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Existence and multiplicity of positive radial solutions to the Dirichlet problem for nonlinear elliptic equations on annular domains

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Abstract. In this paper, we study the existence and nonexistence of monotone positive radial solutions of elliptic boundary value problems on bounded annular domains subject to local boundary condition. By using Krasnoselskii's fixed point theorem of cone expansion-compression type we show that there exists $\lambda^* \geq \lambda_* > 0$ such that the elliptic equation has at least two, one and no radial positive solutions for $0 < \lambda \leq \lambda_*$, $\lambda_* < \lambda \leq \lambda^*$ and $\lambda > \lambda^*$ respectively. We include an example to illustrate our results.

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1. Introduction

In this paper, we are interested in the existence of radial positive solutions to the following, boundary value problem BVP

$$\begin{cases} -\triangle u(x) = \lambda f(|x|, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = \left\{x \in \mathbb{R}^N : a < |x| < b\right\}$ with $1 < a < b$ is an annulus in $\mathbb{R}^N\ (N \geq 3)$, $f \in C([a,b] \times [0,\infty), [0,\infty))$ and λ is a positive parameter.

The study of such problems is motivated by a lot of physical applications starting from the well-known Poisson-Boltzmann equation (see [2, 20, 30]), also they serve as models for some phenomena which arise in fluid mechanics, such as the exothermic chemical reactions or autocatalytic reactions (see [27, Section 5.11.1]). The nonlinearity f in applications always has a special form and here we assume only the continuity of f and some inequalities at some points for the values of this function. However, we know that in the integrand should stay a superposition of u with a given function (usually the exponent of u in applications) instead of u alone, but we treat this paper as the first step in this direction. The method we use is typical for local boundary value problems. We shall formulate an equivalent fixed point problem and look for its solution in the cone of nonnegative function in an appropriate Banach space. The most popular fixed point theorem in a cone is the cone-compression and cone-expansion theorem due to M. Krasnosel'skii [19] which we use in the form taken from [16]. We also point out the fact that problems of type (1.1) when equation does not contain parameter λ, are connected with the classical boundary value theory of Bernstein [1] (see also the studies of Granas, Gunther and Lee [15] for some extensions to nonlinear problems).

The existence and uniqueness of positive radial solutions for equations of type (1.1) when equation does not contain paramete λ, were obtained in [5], [21], [32]. Wang [32] proved that if $f:(0,\infty)\to(0,\infty)$ satisfies $\lim\limits_{z\to 0}\frac{f(z)}{z}=\infty$ and $\lim\limits_{z\to\infty}\frac{f(z)}{z}=0$ then problem (1.1) when equation does not contain paramete λ, has a positive radial solution in $\Omega=\left\{x\in\mathbb{R}^N,\ N>2\right\}$. That result was extended for the systems of elliptic equations by Ma [23]. We quote also the research of Ovono and Rougirel [28] where the diffusion at each point depends on all the values of the solutions in a neighborhood of this point and Chipot et al. [11], [12]. For example in [11] considered the solvability of a class of nonlocal problems which admit a formulation in term of quasi-variational inequalities. There is a wide literature that deals with existence multiplicity results for various second-order, fourth-order and higher-order boundary value problems by different approaches, see [5], [8], [6], [7], [10], [17], [25], [22].

In 2011, Bohneure et al. [4] Studied the existence of positive increasing radial solutions for superlinear Neumann problem in the unit ball B in $\mathbb{R}^N,\ N\geq 2$,

$$\begin{cases} -\Delta u+u=a\left(|x|\right)f\left(u\right), & \text{in} \quad B,\\ u>0, & \text{in} \quad B,\\ \partial_t u=0, & \text{on} \quad \partial B,\end{cases}$$

where $a\in C^1\left([0,1],\mathbb{R}\right),\ a\left(0\right)>0$ is nondecreasing, $f\in C^1\left([0,1],\mathbb{R}\right),\ f\left(0\right)=0$,

$$\lim_{s\to 0^+}\frac{f\left(s\right)}{s}=0 \text{ and } \lim_{s\to+\infty}\frac{f\left(s\right)}{s}>\frac{1}{a\left(0\right)}.$$

In 2011, Hakimi and Zertiti, [17] studied the nonexistence of radial positive solutions for a nonpositone problem when the nonlinearity is superlinear and has more than one zero,

$$\begin{cases} -\triangle u\left(x\right)=\lambda f\left(u\left(x\right)\right), & x\in\mathit{\Omega},\\ u\left(x\right)=0, & x\in\partial\mathit{\Omega},\end{cases}$$

where $f\in C\left([0,+\infty),\mathbb{R}\right)$.

