

Existence and multiplicity of solutions to the Navier boundary value problem for a class of $(p(x), q(x))$ -biharmonic systems

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Abstract. In this article, we study the following problem with Navier boundary conditions.

$$\begin{cases} \Delta(a(x, \Delta u)) = F_u(x, u, v), & \text{in } \Omega \\ \Delta(a(x, \Delta v)) = F_v(x, u, v), & \text{in } \Omega, \\ u = v = \Delta u = \Delta v = 0 & \text{on } \partial\Omega. \end{cases}$$

By using the Mountain Pass Theorem and the Fountain Theorem, we establish the existence of weak solutions of this problem.

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1. Introduction

In recent years, the study of differential equations and variational problems with $p(x)$ -growth conditions was an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. In that context we refer the reader to Ruzicka [15], Zhikov [20] and the reference therein; see also [4, 7, 8, 5].

Fourth-order equations appears in many context. Some of theses problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells (see [10]). In addition, this type of equations can describe the static from change of beam or the sport of rigid body.

In [1] the authors studied a class of $p(x)$ -biharmonic of the form

$$\begin{aligned} \Delta(|\Delta u|^{p(x)-2} \Delta u) &= \lambda |u|^{q(x)-2} u \quad \text{in } \Omega, \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in \mathbb{R}^N , with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \geq 0$.

In [3], A. El Amrouss and A. Ourraoui considered the below problem and using variational methods, by the assumptions on the Carathéodory function f , they establish the existence of Three solutions the problem of the form

$$\begin{aligned} \Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)|u|^{p(x)-2}u &= f(x, u) + \lambda g(x, u) \quad \text{in } \Omega, \\ Bu = Tu = 0 &\quad \text{on } \partial\Omega. \end{aligned}$$

Inspired by the above references, the work of L. Li [11]and [14], the aim of this article is to study the existence and multiplicity of weak solutions for $(p(x), q(x))$ -biharmonic type system

$$\begin{cases} \Delta(a(x, \Delta u)) = F_u(x, u, v), & \text{in } \Omega \\ \Delta(a(x, \Delta v)) = F_v(x, u, v), & \text{in } \Omega, \\ u = \Delta u = 0, v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$, $N \geq 1$,

$$\Delta_{p(x)}^2 u := \Delta(|\Delta u|^{p(x)-2}\Delta u),$$

is the $p(x)$ -biharmonic operator, p, q are continuous functions on $\bar{\Omega}$ with

$$\inf_{x \in \bar{\Omega}} p(x) > \max \left\{ 1, \frac{N}{2} \right\}, \quad \inf_{x \in \bar{\Omega}} q(x) > \max \left\{ 1, \frac{N}{2} \right\}$$

and $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function such that $F(., s, t)$ is continuous in $\bar{\Omega}$, for all $(s, t) \in \mathbb{R}^2$, $F(x, ., .)$ is C^1 in \mathbb{R}^2 for every $x \in \Omega$, and F_u, F_v denote the partial derivative of F , with respect to u, v respectively such that

(F₁) For all $(x, s, t) \in \Omega \times \mathbb{R}^2$, we assume

$$\lim_{|s| \rightarrow 0} \frac{F_s(x, s, t)}{|s|^{p(x)-1}} = 0, \quad \lim_{|t| \rightarrow 0} \frac{F_t(x, s, t)}{|s|^{q(x)-1}} = 0.$$

(F₂) For all $(x, s, t) \in \Omega \times \mathbb{R}^2$, we assume

$$F(x, s, t) = o(|s|^{p(x)-1} + |t|^{q(x)-1}) \text{ as } |(s, t)| \rightarrow \infty.$$

(F₃) There exists $\underline{u} > 0, \underline{v} > 0$ such that $F(x, \underline{u}, \underline{v}) > 0$ for a.e $x \in \Omega$

(F₄) There exist $\lambda > 0$ such that $F(x, s, t) \geq \lambda(|s|^{\alpha(x)} - |t|^{\beta(x)})$ for all $(s, t) \in \mathbb{R}^2$, with

$$\alpha^- > r^+, \quad 1 < \beta^- \leq \beta^+ < r^-.$$

(F₅) For all $(x, s, t) \in \Omega \times \mathbb{R}^2$ $F(x, -s, -t) = -F(x, s, t)$.

