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A class of differential systems of even degree with exact non-algebraic limit cycles

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Abstract. Up until now all the polynomial differential systems for which non-algebraic limit cycles are known explicitly have degree odd. Here we show that that there are polynomial systems of even degree with explicit no-algebraic limit cycles. To our knowledge, there are no such type of examples in the literature.

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1. Introduction and statement of the main results

We consider a polynomial differential system of the form

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q\left(x,y\right), \end{cases} \tag{1.1}$$

where P and Q are real polynomials in the variables x and y. The degree of the system (1.1) is the maximum of the degrees of the polynomials P and Q. As usual the dot denotes derivative with respect to the independent variable t.

A limit cycle of system (1.1) is an isolated periodic solution in the set of all periodic solutions of system (1.1). If a limit cycle is contained in the zero level set of a polynomial function, see for example, [[1], [4], [5], [9], [11]], then we say that it is algebraic, otherwise it is called non–algebraic see for example ([2], [4], [8], [10]). The topic of limit cycles is interesting both in mathematics and in science and many models from physics, engineering, chemistry, biology, economics,..., were displayed as differential systems with limit cycles.

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of form (1.1). We usually only ask for the number of such limit cycles, but their location as orbits of the system is also an interesting problem. And an even more difficult problem is to give an explicit expression of them.

In the chronological order the first examples where explicit non-algebraic limit cycles appeared are those of A. Gasull and all [8] and J. Gine and M. Grau [10] and by Al-Dosary, Khalil I. T.[2] for $n = 5$. In [6], an example of an explicit limit cycle which is not algebraic is given for $n = 3$. Bendjeddou in [3] provide a class of polynomial differential system of degree odd with explicit limit cycle non-algebraic.

In this paper, we consider the family of the polynomial differential system of the form

$$\begin{cases} \dot{x} = x\left(l+wx+vy\right)^{n+1} + n\left(vx^2 - vy^2 - 2ly - 2wxy\right)\left(x^2+y^2\right)^n \\ \qquad + x\left(l+wx+vy\right)\left(a\left(x^2+y^2\right) + 2c\left(x^2-y^2\right) - 4bxy\right)\left(x^2+y^2\right)^{n-1} \\ \dot{y} = y\left(l+wx+vy\right)^{n+1} + n\left(wx^2 - wy^2 + 2lx + 2vxy\right)\left(x^2+y^2\right)^n \\ \qquad + y\left(l+wx+vy\right)\left(a\left(x^2+y^2\right) + 2c\left(x^2-y^2\right) - 4bxy\right)\left(x^2+y^2\right)^{n-1}, \end{cases} \tag{1.2}$$

where a, b, c, w, v, n and l are real constants, n is strictly positive integer ($n \in \mathbb{N}^*$). We prove that these systems are Liouville integrable. Moreover, we determine sufficient conditions for a polynomial differential system (1.2) to possess an explicit non-algebraic limit cycle.

It remains the open question to determine if the polynomial differential systems of degree 2 can exhibit explicit non-algebraic limit cycles (this question is due to Benterki and Llibre [6]).

Thus, our main result is the following one.

Theorem 1.1. *Consider a multi-parameter polynomial differential system* (1.2)*. Then the following statements hold.*
(a) System (1.2) *is Darboux integrable with the Liouvillian first integral*

$$\begin{aligned} H\left(x,y\right) &= \left(\frac{x^2+y^2}{wx+vy+l}\right)^n e^{-\left(a\left(\arctan\frac{y}{x}\right)+\frac{bx^2+2cxy-by^2}{x^2+y^2}\right)} \\ &\quad - \int_0^{\arctan\frac{y}{x}} e^{-as-b\cos 2s-c\sin 2s}\,ds. \end{aligned}$$

(b) If $a < 0$, $w \geq 0$, $l > 0$ and $2a\pi + b \neq 0$ then system (1.2) *has an explicit non algebraic limit cycles, given in polar coordinates (r, θ) by*

$$r^*(\theta) = \frac{1}{2}\left(g\left(\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}} + \sqrt{\left(g\left(\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}}\right)^2 + 4l\rho^*\left(\theta\right)^{\frac{1}{n}}}\right),$$

where

$$\begin{aligned} g\left(\theta\right) &= w\cos\theta + v\sin\theta, \\ f\left(\theta\right) &= \int_0^\theta e^{-as-b\cos 2s-c\sin 2s}\,ds, \\ \rho^*\left(\theta\right) &= e^{a\theta+b\cos 2\theta+c\sin 2\theta}\left(\frac{e^{2\pi a}f\left(2\pi\right)}{1-e^{2\pi a}} + f\left(\theta\right)\right). \end{aligned}$$

Moreover, this limit cycle is hyperbolic.