In 2014, Sfecci [31], obtained the existence result by introduced the lim sup and lim inf types of nonresonance condition below the first positive eigenvalue for the following Neumann problems defined on the ball $B_R=\left\{x\in\mathbb{R}^N,\ |x|<R\right\}$,

$$\begin{cases} -\triangle u\left(x\right)=f\left(u\left(x\right)\right)+e\left(|x|\right), & \text{in} \quad B_R,\\ u\left(x\right)=0, & \text{on} \quad \partial B_R,\end{cases}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ and $e \in C([0, R], \mathbb{R})$.
In 2014, Butler et. al, [9] studied the positive radial solutions to the boundary value problem

$$\begin{cases} -\Delta u + u = \lambda a(|x|) f(u), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \overline{c}(u) u = 0, & |x| = r_0, \\ u(x) \to 0, & |x| \to \infty, \end{cases}$$

where $f \in C([0, \infty), \mathbb{R}), \Omega = \left\{x \in \mathbb{R}^N \,:\, N > 2,\, |x| > r_0 \text{ with } r_0 > 0\right\}$, λ is a positive parameter, $a \in C([r_0, \infty), \mathbb{R}^+)$ such that $\lim_{r \to \infty} a(r) = 0$, $\frac{\partial}{\partial u}$ is the outward normal derivative and $\overline{c} \in C([0, \infty), (0, \infty))$.
Instead of working directly with (1.1), we note that the change of variable

$$u(x) = u(|x|),\ t = |x|$$

transforms (1.1) into the following boundary value problem (for details, see [14]:

$$\begin{cases} -u''(t) - \frac{N-1}{t} u(t) = \lambda f(t, u(t)),\ t \in (a, b), \\ u(a) = u(b) = 0, \end{cases}$$

where $\lambda \geq 0$ is a positive parameter and $f \in C([a, b] \times [0, \infty), [0, \infty))$.

Inspired and motivated by the works mentioned above, we deal with existence and nonexistence of radial positive solutions to the BVP (1.1) i.e., an equivalant problem (2.1) by using of the fixed point theorem together with the properties of Green's function and we impose certain conditions on f. The paper is organized as follows. In Section 2, we present that a nontrivial and nonnegative solution of BVP (2.1) is monotone positive solution. In Section 3, we obtain some results of the existence, multiplicity and nonexistence positive solutions for BVP (2.1) depends on the parameter λ and we give an example to illustrate our results.

2. Preliminaries

We shall consider the Banach space $E = C[a, b]$ equipped with sup norm

$$\|u\| = \max_{a \leq t \leq b} |u(t)|,$$

and $C^+[a, b]$ is the cone of nonnegative functions in $C[a, b]$, where $1 < a < b$.

Definition 2.1. A nonempty closed and convex set $P \subset E$ is called a cone of E if it satisfies
(i) $u \in P,\ r > 0$ implies $ru \in P$,
(ii) $u \in P,\ -u \in P$ implies $u = \theta$, where θ denote the zero element of E.

Definition 2.2. A cone P is said to be normal if there exists a positive number N called the normal constant of P, such that $\theta \leq u \leq v$ implies $\|u\| \leq N \|v\|$.

We are interested in finding radial solutions for problem (1.1). We proceed as in introduction, setting $u(x) = u(|x|)$, $t = |x|$, we have the following equivalent boundary value problem

$$\begin{cases} -u''(t) - \frac{N-1}{t}u(t) = \lambda f(t, u(t)), \ t \in (a, b), \\ u(a) = u(b) = 0. \end{cases} \tag{2.1}$$

We observe that the existence and nonexistence of radial positive solutions of (1.1) is equivalent to the existence and nonexistence of positive solutions of the problem (2.1).