Let $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ to be a continuous potential derivative with respect to ξ of the mapping $A : \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ where $a = DA = A'$, with the assumption as below

(A₁) $A(x, 0) = 0$, for all $x \in \Omega$.

(A₂) $a(x, \xi) \leq C_1(1 + |\xi|^{r(x)-1})$, $C_1 > 0$ and $r^- > p^+, r^- > q^+$.

(A₃) A is $r(x)$ -uniformly convex: there exists a constant $k > 0$ such that

$$A\left(x, \frac{\xi + \eta}{2}\right) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \eta) - k|\xi - \eta|^{r(x)},$$

for all $x \in \Omega, \xi, \eta \in \mathbb{R}^N$.

(A₄) A is $r(x)$ -subhomogeneous, for all $(x, \xi) \in \Omega \times \mathbb{R}^N$,

$$|\xi|^{r(x)} \leq a(x, \xi) \leq r(x)A(x, \xi).$$

(A₅) For all $(x, s) \in \Omega \times \mathbb{R}^N$ $a(x, -s) = -a(x, s)$.

The main results of this paper are the following theorems.

Theorem 1.1. *Assume that (A₁) – (A₄) and (F₁) – (F₃) hold. Then the problem (1.1) has two weak solutions.*

Theorem 1.2. *Assume that (A₁) – (A₅) and (F₁) – (F₅) hold. Then the problem (1.1) has a sequence of weak solutions such that $\phi(\pm(u_k, v_k)) \rightarrow +\infty$, as $k \rightarrow +\infty$ with ϕ is a energy associated of the problem (1.1) defined in (2.2).*

This paper is organized as three sections. In Section 2, we recall some basic properties of the variable exponent Lebesgue-Sobolev spaces. In Section3 we give the proof of main results.

2. Preliminaries

To study $p(x)$ -Laplacian problems, we need some results on the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, and properties of $p(x)$ -Laplacian, which we use later. Let Ω be a bounded domain of \mathbb{R}^N , denote

$$C_+(\overline{\Omega}) = \{h(x); h(x) \in C(\overline{\Omega}), h(x) > 1, \forall x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max\{h(x); x \in \overline{\Omega}\}, \quad h^- = \min\{h(x); x \in \overline{\Omega}\};$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{ u; u \text{ is a measurable real-valued function such that } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the so-called *Luxemburg norm*

$$\|u\|_{p(x)} = \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Then $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ becomes a Banach space.

Proposition 2.1 ([9]). *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \leq 2 |u|_{p(x)} |v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^{\alpha}u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where

$$D^{\alpha}u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} u,$$

with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^{\alpha}u|_{p(x)},$$

also becomes a separable and reflexive Banach space. For more details, we refer the reader to [6, 9, 13]. Denote

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N \end{cases}$$

for any $x \in \bar{\Omega}$, $k \geq 1$.

Proposition 2.2 ([9]). *For $p, r \in C_+(\bar{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \bar{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace \leq with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$. Then the function space $\left(\left(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \right), \|u\|_{p(x)} \right)$ is a separable and reflexive Banach space, where

$$\|u\|_{p(x)} = \inf \left\{ \mu > 0 : \int_{\Omega} \left(\left| \frac{\Delta u(x)}{\mu} \right|^{p(x)} \leq 1 \right) \right\}.$$

Remark 2.3. According to [[18] Theorem 4.4.], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $\|\cdot\|_{p(x)}$ in the space X . Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $\|\cdot\|_{p(x)}$ are equivalent.