2. Proof of Theorem 1.1

Firstly, we have

$$x\dot{y} - y\dot{x} = n\left(2l + wx + vy\right)\left(x^2 + y^2\right)^{n+1},$$

thus, the equilibrium points of system (1.2) are present in the equation curve's

$$\left(x^2 + y^2\right)^{n+1}\left(2l + wx + vy\right) = 0, \tag{2.1}$$

we deduce that the origin is an equilibrium point, and any other, if exists must lies on the straight line

$$(\Delta) : (2l + wx + vy) = 0.$$

Let $(x_0, y_0) \neq (0, 0)$ be such a point. Then form the remark above, x_0 and y_0 must satisfy

$$\begin{cases} x_0(-l)^{n+1} + n(vx_0^2 - wx_0y_0)(x_0^2 + y_0^2) + x_0(-l)(a(x_0^2 + y_0^2) \\ \quad +2c(x_0^2 - y_0^2) - 4bx_0y_0) = 0, \\ y_0(-l)^{n+1} + n(vx_0y_0 - wy_0^2)(x_0^2 + y_0^2) + y_0(-l)(a(x_0^2 + y_0^2) \\ \quad +2c(x_0^2 - y_0^2) - 4bx_0y_0) = 0, \\ vy_0 + wx_0 + 2l = 0, \end{cases}$$

$$\begin{cases} (-l)^{n+1} + n\left(vx_0 - wy_0\right)\left(x_0^2 + y_0^2\right) + (-l)\left(a\left(x_0^2 + y_0^2\right) + 2c\left(x_0^2 - y_0^2\right) - 4bx_0y_0\right) = 0, \\ y_0 = -\frac{1}{v}\left(2l + wx_0\right), \end{cases}$$

this system can be written as

$$\begin{aligned} &-l\left(av^3 + 2cv^3 - 6nw^3 + avw^2 + 4bv^2w - 2cvw^2 - 6nv^2w\right)x_0^2 \\ &-4l^2\left(2bv^2 - nv^2 - 3nw^2 + avw - 2cvw\right)x_0 + n\left(v^2 + w^2\right)^2 x_0^3 \\ &-\left(4al^3v - (-l)^{n+1}v^3 - 8cl^3v - 8l^3nw\right) = 0, \end{aligned} \tag{2.2}$$

then, the equilibrium points of system (1.2) are $\left\{(0,0), \left(x_0, -\frac{1}{v}\left(2l + wx_0\right)\right)\right\}$, where x_0 is a real root of the equation (2.2).

Note that, the origin of coordinates which is an unstable node because its eigenvalues are $l^{n+1} > 0$ with multiplicity two, for more details see for instance [[7], Theorem 2.15].

Proof of statement (a).

To prove our results (a) and (b) we write the polynomial differential system (1.2) in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$. Then the system (1.2) become

$$\begin{cases} \dot{r} = r\left(l + wr\cos\theta + vr\sin\theta\right)^{n+1} + l\left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)r^{2n+1} \\ \quad + \left(n\left(v\cos\theta - w\sin\theta\right) + \left(w\cos\theta + v\sin\theta\right)\left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)\right)r^{2n+2}, \\ \dot{\theta} = 2lnr^{2n} + n\left(v\sin\theta + w\cos\theta\right)r^{2n+1}. \end{cases}$$

Taking θ as an independent variable, we obtain the equation

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{\left(l + wr\cos\theta + vr\sin\theta\right)^{n+1} r + l\left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)r^{2n+1}}{2lnr^{2n} + n\left(v\sin\theta + w\cos\theta\right)r^{2n+1}} \\ &+ \frac{\left(n\left(v\cos\theta - w\sin\theta\right) + \left(w\cos\theta + v\sin\theta\right)\left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)\right)r^{2n+2}}{2lnr^{2n} + n\left(v\sin\theta + w\cos\theta\right)r^{2n+1}}. \end{aligned} \tag{2.3}$$

Via the change of variables

$$\rho = \frac{r^{2n}}{\left(\left(w\cos\theta + v\sin\theta\right)r + l\right)^{n}},$$

the equation (2.3) is transformed into the linear differential equation

$$\frac{d\rho}{d\theta} = \left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)\rho + 1. \tag{2.4}$$

The general solution of linear equation (2.4) is

$$\rho\left(\theta, k\right) = e^{a\theta + b\cos 2\theta + c\sin 2\theta}\left(k + \int_0^\theta e^{-as - b\cos 2s - c\sin 2s}ds\right), \tag{2.5}$$

where $k \in \mathbb{R}$. Going back through the changes of variables we obtain the first integral of the statement (a) of Theorem 1. Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and consequently system (1.2) is Darboux integrable.