In arriving our results, we need the following six preliminary lemmas. The first one is well known.

Lemma 2.3. (see [13]) *Let $y(\cdot) \in C[a, b]$. If $u \in C^4[a, b]$, then the BVP*

$$\begin{cases} -u''(t) - \frac{N-1}{t}u(t) = y(t), \ t \in (a, b), \\ u(a) = u(b) = 0, \end{cases}$$

has a unique solution

$$u(t) = \int_a^b s^{N-1} G(t, s)\, y(s)\, ds, \ N > 2,$$

where

$$G(t, s) = \begin{cases} \frac{\left(1 - \left(\frac{a}{s}\right)^{N-2}\right)\left(\left(\frac{b}{t}\right)^{N-2} - 1\right)}{(N-2)(b^{N-2} - a^{N-2})}, & a \le t \le s \le b, \\ \frac{\left(1 - \left(\frac{a}{t}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2} - 1\right)}{(N-2)(b^{N-2} - a^{N-2})}, & a \le s \le t \le b. \end{cases} \tag{2.2}$$

Lemma 2.4. *For any $(t, s) \in [a, b] \times [a, b]$, we have*

$$\frac{\left(1 - \left(\frac{a}{t}\right)^{N-2}\right)}{(N-2)(b^{N-2} - a^{N-2})} \le G(t, s) \le \frac{\left(\left(\frac{b}{t}\right)^{N-2} - 1\right)}{(N-2)(b^{N-2} - a^{N-2})}, \tag{2.3}$$

and

$$0 \le \frac{\partial G}{\partial t}(t, s) \le \frac{\left(\left(\frac{b}{s}\right)^{N-2} - 1\right)\left(\frac{(N-2)b}{a^{N-1}}\right)}{(N-2)(b^{N-2} - a^{N-2})}, \ (t, s) \in [a, b] \times [a, b]. \tag{2.4}$$

Proof. The proof is evident, we omit it. □

Lemma 2.5. (see [10]) *For $y(\cdot) \in C^+[a, b]$. Then the unique solution $u(t)$ of BVP*

$$\begin{cases} -u''(t) - \frac{N-1}{t}u(t) = y(t), \ t \in (a, b), \\ u(a) = u(b) = 0. \end{cases}$$

is nonnegative and satisfies

$$\min_{a_1 \le t \le b_1} u(t) \ge c\|u\|,$$

where $c = \frac{\min\left\{\left(\frac{b}{b_1}\right)^{N-2} - 1, 1 - \left(\frac{a}{a_1}\right)^{N-2}\right\}}{\max\left\{\left(\frac{b}{a}\right)^{N-2} - 1, 1 - \left(\frac{a}{b}\right)^{N-2}\right\}}$ *and $a_1, b_1 \in (a, b)$ with $a_1 < b_1$.*

If we let

$$P=\left\{u\in C^{+}\left[a,b\right]:\quad \min_{a_1\le t\le b_1} u\left(t\right)\ge c\left\|u\right\|\right\}, \tag{2.5}$$

then it is easy to see that P is a cone in $C\left[a,b\right]$. It is evident that BVP (2.1) has an integral formulation given by

$$u\left(t\right)=\lambda\int_a^b s^{N-1}G\left(t,s\right)f\left(s,u\left(s\right)\right)ds,$$

where G defined in (2.2).
Now, we define an integral operator $T_\lambda : P\to C\left[a,b\right]$ by

$$\left(T_\lambda u\right)\left(t\right)=\lambda\int_a^b s^{N-1}G\left(t,s\right)f\left(s,u\left(s\right)\right)ds.$$

Lemma 2.6. *Let $y\in C^{+}\left[a,b\right]$. If $u\in C^2\left[a,b\right]$ satisfies*

$$\begin{cases} -u''\left(t\right)-\frac{N-1}{t}\ u(t)=y\left(t\right),\ t\in(a,b),\\ u\left(a\right)=0,\ u\left(b\right)=0,\end{cases}$$

then
(i) $u\left(t\right)\ge 0$ for $t\in\left[a,b\right]$,
(ii) $u'\left(t\right)\ge 0$ for $t\in\left[a,b\right]$.