Proposition 2.4 ([2]). *If we denote $\rho(u) = \int_{\Omega} |\Delta u|^{p(x)} dx$, then for $u, u_n \in X$, we have*
 (1) $\|u\|_p < 1$ (respectively $=1; > 1$) $\iff \rho(u) < 1$ (respectively $= 1; > 1$);

- (2) $\|u\|_p \leq 1 \Rightarrow \|u\|_p^{p^+} \leq \rho(u) \leq \|u\|_p^{p^-}$;
- (3) $\|u\|_p \geq 1 \Rightarrow \|u\|_p^{p^-} \leq \rho(u) \leq \|u\|_p^{p^+}$;
- (4) $\|u_n\|_p \rightarrow 0$ (respectively $\rightarrow \infty$) $\iff \rho(u_n) \rightarrow 0$ (respectively $\rightarrow \infty$).

Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space

$$X = \left(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega) \right) \times \left(W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega) \right)$$

equipped with the norm

$$\|(u, v)\| = \max\{\|u\|_{p(x)}, \|u\|_{q(x)}\}.$$

Remark 2.5 (see [19]). As the Sobolev space X is a reflexive and separable Banach space, there exist $(e_n)_{n \in \mathbb{N}^*} \subseteq X$ and $(f_n)_{n \in \mathbb{N}^*} \subseteq X^*$ such that $f_n(e_l) = \delta_{nl}$ for any $n, l \in \mathbb{N}^*$ and

$$X = \overline{\text{span}\{e_n : n \in \mathbb{N}^*\}}, \quad X^* = \overline{\text{span}\{f_n : n \in \mathbb{N}^*\}}^{w^*}.$$

For $k \in \mathbb{N}^*$, denote by

$$X_k = \text{span}\{e_k\}, \quad Y_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \overline{\bigoplus_k X_j}.$$

For every $m > 1$, $u, v \in L^m(\Omega)$, we define

$$|(u, v)|_m := \max\{|u|_m, |v|_m\}.$$

Lemma 2.6 (See [8]). *Define*

$$\beta_k := \sup\{|(u, v)|_m; \|(u, v)\| = 1, (u, v) \in Z_k\},$$

where $m := \max_{x \in \overline{\Omega}}(p(x), q(x))$. Then, we have

$$\lim_{k \rightarrow \infty} \beta_k = 0.$$

2.1. Existence and multiplicity of weak solutions

Definition 2.7. We say that $(u, v) \in X$ is weak solution of (1.1) if

$$\int_{\Omega} a(x, \Delta u) \Delta \varphi dx + \int_{\Omega} a(x, \Delta v) \Delta \varphi dx = \int_{\Omega} F_u(x, u, v) \varphi dx + \int_{\Omega} F_v(x, u, v) \varphi dx, \tag{2.1}$$

for all $\varphi \in X$.

The functional associated to (1.1) is given by

$$\phi(u, v) = \int_{\Omega} A(x, \Delta u) dx + \int_{\Omega} A(x, \Delta v) dx - \int_{\Omega} F(x, u, v) dx. \tag{2.2}$$

It should be noticed that under the condition $(F_1) - (F_2)$ the functional ϕ is of class $C^1(X, \mathbb{R})$ and

$$\begin{aligned} \phi'(u, v)(\psi, \varphi) &= \int_{\Omega} a(x, \Delta u) \Delta \psi dx + \int_{\Omega} a(x, \Delta v) \Delta \varphi dx \\ &\quad - \int_{\Omega} F_u(x, u, v) \psi dx - \int_{\Omega} F_v(x, u, v) \varphi dx, \quad \forall (\psi, \varphi) \in X. \end{aligned} \tag{2.3}$$

Then, we know that the weak solution of (1.1) corresponds to critical point of the functional ϕ .

Definition 2.8. We say that

- (1) The C^1 -functional ϕ satisfies the Palais-Smale condition (in short (PS) condition) if any sequence $(u_n)_{n \in \mathbb{N}} \subseteq X$ for which, $(\phi(u_n))_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded and $\phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$, has a convergent subsequence.
- (2) The C^1 -functional ϕ satisfies the Palais-Smale condition at the level c (in short $(PS)_c$ condition) for $c \in \mathbb{R}$ if any sequence $(u_n)_{n \in \mathbb{N}} \subseteq X$ for which, $\phi(u_n) \rightarrow c$ and $\phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$, has a convergent subsequence.
- (3) The C^1 -functional ϕ satisfies the $(PS)_c^*$ condition for $c \in \mathbb{R}$ if any sequence $(u_n)_{n \in \mathbb{N}} \subseteq X$ for which, $u_n \in Y_n$ for each $n \in \mathbb{N}$, $\phi(u_n) \rightarrow c$ and $\phi'_{|Y_n}(u_n) \rightarrow 0$ as $n \rightarrow \infty$ with $Y_n, n \in \mathbb{N}$ as defined in Remark 2.5, has a subsequence convergent to a critical point of ϕ .