Proof of statement (b) of Theorem 1.

In (2.5) let $\theta \to \rho(\theta, k^*)$ be the solution taking the value of $k^* \in \mathbb{R}$ for $\theta = 0$. To be a periodic solution, it must satisfy at first the condition

$$\rho\left(0, k^*\right) = \rho\left(2\pi, k^*\right),$$

providing the value of k^* is

$$k^* = \frac{e^{2\pi a} f\left(2\pi\right)}{1 - e^{2\pi a}} > 0,$$

because $a < 0$ and $f\left(\theta\right) = \displaystyle\int_0^\theta e^{-as - b\cos 2s - c\sin 2s}ds > 0$ for all $\in \mathbb{R}$.

After the substitution of the value k^* into $\rho\left(\theta, k\right)$ we obtain

$$\rho\left(\theta, k^*\right) = \rho^*\left(\theta\right) = e^{a\theta + b\cos 2\theta + c\sin 2\theta}\left(k^* + \int_0^\theta e^{-as - b\cos 2s - c\sin 2s}ds\right). \tag{2.6}$$

Note that, since

$$f\left(\theta\right) = \int_0^\theta e^{-as - b\cos 2s - c\sin 2s}ds > 0$$

for all $\in \mathbb{R}$ and $k^* > 0$, consequently,$\rho^*\left(\theta\right) > 0$ for all $\theta \in \mathbb{R}$.
Note that, since $\rho^*\left(\theta\right) > 0$ for all $\theta \in \mathbb{R}$, from the expression of the change of variable that transform (2.3) into (2.4), one gets a unique $r^*\left(\theta\right) > 0$ for all $\theta \in \mathbb{R}$ and it has the expression

$$r^*\left(\theta\right) = \frac{1}{2}\left(g\left(\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}} + \sqrt{\left(g\left(\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}}\right)^2 + 4l\rho^*\left(\theta\right)^{\frac{1}{n}}}\right). \tag{2.7}$$

Moreover, since $l > 0$ and $\rho^*\left(\theta\right) > 0$ for all $\theta \in \mathbb{R}$, then $r^*\left(\theta\right) > 0$, one can see that it is $2\pi-$periodic, since g and ρ^* are $2\pi-$ periodic.

In order to prove that the periodic orbit is hyperbolic limit cycles, we consider (2.6), and introduce the Poincaré return map

$$\lambda \mapsto \Pi(2\pi, \lambda) = \rho\left(\theta, \lambda\right).$$

Therefore, a limit cycle of system (1.2) is hyperbolic if and only if

$$\left.\frac{d\rho(2\pi, \lambda)}{d\lambda}\right|_{\lambda=k^*} \neq 1.$$

An easy computation shows that:

$$\left.\frac{d\rho(2\pi, \lambda)}{d\lambda}\right|_{\lambda=k^*} = \frac{d\rho^*\left(\theta\right)}{dk^*} = e^{2\pi a+b} \neq 1.$$

Therefore the limit cycle of the differential equation (2.4) is hyperbolic, for more details see [12]. Consequently 2.7 is hyperbolic limit cycle of the differential equation (2.3).

Clearly the curve $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ in the (x, y) plane with

$$\frac{r^{2n}}{\left(g\left(\theta\right)r + l\right)^n} - e^{a\theta+b\cos 2\theta+c\sin 2\theta}\left(\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}} + f\left(\theta\right)\right) = 0, \tag{2.8}$$

is not algebraic, due to the expression $\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}}e^{a\theta+b\cos 2\theta+c\sin 2\theta}$. More precisely, in Cartesian coordinates $r^2 = x^2+y^2$, $\theta = \arctan\frac{y}{x}$, the curve defined by this limit cycle is

$$\begin{aligned} F\left(x, y\right) &= \left(\frac{x^2+y^2}{wx+vy+l}\right)^n - e^{a\left(\arctan\frac{y}{x}\right)+\frac{bx^2+2cxy-by^2}{x^2+y^2}} \\ &\quad\times\left(\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}} + \int_0^{\arctan\frac{y}{x}} e^{-as-b\cos 2s-c\sin 2s}\,ds\right). \end{aligned}$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $F(x, y)$ in the variables x and y satisfies that there is a positive integer n such that $\frac{\partial^{(n)}F}{\partial x^n} = 0$, and this is not the case because in the derivative