Proof. From Lemma 2.4, we obtain $u\left(t\right)\ge 0$ and $u'\left(t\right)\ge 0$ for $t\in\left[a,b\right]$. □

Lemma 2.7. *$T_\lambda\left(P\right)\subset P$.*

Proof. For any $u\in P$, we have

$$\begin{aligned}\min_{a_1\le t\le b_1} T_\lambda u\left(t\right)=\frac{\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}&\left\{\int_a^t\left(1-\left(\frac{a}{s}\right)^{N-2}\right)s^{N-1}f\left(s,u\left(s\right)\right)\right.\\ \times\left(\left(\frac{b}{t}\right)^{N-2}-1\right)ds+\int_t^b\left(1-\left(\frac{a}{t}\right)^{N-2}\right)&\left.\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}\\ \ge\frac{\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}&\left\{\int_a^t\left(1-\left(\frac{a}{s}\right)^{N-2}\right)\left(\left(\frac{b}{b_1}\right)^{N-2}-1\right)\right.\\ \times s^{N-1}f\left(s,u\left(s\right)\right)ds+\int_t^b\left(1-\left(\frac{a}{a_1}\right)^{N-2}\right)&\left.\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}\\ \ge\frac{\lambda\min\left\{\left(\frac{b}{b_1}\right)^{N-2}-1,1-\left(\frac{a}{a_1}\right)^{N-2}\right\}}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}&\left\{\int_a^t s^{N-1}f\left(s,u\left(s\right)\right)\right.\end{aligned}$$

$$\times\left(1-\left(\frac{a}{s}\right)^{N-2}\right)ds+\int_t^b\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\Bigg\}$$

$$=\frac{\lambda\min\left\{\left(\frac{b}{b_1}\right)^{N-2}-1,1-\left(\frac{a}{a_1}\right)^{N-2}\right\}}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}\left\{\int_a^t\frac{\left(\frac{b}{s}\right)^{N-2}-1}{\left(\frac{b}{s}\right)^{N-2}-1}s^{N-1}f\left(s,u\left(s\right)\right)\right.$$

$$\left.\times\left(1-\left(\frac{a}{s}\right)^{N-2}\right)ds+\int_t^b\frac{1-\left(\frac{a}{s}\right)^{N-2}}{1-\left(\frac{a}{s}\right)^{N-2}}\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}$$

$$\ge\frac{\lambda\min\left\{\left(\frac{b}{b_1}\right)^{N-2}-1,1-\left(\frac{a}{a_1}\right)^{N-2}\right\}}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}\left\{\int_a^t\frac{\left(\frac{b}{s}\right)^{N-2}-1}{\left(\frac{b}{a}\right)^{N-2}-1}s^{N-1}f\left(s,u\left(s\right)\right)\right.$$

$$\left.\times\left(1-\left(\frac{a}{s}\right)^{N-2}\right)ds+\int_t^b\frac{1-\left(\frac{a}{s}\right)^{N-2}}{1-\left(\frac{a}{b}\right)^{N-2}}\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}$$

$$\ge\frac{c\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}\left\{\int_a^t\left(\left(\frac{b}{s}\right)^{N-2}-1\right)\left(1-\left(\frac{a}{s}\right)^{N-2}\right)\right.$$

$$\left.s^{N-1}f\left(s,u\left(s\right)\right)ds+\int_t^b\left(1-\left(\frac{a}{s}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}$$

$$=\frac{c\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\int_a^b\left(\left(\frac{b}{s}\right)^{N-2}-1\right)\left(1-\left(\frac{a}{s}\right)^{N-2}\right)s^{N-1}f\left(s,u\left(s\right)ds\right)$$

$$\ge\frac{c\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\max_{a\le t\le b}\left\{\int_a^t\left(\left(\frac{b}{s}\right)^{N-2}-1\right)\left(1-\left(\frac{a}{s}\right)^{N-2}\right)\right.$$

