, \tag{4.26}$$

we can show that for ϵ small enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \le E_\epsilon(t) \le \beta_2 E(t). \tag{4.27}$$

In fact

$$\begin{aligned} E_\epsilon(t) \quad &\le E(t) + \tfrac{\epsilon}{2}\|u_t\|_2^2 + \tfrac{\epsilon}{2}\|u\|_2^2 \le (1+\epsilon)E(t) + \tfrac{\epsilon}{2}B_1^2\|\nabla u\|_2^2 \\ &\le (1+\epsilon)E(t) + \tfrac{\epsilon}{2}B_1^2\left(\tfrac{2p}{l_1(p-2)}\right)E(t) \le \beta_2 E(t), \end{aligned} \tag{4.28}$$

and

$$E_\epsilon(t) \geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 - \epsilon\delta\|u\|_2^2 \geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 - \epsilon\delta B_1^2\|\nabla u\|_2^2. \tag{4.29}$$

By choosing δ small enough, we have

$$E_\epsilon(t) \geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 \geq J\big(u(t)\big) + \left(\frac{1}{2} - \frac{\epsilon}{4\delta}\right)\|u_t\|_2^2. \tag{4.30}$$

Once δ is chosen, we take ϵ so small that

$$E_\epsilon(t) \geq J\big(u(t)\big) + \frac{\beta_1}{2}\|u_t\|_2^2 \geq \beta_1 E(t), \tag{4.31}$$

where $\frac{\beta_1}{2} \leq \frac{1}{2} - \frac{\epsilon}{4\delta}$. Now taking the derivative of $\varphi(t)$ defined in (4.26) and substituting

$$\begin{aligned} u_{tt} \quad &= \psi\Big(\big\|\nabla u\big\|_2^2\Big)\nabla u + \alpha\Delta u_t - \int_0^t g(t-\tau)\Delta u(\tau)d\tau \\ &-M\Big(\big\|\nabla u\big\|_2^2\Big)u_t + |u|^{p-2}u, \end{aligned} \tag{4.32}$$

in the obtained expression, it results that

$$\begin{aligned} \varphi'(t) \quad &= \|u_t\|_2^2 - \|\nabla u\|_2^2 - b\|\nabla u\|_2^{2(\gamma+1)} \\ &+ \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau - \alpha\big(\nabla u_t, \nabla u\big) \\ &-M\Big(\big\|\nabla u\big\|_2^2\Big)\big(u_t, u\big) + \|u\|_p^p. \end{aligned} \tag{4.33}$$

Adding and subtracting $2E(t)$, and taking (4.5) into account, from (4.33) we infer

$$\begin{aligned} \varphi'(t) \quad &= -2E(t) + 2\|u_t\|_2^2 - \left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &+\big(g\circ\nabla u\big)(t) - b\left(1 - \tfrac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)} \\ &+\left(1 - \tfrac{2}{p}\right)\|u\|_p^p - \alpha\big(\nabla u_t, \nabla u\big) - M\Big(\big\|\nabla u\big\|_2^2\Big)\big(u_t, u\big) \\ &+\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau. \end{aligned} \tag{4.34}$$

Estimate for $J_1 = \alpha\big(\nabla u_t, \nabla u\big)$. Considering Cauchy-Schwartz inequality, we have

$$|J_1| \leq \frac{\alpha^2}{2}\|\nabla u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2. \tag{4.35}$$

Let us estimate $J_2 = M\Big(\big\|\nabla u\big\|_2^2\Big)\big(u_t, u\big)$. Noting that $\|\nabla u(t)\|_2^2 \leq \frac{2p}{l_1(p-2)}E(0) = \beta_3$ for all $t \geq 0$, we have that

$$M\Big(\big\|\nabla u\big\|_2^2\Big) \leq \xi, \quad \forall t \geq 0, \tag{4.36}$$

where $\xi = \max\{M(r);\ r \in [0, \beta_3]\}$. From (4.36) we conclude that

$$|J_2| \leq \frac{\xi^2}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u(t)\|_2^2 \leq \frac{\xi^2}{2}\|u_t(t)\|_2^2 + \frac{1}{2}B_1^2\|\nabla u(t)\|_2^2. \tag{4.37}$$

Estimate $J_3 = \displaystyle\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau$. From assumption $(A1)$ and making use of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} J_3 &= \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau \\ &= \int_0^t g(t-\tau)\int_\Omega \Big[\nabla u(\tau)-\nabla u(t)+\nabla u(t)\Big]\cdot\nabla u(t)dxd\tau \\ &\le \int_0^t g(t-\tau)\big|\nabla u(t)-\nabla u(\tau)\big|\big|\nabla u(t)\big|dxd\tau \\ &+ \left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &\le \|\nabla u(t)\|_2^2\int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^2 d\tau \\ &+ \left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &\le \tfrac{1}{2}\|\nabla u(t)\|_2^2+\tfrac{1}{2}\|g\|_{L^1(0,\infty)}\big(g\circ\nabla u\big)(t)+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &\le \tfrac{1}{2}\|\nabla u(t)\|_2^2+\tfrac{1}{2}\big(g\circ\nabla u\big)(t)+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2. \end{aligned} \tag{4.38}$$