Remark 2.9. It is easy to see that if ϕ satisfies the (PS) condition, then ϕ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.

Proof of Theorem 1.1. To prove Theorem 1.1, we shall use the Mountain Pass theorem [16]. We first start with the following lemmas.

Lemma 2.10. *Under the assumptions (F_1) - (F_3) and (A_1) - (A_3) ϕ is sequentially weakly lower semi continuous and coercive .*

Proof. By (F_1) - (F_2) , we see that

$$|F(x, s, t)| \leq C_3(1 + |s|^{p(x)} + |t|^{q(x)}), \quad \forall (s, t) \in \mathbb{R}^2. \tag{2.4}$$

By the compact embeddings

$$X \hookrightarrow L^{p(x)}(\Omega), \quad X \hookrightarrow L^{q(x)}(\Omega),$$

we deduce that $w \mapsto \int_{\Omega} F(x, w)dx$ is sequentially lower semi continuous $\forall w \in \mathbb{R}^2$.

Since

$$w \mapsto \int_{\Omega} A(x, \Delta u)dx + \int_{\Omega} A(x, \Delta v)dx$$

is convex uniformly, so it is sequentially lower semi continuous.

Now we prove that ϕ is coercive. From (F_2) for ε small enough, there exist $\delta > 0$ such that

$$|F(x, s, t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}), \quad \text{for } |(s, t)| > \delta,$$

and thus we have

$$|F(x, s, t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}) + \max_{|(s,t)| \leq \delta} |F(x, s, t)| |(s, t)|, \quad \forall (s, t) \in \mathbb{R}^2,$$

for a.e $x \in \Omega$. Consequently, for $\|(u, v)\| > 1$ we obtain

$$\begin{aligned} \phi(u, v) &\geq \int_{\Omega} A(x, \Delta u) dx + \int_{\Omega} A(x, \Delta v) dx \\ &- \varepsilon \int_{\Omega} |u|^{p(x)} dx - \varepsilon \int_{\Omega} |v|^{q(x)} dx - \max_{|(u,v)| \leq \delta} |F(x, u, v)| \int_{\Omega} |(u, v)| dx \\ &\geq \int_{\Omega} \frac{1}{r(x)} |\Delta u|^{r(x)} dx + \int_{\Omega} \frac{1}{r(x)} |\Delta v|^{r(x)} dx \\ &- C\varepsilon \int_{\Omega} |u|^{p(x)} dx - C\varepsilon \int_{\Omega} |v|^{q(x)} dx - \max_{|(u,v)| \leq \delta} |F(x, u, v)| \int_{\Omega} |(u, v)| dx \\ &\geq \frac{1}{r^+} \max \left(\|u\|_{r(x)}^-, \|v\|_{r(x)}^- \right) - 2C\varepsilon \max \left(\|u\|_{p(x)}^+, \|v\|_{q(x)}^+ \right) \\ &- C\varepsilon |\Omega| \max_{|(u,v)| \leq \delta} |F(x, u, v)| \max \left(\|u\|_{p(x)}^+, \|v\|_{q(x)}^+ \right). \end{aligned}$$

Therefore, ϕ is coercive and has a global minimizer (\bar{u}_1, \bar{v}_1) which is a nontrivial because by (F_3)

$$\phi(\bar{u}_1, \bar{v}_1) \leq \phi(\underline{u}, \underline{v}) < 0.$$

Lemma 2.11. *Under the assumptions (F_1) - (F_3) and (A_1) - (A_4) . Then ϕ satisfies the Palais-smale condition.*