$$\left.\times s^{N-1}f\left(s,u\left(s\right)\right)ds+\int_t^b\left(1-\left(\frac{a}{s}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}$$

$$\ge\frac{c\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\max_{a\le t\le b}\left\{\int_a^t\left(\left(\frac{b}{t}\right)^{N-2}-1\right)\left(1-\left(\frac{a}{s}\right)^{N-2}\right)\right.$$

$$\left.\times s^{N-1}f\left(s,u\left(s\right)\right)ds+\int_t^b\left(1-\left(\frac{a}{t}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}$$

$$=c\max_{a\le t\le b}T_\lambda u\left(t\right)=c\left\|T_\lambda u\right\|.$$

In other words, we find,

$$\max_{a_1 \leq t \leq b_1} T_\lambda u(t) = \|T_\lambda u\|, \ \forall u \in P.$$

Thus, we get that $T_\lambda : P \to P$ is well defined. Moreover, it is easy to show that T_λ is completely continuous. □

If we let

$$K = \{u \in P / u(t) \ \textit{is nondecreasing}\},$$

then, it is easy to show that $K \subset P$ is also a cone in E.

Lemma 2.8. $T_\lambda(P) \subset K$.

Proof. It follows from Lemma 2.6 (ii) and Lemma 2.7. □

Lemma 2.9. $T_\lambda : K \to K$ *is completely continuous.*

Proof. Let $D \subset K$ is a bounded subset. Then there exists a positive constanty M_1 such that

$$\|u\| \leq M_1, \ \forall u \in D$$

Now, we shall prove that $T_\lambda(D)$ is relatively compact in K.
Suppose that $(y_k)_{k\in\mathbb{N}^\star} \subset T_\lambda(D)$. Then there exist $(x_k)_{k\in\mathbb{N}^\star} \subset D$, such that

$$y_k = Ax_k$$

Let $M_2 = \underset{a \leq t \leq b}{sup} |f(t, u(t))|$ for all $(t, u) \in [a, b] \times [0, M_1]$. For any $k \in \mathbb{N}^*$, by Lemma 2.2, we have

$$\begin{aligned}
|y_k(t)| = |(T_n x_k)(t)| &= \lambda \left| \int_a^b s^{N-1} G(t,s) f(s, x_k(s))\, ds \right| \\
&\leq \lambda M_2 \int_a^b s^{N-1} G(t,s)\, ds \\
&\leq \frac{1}{(N-2)(b^{N-2} - a^{N-2})} \lambda M_2 \left(\left(\frac{b}{t}\right)^{N-2} - 1 \right) \int_a^b s^{N-1} ds \\
&\leq \frac{b^N - a^N}{N(N-2)(b^{N-2} - a^{N-2})} \lambda M_2 \left(\left(\frac{b}{a}\right)^{N-2} - 1 \right),
\end{aligned}$$

which implies that $(y_k(t))_{k\in\mathbb{N}^\star}$ is uniformly bounded.

Now, we show that T_λ is equicontinuous. For any $u \in K$, $n \geq 2$, and $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
|y_k(t_1) - y_k(t_2)| &= |T_\lambda u(t_1) - T_\lambda u(t_2)| \\
&\leq \left| \lambda \int_a^b s^{N-1} (G(t_1, s) - G(t_2, s)) f(s, x_k(s))\, ds \right|
\end{aligned}$$

$$\leq \lambda M_2 \int_a^b s^{N-1} \left| G(t_1, s) - G(t_2, s) \right| ds.$$

It follows from the uniform continuity of Green's function G on $[a,b] \times [a,b]$, that for any $\varepsilon > 0$, we have

$$\left| G(t_1, s) - G(t_2, s) \right| \leq \frac{\varepsilon N}{\lambda (b^N - a^N) M_2}, \quad for\ t_1, t_2, s \in [a,b], \ |t_1 - t_2| < \delta.$$

Then

$$\begin{aligned} |y_k(t_1) - y_k(t_2)| &= |T_\lambda u(t_1) - T_\lambda u(t_2)| \\ &\leq \lambda M_2 \int_a^b s^{N-1} \left| G(t_1, s) - G(t_2, s) \right| ds \\ &\leq \varepsilon. \end{aligned}$$

Therefore, T_λ is equicontinuous. By the Ascoli-Arzela Theorem, we know that T_λ is completely continuous. □

By Lemmas 2.8 and 2.9, we know that if $u \in P \setminus \theta$ is solution for BVP (2.1), then u is positive solution for BVP (2.1) and it is obvious from Lemma 2.8 that if $u \in P \setminus \{\theta\}$ is a solution for BVP (2.1) then $u \in K \setminus \{\theta\}$.