Utilizing Lemma 4.3 and inserting (4.35), (4.38) and (4.37) in (4.34), we have

$$\begin{aligned} \varphi'(t) &\le \left(\tfrac{\xi^2}{2}+2\right)\|u_t\|_2^2+\left(1+\tfrac{B_1^2}{2}\right)\|\nabla u\|_2^2 \\ &+\left[\left(1-\tfrac{2}{p}\right)B-2\right]E(t)-b\left(1-\tfrac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)} \\ &+\tfrac{\alpha^2}{2}\|\nabla u_t(t)\|_2^2+\tfrac{3}{2}\Big(g\circ\nabla u\Big)(t). \end{aligned} \tag{4.39}$$

Then, from (4.6), (4.25), (4.26) and (4.39) we arrive at

$$\begin{aligned} E'_\epsilon(t) &= E'(t)+\epsilon\varphi'(t)\le -\Big(m_1-\lambda_1\epsilon\Big)\|u_t\|_2^2+\lambda_2\epsilon\|\nabla u\|_2^2 \\ &-\left(\tfrac{K_2}{2}-\tfrac{3}{2}\epsilon\right)\Big(g\circ\nabla u\Big)(t)-\left(\alpha-\tfrac{\alpha^2}{2}\epsilon\right)\|\nabla u_t(t)\|_2^2 \\ &-\epsilon(-\lambda_3)E(t)-b\epsilon\left(1-\tfrac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)}-\tfrac{1}{2}g(t)\|\nabla u(t)\|_2^2, \end{aligned} \tag{4.40}$$

where

$$\lambda_1=\frac{\xi^2}{2}+2>0,\ \lambda_2=\frac{B_1^2}{2}+1>0$$

and

$$\lambda_3=\left(1-\frac{2}{p}\right)B-2=\left(1-\frac{2}{p}\right)\left(\frac{2p}{p-2}\right)\sigma-2=2\sigma-2<0.$$

On the other hand, since

$$\int_0^t g'(\tau)d\tau=g(t)-g(0),$$

then

$$-g(t)\|\nabla u(t)\|_2^2=-g(0)\|\nabla u(t)\|_2^2-\left(\int_0^t g'(\tau)d\tau\right)\|\nabla u(t)\|_2^2.$$

From $(A1)$ the last inequality yields

$$-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \leq -\frac{1}{2}g(0)\|\nabla u(t)\|_2^2 + \frac{K_1}{2}\|g\|_{L^1(0,\infty)}\|\nabla u(t)\|_2^2. \tag{4.41}$$

Combining (4.40) and (4.41) we conclude that

$$\begin{aligned} E'_\epsilon(t) &\leq -\Big(m_1 - \lambda_1\epsilon\Big)\|u_t\|_2^2 - \left(\tfrac{K_2}{2} - \tfrac{3}{2}\epsilon\right)\Big(g \circ \nabla u\Big)(t) \\ &- \left(\alpha - \tfrac{\alpha^2}{2}\epsilon\right)\|\nabla u_t(t)\|_2^2 - b\epsilon\left(1 - \tfrac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)} - \epsilon(-\lambda_3)E(t) \\ &-\tfrac{1}{2}\left[g(0) - K_1\|g\|_{L^1(0,\infty)} - 2\lambda_2\epsilon\right]\|\nabla u(t)\|_2^2. \end{aligned} \tag{4.42}$$

From (2.1) we have $l_2 = g(0) - K_1\|g\|_{L^1(0,\infty)} > 0$. Defining

$$\epsilon_1 = \min\left\{\frac{m_1}{\lambda_1}, \frac{K_2}{3}, \frac{2}{\alpha}, \frac{l_2}{2\lambda_2}\right\}, \tag{4.43}$$

we conclude by taking $\epsilon \in (0, \epsilon_1]$ in (4.42) that

$$E'_\epsilon(t) \leq -\epsilon(-\lambda_3)E(t). \tag{4.44}$$

Thus, we see that $\forall t \geq 0$ and $\epsilon \in (0, \epsilon_1]$

$$E'_\epsilon(t) \leq -\epsilon(-\lambda_3)E(t) \leq -\frac{-\lambda_3}{\beta_2}\epsilon E_\epsilon(t). \tag{4.45}$$

By the Gronwall inequality, we see that

$$E_\epsilon(t) \leq E_\epsilon(0)e^{-\kappa\epsilon t}, \quad \forall t \geq 0 \text{ and } \quad \epsilon \in (0, \epsilon_1], \tag{4.46}$$

where $\kappa = \frac{-\lambda_3}{\beta_2}$. Combining with (4.27), we obtain

$$\beta_1 E(t) \leq E_\epsilon(t) \leq E_\epsilon(0)e^{-\kappa\epsilon t}, \quad \forall t \geq 0 \text{ and } \quad \epsilon \in (0, \epsilon_1], \tag{4.47}$$

and

$$E(t) \leq ce^{-\kappa\epsilon t}, \quad \forall t \geq 0 \text{ and } \quad \epsilon \in (0, \epsilon_1], \tag{4.48}$$

where $c = \frac{E_\epsilon(0)}{\beta_1}$. Thus, the proof of the theorem is completed. □

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