Proof. Let $w_n = (u_n, v_n) \subset X$ be a Palais-smale sequence, then

$$\phi'(w_n) \rightarrow 0 \text{ in } X^*, \quad \phi(w_n) \rightarrow l \in \mathbb{R}.$$

We show that (w_n) is bounded. By (A_5) we have

$$\begin{aligned} \phi(w_n) &= \int_{\Omega} A(x, \Delta u_n) dx + \int_{\Omega} A(x, \Delta v_n) dx - \int_{\Omega} F(x, u_n, v_n) dx \\ &\geq \int_{\Omega} \frac{1}{r(x)} |\Delta u_n|^{r(x)} dx + \int_{\Omega} \frac{1}{r(x)} |\Delta v_n|^{r(x)} dx - \int_{\Omega} F(x, u_n, v_n) dx, \end{aligned}$$

and we get

$$\begin{aligned} &\phi'(u_n, v_n)(u_n, v_n) \\ &= \int_{\Omega} a(x, \Delta u_n) \Delta u_n dx + \int_{\Omega} a(x, \Delta v_n) \Delta v_n dx \\ &- \int_{\Omega} F_{u_n}(x, u_n, v_n) u_n dx - \int_{\Omega} F_{v_n}(x, u_n, v_n) v_n dx \\ &\leq \int_{\Omega} r(x) A(x, \Delta u_n) dx + \int_{\Omega} r(x) A(x, \Delta v_n) dx \\ &- \int_{\Omega} F_{u_n}(x, u_n, v_n) u_n dx - \int_{\Omega} F_{v_n}(x, u_n, v_n) v_n dx. \end{aligned}$$

Using the fact that $F_s, F_t \in C(\Omega \times \mathbb{R}^2, \mathbb{R})$ and with $(F_1) - (F_2)$, for $\varepsilon > 0$ there exists $\delta > 0$ and $\eta > 0$ such that

$$|F_s(x, s, t)| \leq \varepsilon |s|^{p(x)-1}, \quad |F_t(x, s, t)| \leq \varepsilon |t|^{q(x)-1},$$

and

$$|F(x, s, t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}),$$

for all $|s, t| \leq \delta$, and for all $|s, t| \geq \eta$.

Then we have

$$|F_s(x, s, t)s| \leq \varepsilon|s|^{p(x)}, \quad |F_t(x, s, t)t| \leq \varepsilon|t|^{q(x)}, \tag{2.5}$$

and

$$|F(x, s, t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}),$$

for all $|s, t| \leq \delta$, and for all $|s, t| \geq \eta$.

It yields,

$$\begin{aligned} & -\frac{1}{2r^+}\phi'(u_n, v_n)(u_n, v_n) \\ \geq & -\frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta u_n)dx - \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta v_n)dx \\ & + \frac{1}{2r^+} \left[\int_{\Omega} F_{u_n}(x, u_n, v_n)u_n dx + \int_{\Omega} F_{v_n}(x, u_n, v_n)v_n dx \right] \\ \geq & -\frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta u_n)dx - \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta v_n)dx \\ & + \frac{1}{2r^+} \left[\int_{\Omega} F_{u_n}(x, u_n, v_n)u_n dx + \int_{\Omega} F_{v_n}(x, u_n, v_n)v_n dx \right]. \end{aligned}$$

Thus,

$$\begin{aligned} & \phi(u_n, v_n) - \frac{1}{2r^+}\phi'(u_n, v_n)(u_n, v_n) \\ \geq & \int_{\Omega} A(x, \Delta u_n)dx + \int_{\Omega} A(x, \Delta v_n)dx - \int_{\Omega} F(x, u_n, v_n)dx \\ & - \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta u_n)dx - \frac{1}{2r^+} \int_{\Omega} r(x)A(x, \Delta v_n)dx \\ & - \int_{\Omega} F(x, u_n, v_n)dx + \frac{1}{2r^+} \left[\int_{\Omega} F_{u_n}(x, u_n, v_n)u_n dx + \int_{\Omega} F_{v_n}(x, u_n, v_n)v_n dx \right] \\ \geq & \frac{1}{2} \left[\int_{\Omega} |\Delta u_n|^{r(x)}dx + \int_{\Omega} |\Delta v_n|^{r(x)}dx \right] - \int_{\Omega} F(x, u_n, v_n)dx \\ & + \frac{1}{2r^+} \left[\int_{\Omega} F_{u_n}(x, u_n, v_n)u_n dx + \int_{\Omega} F_{v_n}(x, u_n, v_n)v_n dx \right] \\ \geq & \frac{1}{2} \max \left(\|u_n\|_{r(x)}^+, \|v_n\|_{r(x)}^+ \right) - (C\varepsilon + \varepsilon) \int_{\Omega} |u_n|^{p(x)}dx - (C\varepsilon + \varepsilon) \int_{\Omega} |v_n|^{q(x)}dx. \end{aligned}$$