3. Existence and nonexistence results

In this section we will apply theorem due Krasnoselskii to study the existence, multiplicity and nonexistence of solutions for BVP (2.1) in $K \setminus \{\theta\}$.

Theorem 3.1. (see [19]) *Let E be a Banach space and $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subset of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$, $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*
(A) $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_2$; or
(B) $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$
Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

We adopt the following assumptions:
(H_1) $f(t, u(t)) \in C((a,b), [0,\infty))$ is nondecreasing in $u \in [0,\infty)$ for fixed $t \in [a,b]$.

(H_2) $F_a = \displaystyle\int_a^b s^{N-1} f(s, 0)\, ds > 0,$

(H_3) $f_\infty = \lim\limits_{u \to \infty} \min\limits_{t \in \left[\frac{a}{a+b}, b\right]} \frac{f(t,u)}{u} = +\infty.$

Set

$$\Lambda = \{\lambda > 0 / there\ exists\ u_\lambda \in K \setminus \{\theta\}\ such\ that\ T_\lambda u_\lambda = u_\lambda\},$$

and

$$\lambda^* = sup \Lambda.$$

Lemma 3.2. *Suppose that $(H_1) - (H_3)$ hold. If $\lambda' \in \Lambda$, then $(0, \lambda'] \subset \Lambda$.*

Proof. $\lambda^{'} \in \Lambda$ means that there exists $u_{\lambda'} \in K\backslash\{\theta\}$ such that $T_{\lambda'}u_{\lambda'} = u_{\lambda'}$. Therefore, for any $\lambda \in (0, \lambda']$ we have

$$T_\lambda u_{\lambda'} \leq T_{\lambda'} u_{\lambda'} = u_{\lambda'},$$

Set

$$w_0 = u_{\lambda'},\ w_n = T_\lambda w_{n-1},\ n = 1, 2, ...$$

From (H_1), we obtain

$$w_0\left(t\right) \geq w_1\left(t\right) \geq ... \geq w_n\left(t\right) \geq ... \geq \frac{F_a \lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right),$$

by Lemma 2.9 and (H_2), $\{w_n\}$ converges to fixed point of T_λ in $K \setminus \{\theta\}$. Thus $(0, \lambda'] \subset \Lambda$. The proof is complete. □

Let

$$\lambda_* < \frac{\left(b^{N-2}-a^{N-2}\right)}{F_b}, \quad F_b = \int_a^b s^{N-1} f\left(s, \left(\frac{b}{a}\right)^{N-2} - 1\right) ds,$$

$$u_0\left(t\right) = \frac{\lambda F_a}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right),\ v_0\left(t\right) = \left(\left(\frac{b}{t}\right)^{N-2} - 1\right),$$

and

$$F_\infty = \lim_{u\to\infty} sup \max_{a\leq t\leq b} \frac{f\left(t,u\right)}{u}.$$

Theorem 3.3. *Suppose that $(H_1) - (H_3)$ hold. Then T_λ has minimal and maximal fixed point in $[u_0, v_0]$ for $\lambda \in (0, \lambda_*]$. Moreover, there exists $\lambda^* \geq \lambda_* > 0$ such that T_λ has at least one and has no fixed points in $K \setminus \{\theta\}$ for $0 < \lambda < \lambda^*$ and $\lambda > \lambda^*$, respectively.*