$$\begin{aligned} \frac{d}{dx}F\left(x, y\right) &= n\frac{\left(wx^2+2vxy+2lx-wy^2\right)\left(\frac{x^2+y^2}{l+vy+wx}\right)^{n-1}}{\left(l+vy+wx\right)^2} \\ &\quad - \frac{y}{x^2+y^2}\left(\begin{array}{c} 1+\left(\frac{ax^2+ay^2+2cx^2-2cy^2-4bxy}{x^2+y^2}\right)e^{a\left(\arctan\frac{y}{x}\right)+\frac{bx^2+2cxy-by^2}{x^2+y^2}} \\ \times\left(\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}} + \int_0^{\arctan\frac{y}{x}} e^{-as-b\cos 2s-c\sin 2s}\,ds\right) \end{array}\right), \end{aligned}$$

it appears again the expression

$$e^{a\left(\arctan\frac{y}{x}\right)+\frac{bx^2+2cxy-by^2}{x^2+y^2}}\left(\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}} + \int_0^{\arctan\frac{y}{x}} e^{-as-b\cos 2s-c\sin 2s}\,ds\right),$$

which already appears in $F(x, y)$, and this expression will appear in the partial derivative at any order. This completes the proof of theorem.

3. Example

If we take $b = \frac{1}{2}, c = 0, a = -1, v = w = l = 1$, then system (1.2) reads

$$\begin{cases} \dot{x} = x\left(1+x+y\right)^{n+1} + n\left(x^2 - y^2 - 2y - 2xy\right)\left(x^2+y^2\right)^n \\ \qquad + x\left(1+x+y\right)\left(-\left(x^2+y^2\right) + -2xy\right)\left(x^2+y^2\right)^{n-1} \\ \dot{y} = y\left(1+x+y\right)^{n+1} + n\left(x^2 - y^2 + 2x + 2xy\right)\left(x^2+y^2\right)^n \\ \qquad + y\left(1+x+y\right)\left(-\left(x^2+y^2\right) + -2xy\right)\left(x^2+y^2\right)^{n-1}, \end{cases} \tag{3.1}$$

has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r^*\left(\theta\right) = \frac{1}{2}\left(\left(\cos\theta + \sin\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}} + \sqrt{\left(\cos\theta+\sin\theta\right)^2\rho^*\left(\theta\right)^{\frac{2}{n}} + 4l\rho^*\left(\theta\right)^{\frac{1}{n}}}\right),$$

where

$$\rho^*\left(\theta\right) = e^{-\theta+\frac{1}{2}\cos 2\theta}\left(\frac{e^{-2\pi}f(2\pi)}{1-e^{-2\pi}} + f\left(\theta\right)\right) \text{ and } f\left(\theta\right) = \int_0^\theta e^{s-\frac{1}{2}\cos 2s}\,ds.$$

For $n = 1$**:** The system (3.1) is a quartic system and that has a non algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r^*(\theta) = \frac{1}{2}\left(\left(\cos\theta+\sin\theta\right)\rho^*\left(\theta\right) + \sqrt{\left(\cos\theta+\sin\theta\right)^2\rho^*\left(\theta\right)^2 + 4l\rho^*\left(\theta\right)}\right).$$

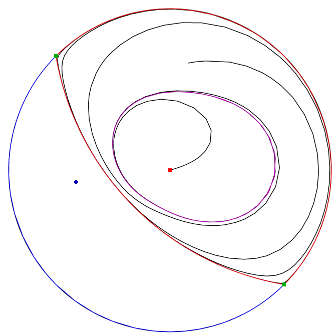

FIGURE 1. Limit cycle of system (3.1) for $n = 1$

For $n = 2$**:** The system (3.1) is of degree 6 and that has a non algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r^*(\theta) = \frac{1}{2}\left(\left(\cos\theta+\sin\theta\right)\rho^*\left(\theta\right)^{\frac{1}{2}} + \sqrt{\left(\left(\cos\theta+\sin\theta\right)\rho^*\left(\theta\right)^{\frac{1}{2}}\right)^2 + 4l\rho^*\left(\theta\right)^{\frac{1}{2}}}\right).$$

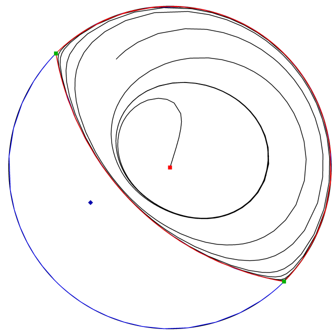

FIGURE 2. Limit cycle of system (3.1) for $n = 2$

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