Since $r^- > p^+ > 1$, $r^- > q^+ > 1$, by the compact embeddings

$$X \hookrightarrow L^{p(x)}(\Omega), \quad X \hookrightarrow L^{q(x)}(\Omega),$$

we deduce

$$\begin{aligned} & \phi(u_n, v_n) - \frac{1}{2r^+} \phi'(u_n, v_n)(u_n, v_n) \\ & \geq \frac{1}{2} \max \left(\|u_n\|_{r(x)}^{r^+}, \|v_n\|_{r(x)}^{r^+} \right) - 2(C' \varepsilon + \varepsilon) \|(u_n, v_n)\| \\ & \geq \left[\frac{1}{2} - 2(C' \varepsilon + \varepsilon) \right] \|(u_n, v_n)\|, \end{aligned}$$

where C' is positive constant.

For ε small enough with $R = \frac{1}{2} - 2(C' \varepsilon + \varepsilon) > 0$, we get

$$\|(u_n, v_n)\| \leq \frac{1}{R} \left(\phi(u_n, v_n) - \frac{1}{2r^+} \phi'(u_n, v_n)(u_n, v_n) \right).$$

Since $\phi(u_n, v_n)$ is bounded and $\phi'(u_n, v_n)(u_n, v_n) \rightarrow 0$ as $n \rightarrow \infty$, then (u_n, v_n) is bounded in X , passing to a subsequence, so $(u_n, v_n) \rightharpoonup (u, v)$ in X and $(u_n, v_n) \rightarrow L^{p(x)}(\Omega) \times L^{q(x)}(\Omega)$. We show that $(u_n, v_n) \rightarrow (u, v)$ in X .

$$\begin{aligned} & \phi'(u_n, v_n) ((u_n, v_n) - (u, v)) \\ & = \int_{\Omega} a(x, \Delta u_n) \Delta(u_n - u) dx + \int_{\Omega} a(x, \Delta v_n) \Delta(v_n - v) dx \\ & - \int_{\Omega} F_{u_n}(x, u_n, v_n)(u_n - u) dx - \int_{\Omega} F_{v_n}(x, u_n, v_n)(v_n - v) dx. \end{aligned}$$

Since

$$\begin{aligned} & \left| \int_{\Omega} a(x, \Delta u_n) \Delta(u_n - u) dx + \int_{\Omega} a(x, \Delta v_n) \Delta(v_n - v) dx \right| \\ & = |\phi'(u_n, v_n) ((u_n, v_n) - (u, v)) + \int_{\Omega} F_{u_n}(x, u_n, v_n)(u_n - u) dx \\ & + \int_{\Omega} F_{v_n}(x, u_n, v_n)(v_n - v) dx| \\ & \leq \|\phi'(u_n, v_n)\|_{X^*} \|(u_n, v_n) - (u, v)\| \\ & + \int_{\Omega} |F_{u_n}(x, u_n, v_n)| |u_n - u| dx + \int_{\Omega} |F_{v_n}(x, u_n, v_n)| |v_n - v| dx. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} & \int_{\Omega} |F_{u_n}(x, u_n, v_n)| |u_n - u| dx + \int_{\Omega} |F_{v_n}(x, u_n, v_n)| |v_n - v| dx \\ & \leq \varepsilon \int_{\Omega} \left(|u_n - u|^{p(x)} + |v_n - v|^{q(x)} \right) dx, \end{aligned}$$

we get

$$\limsup_{n \rightarrow +\infty} \left(\int_{\Omega} a(x, \Delta u_n) \Delta(u_n - u) dx + \int_{\Omega} a(x, \Delta v_n) \Delta(v_n - v) dx \right) \leq 0.$$

Since $a(x, \xi)$ is of (S_+) type, we see that $(u_n, v_n) \rightarrow (u, v)$ in X .