Proof. From $(H_1) - (H_3)$ and (2.3), we have $\lambda_* > 0$. For any $\lambda \in (0, \lambda_*]$, we obtain

$$\begin{aligned}
\left(T_\lambda u_0\right)\left(t\right) &= \lambda \int_a^b s^{N-1} G\left(t,s\right) f\left(s, u_0\left(s\right)\right) ds \\
&\geq \lambda \int_a^b s^{N-1} G\left(t,s\right) f\left(s, u_0\left(a\right)\right) ds \\
&\geq \frac{\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s,0\right) ds \\
&\geq \frac{\lambda F_a}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right) = u_0\left(t\right),
\end{aligned}$$

and

$$\left(T_\lambda v_0\right)\left(t\right) = \lambda \int_a^b s^{N-1} G\left(t,s\right) f\left(s, v_0\left(s\right)\right) ds$$

$$\leq \lambda_* \int_a^b s^{N-1} G(t,s) f(s, v_0(b))\, ds$$

$$\leq \frac{\lambda_*}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \left(\left(\frac{b}{t}\right)^{N-2} - 1 \right) \int_a^b s^{N-1} f(s, v_0(b))\, ds$$

$$\leq \frac{\lambda_* F_b}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \left(\left(\frac{b}{t}\right)^{N-2} - 1 \right) \leq v_0(t),$$

Set

$$u_n = T_\lambda u_{n-1},\ v_n = T_\lambda v_{n-1},\ n = 1, 2, ...,$$

then from (H_1), we have

$$u_0(t) \leq u_1(t) \leq ... \leq u_n(t) \leq ... \leq v_1(t) \leq v_0(t). \tag{3.1}$$

Lemma 2.9 implies that $\{u_n\}$ and $\{v_n\}$ converge to fixed points u_λ and v_λ of T_λ, respectively.
From (3.1) it is evident that $u_\lambda, v_\lambda \in K \setminus \{\theta\}$ are the minimal fixed point and maximal fixed point of T_λ in $[u_0, v_0]$, respectively.
By the definition of λ^*, there exists a nondecreasing sequence $\{\lambda_n\}_1^{+\infty}$ such that $\lim\limits_{n\to+\infty} \lambda_n = \lambda^*$. Let $\{u_{\lambda_n}\}_1^{+\infty}$ is bounded subset in K. Then there exists a constant $M > 0$ such that

$$\|u_{\lambda_n}\| \leq M,\ for\ n \in \mathbb{N}^*,$$

which implies that $\{u_{\lambda_n}\}_1^{+\infty}$ is uniformly bounded.
Now, we show that $\{u_{\lambda_n}\}_1^{+\infty}$ is equicontinuous. For any $u_{\lambda_n} \in K$, $n \in \mathbb{N}^*$ and $t_1, t_2 \in [a, b]$, with $|t_1 - t_2| < \delta$, we have

$$|x_{\lambda_n}(t_1) - x_{\lambda_n}(t_2)| \leq \lambda^* \int_a^b s^{N-1} |G(t_1,s) - G(t_2,s)| f(s, M)\, ds$$

$$\leq \lambda^* \int_a^b s^{N-1} |G(t_1,s) - G(t_2,s)| f(s, M)\, ds,$$

which implies that $\{x_{\lambda_n}\}_1^{+\infty}$ is equicontinuous subset in K. Consequently, by an application of the Arzela-Ascoli theorem we conclude that $\{x_{\lambda_n}\}_1^{+\infty}$ is a relatively compact set in K. So, there exists a subsequence $\left\{x_{\lambda_{n_i}}\right\} \subset \{x_{\lambda_n}\}$ converging to $x^* \in K$. Note that

$$\left(x_{\lambda_{n_i}}\right)(t) = \lambda_{n_i} \int_a^b s^{N-1} G(t,s) f\left(s, x_{\lambda_{n_i}}(s)\right) ds.$$

By taking the limit we have $x^*(t) = (T_{\lambda^*} x^*)(t)$. Therefore T_λ has at least one fixed point for $0 < \lambda < \lambda^*$. Finaly, for T_λ has no fixed point for $\lambda > \lambda^*$. The proof is complete. □