Now, we verified the conditions of Mountain Pass Theorem. By Hölder’s inequality, from (F_1) there exists $\delta > 0$ such that

$$\begin{aligned} |F(x, u, v)| &\leq \left| \int_0^u F_s(x, s, v)dx + \int_0^v F_t(x, 0, t)dx + F(x, 0, 0) \right| \\ &\leq \varepsilon \left| \int_0^u |s|^{p(x)-1}dx + \int_0^v |t|^{q(x)-1}dx \right| + |F(x, 0, 0)| \\ &\leq \varepsilon(|u|^{p(x)} + |v|^{q(x)}) + M, \end{aligned}$$

for all $|u, v| \leq \delta$, with $M := \max_{x \in \Omega} F(x, 0, 0)$ and by (F_2) , there exists $M(\delta) > 0$ such that

$$|F(x, u, v)| \leq M(\delta)(|u|^{p(x)} + |v|^{q(x)}), \text{ for } |(u, v)| > \delta.$$

Therefore, for $\|(u, v)\| = \varrho$ small enough, we have

$$\begin{aligned} \phi(u, v) &\geq \int_{\Omega} A(x, \Delta u)dx + \int_{\Omega} A(x, \Delta v)dx - \varepsilon \int_{|(u,v)| < \delta} (|u|^{p(x)} + |v|^{q(x)}) dx \\ &\quad - M(\delta) \int_{|(u,v)| > \delta} (|u|^{p(x)} + |v|^{q(x)}) - M \text{meas}\{|(u, v)| < \delta\} \\ &\geq \frac{1}{r^+} \max \left(\|u\|_{r(x)}^{r^+}, \|v\|_{r(x)}^{r^+} \right) \\ &\quad - \min(\varepsilon C, M(\delta)C') \max \left(\|u\|_{p(x)}^{p^-}, \|v\|_{q(x)}^{q^-} \right) - M \text{meas}\{|(u, v)| < \delta\} \\ &= g(\varrho). \end{aligned}$$

There exists $\theta > 0$ such that $g(\varrho) > \theta > 0$. Since $\phi(0, 0) = 0$, we conclude that ϕ satisfies the conditions of Mountain Pass Theorem. Then there exists (\bar{u}_2, \bar{v}_2) such that $\phi'(\bar{u}_2, \bar{v}_2) = 0$.

Proof of Theorem 1.2. To prove Theorem 1.2, above, will be based on a variational approach, using the critical points theory, we shall prove that the C^1 -functional ϕ has a sequence of critical values. The main tools for this end are “Fountain theorem” (see Willem [16, Theorem 6.5]) which we give below.

Theorem 2.12 (“Fountain theorem”, [16]). *Let X be a reflexive and separable Banach space, $\phi \in C^1(X, \mathbb{R})$ be an even functional and the subspaces X_k, Y_k, Z_k as defined in remark 2.5. If for each $k \in \mathbb{N}^*$ there exist $\rho_k > r_k > 0$ such that*

- (1) $\inf_{x \in Z_k, \|x\| = r_k} \phi(x) \rightarrow \infty$ as $k \rightarrow \infty$,
- (2) $\max_{x \in Y_k, \|x\| = \rho_k} \phi(x) \leq 0$,
- (3) I satisfies the $(PS)_c$ condition for every $c > 0$.

Then I has a sequence of critical values tending to $+\infty$.

According to Lemma 2.6, (F_5) and (A_5) , $\Phi \in C^1(X, \mathbb{R})$ is an even functional. We will prove that if k is large enough, then there exist $\rho_k > \nu_k > 0$ such that

$$b_k := \inf \{ \Phi(u) / u \in Z_k, \|u\| = \nu_k \} \rightarrow +\infty \text{ as } k \rightarrow +\infty; \tag{2.6}$$

$$a_k := \max \{ \Phi(u) / u \in Y_k, \|u\| = \rho_k \} \rightarrow 0 \text{ as } k \rightarrow +\infty. \tag{2.7}$$

For any $(u, v) \in Z_k$, $\|v\|_{q(x)} > 1$, $\|u\|_{p(x)} > 1$ and $\|(u, v)\| = \eta_k$, (η_k will be specified later), by (2.4) we have

$$\begin{aligned} \phi(u, v) &= \int_{\Omega} A(x, \Delta u) dx + \int_{\Omega} A(x, \Delta v) dx - \int_{\Omega} F(x, u, v) dx \\ &\geq \frac{1}{r^+} \max \left(\|u\|_{r(x)}^-, \|v\|_{r(x)}^- \right) - \int_{\Omega} C_3(1 + |u|^{p(x)} + |v|^{q(x)}) dx \\ &\geq \frac{1}{r^+} \max \left(\|u\|_{r(x)}^-, \|v\|_{r(x)}^- \right) - C_3 \int_{\Omega} dx - C_3 \int_{\Omega} |u|^{p(x)} dx - C_3 \int_{\Omega} |v|^{q(x)} dx \\ &\geq \frac{1}{r^+} \|(u, v)\|^{r^-} - C_3(\beta_k \|(u, v)\|)^{p^+} - C_3(\beta_k \|(u, v)\|)^{q^+} - C_3|\Omega| \\ &\geq \frac{1}{r^+} \|(u, v)\|^{r^-} - C_4\beta_k \|(u, v)\|^m - C_3|\Omega|, \end{aligned}$$

where m is defined in Lemma 2.6. We fix

$$\eta_k = \left(\frac{1}{r^+ C_4 \beta_k^b} \right)^{\frac{1}{m-r^-}} \rightarrow +\infty \text{ as } k \rightarrow +\infty.$$

Consequently

$$\phi(u, v) \geq \eta_k \left[\frac{1}{r^+} \eta_k^{r^- - 1} - C_4 \beta_k^b \eta_k^{m-1} \right] - C_3|\Omega|.$$

Then,

$$\phi(u, v) \rightarrow +\infty \text{ as } k \rightarrow +\infty.$$

Proof of (2.7). From (F_4) , there exists $\lambda > 0$ such that

$$F(x, s, t) \geq \lambda(|s|^{\alpha(x)} - |t|^{\beta(x)}),$$

with $\alpha^- > r^+, \beta^+ < r^-$.

Therefore, by Lemma 2.1 [12] and Lemma 3.1 [17], for any $\omega := (u, v) \in Y_k$ with $\|\omega\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned} \phi(t\omega) &= \int_{\Omega} A(x, t\Delta u) dx + \int_{\Omega} A(x, t\Delta v) dx - \int_{\Omega} F(x, t\omega) dx \\ &\leq \int_{\Omega} t^{r(x)} A(x, \Delta u) dx + \int_{\Omega} t^{r(x)} A(x, \Delta v) dx \\ &\quad - \lambda \int_{\Omega} |tu|^{\alpha(x)} dx + \lambda \int_{\Omega} |tv|^{\beta(x)} dx \\ &\leq t^{r^+} \left[\int_{\Omega} A(x, \Delta u) dx + \int_{\Omega} A(x, \Delta v) dx \right] \\ &\quad - \lambda t^{\alpha^-} \int_{\Omega} |u|^{\alpha(x)} dx + \lambda t^{\beta^-} \int_{\Omega} |v|^{\beta(x)} dx. \end{aligned}$$

By $\alpha^- > r^+ > \beta^-$ and $\dim Y_k < \infty$, we conclude that $\phi(tu, tv) \rightarrow -\infty$ as $\|t\omega\| \rightarrow +\infty$ for $\omega \in Y_k$. By applying the fountain Theorem, we achieved the proof of Theorem 1.2.

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