Theorem 3.4. *Suppose that* $(H_1),(H_3)$ *and* (2.3) *hold. If* $(F_{+\infty} < +\infty)$*, then when* $F_\infty > 0$*, there exists* $\lambda^* \geq \frac{N(N-2)\left(b^{N-2}-a^{N-2}\right)\left(b^N-a^N\right)}{F_\infty} > 0$ *such that* T_λ *has at least one and has no fixed points in* $K\setminus\{\theta\}$ *for* $0<\lambda<\lambda^*$ *and* $\lambda>\lambda^*$*, respectively. When* $F_\infty = 0$*,* T_λ *has at least one fixed points in* $K\setminus\{\theta\}$ *for* $\lambda>0$*.*

Proof. Since $F_\infty < \infty$, for any $\varepsilon>0$, there exists $N_0>0$ such that

$$f(t,u) \leq (F_\infty+\varepsilon)\,u$$

for $u>N_0,\ t\in[a,b]$.
Let $w_0(t) = N_0\left(\left(\frac{b}{t}\right)^{N-2}-1\right)$ and $\lambda_0 = \frac{N(N-2)\left(b^{N-2}-a^{N-2}\right)\left(b^N-a^N\right)}{\left(\left(\frac{b}{a}\right)^{N-2}-1\right)(F_\infty+\varepsilon)}$, then $\lambda_0>0$ and

$$\left(T_{\lambda_0}w_0\right)(t) = \lambda_0\int_a^b s^{N-1}G(t,s)\,f(s,w_0(s))\,ds$$

$$\leq \frac{\lambda_0}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{t}\right)^{N-2}-1\right)\int_a^b s^{N-1}\left(F_\infty+\epsilon\right)w_0(t)\,ds$$

$$\leq \frac{\lambda_0 w_0(t)\left(F_\infty+\epsilon\right)}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{t}\right)^{N-3}-1\right)\int_a^b s^{N-1}ds$$

$$\leq \frac{\lambda_0 w_0(t)\left(F_\infty+\epsilon\right)}{N(N-2)\left(b^{N-2}-a^{N-2}\right)\left(b^N-a^N\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)$$

$$\leq w_0(t),$$

Now, set $w_0(t) = N_0\left(\left(\frac{b}{t}\right)^{N-2}-1\right)$,

$$w_n = T_{\lambda_{n-1}}w_{n-1},\ n=1,2,....$$

From (H_1), we obtain

$$w_0(t)\geq w_1(t)\geq ...\geq w_n(t)\geq ...\geq \frac{F_a\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right). \quad (3.2)$$

Therefore, the sequence $\{w_n\}$ is bounded in $K\setminus\{\theta\}$. By Lemma 2.9 and the definition of λ^*, the operator T_{λ_n} completely continuous. Hence the sequence $\{w_n\}$ is compact in $K\setminus\{\theta\}$, its also monotone. Then it is uniformly convergent to fixed points u^* of T_{λ_n} in $K\setminus\{\theta\}$. When we pass to the limit we get

$$u^* = T_{\lambda^*}u^*$$

For $\lambda>\lambda^*$, there exists $\{\lambda_n\}_1^\infty$, *with* $\lim_{n\to\infty}\lambda_n = \lambda$, we prove that problem has no positive solution. suppose the contrary that the problem has a positive solution x_{λ_n}, then we get

$$\|u_{\lambda_n}\| = \left(T_{\lambda_n}u_{\lambda_n}\right)\left(\left(\frac{b}{a}\right)^{N-2}-1\right)$$

$$\leq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f\left(s, u_{\lambda_n}(s)\right) ds$$

$$\leq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1}\left(F_\infty+\epsilon\right) u_{\lambda_n}(b)\, ds$$

$$\leq \frac{\lambda_n\left(b^N-a^N\right)}{N(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\left(F_\infty+\epsilon\right) u_{\lambda_n}(b)$$

$$\leq \frac{\lambda_n\left(b^N-a^N\right)}{N(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\left(F_\infty+\epsilon\right)\left\|u_{\lambda_n}\right\| < \left\|u_{\lambda^*}\right\|.$$

Taking the limit we obtain

$$\left\|u_\lambda\right\| < \left\|u_\lambda\right\|,$$

which is a contradiction. The proof is complete. □

Lemma 3.5. *Assume that* $(H_1), (H_2)$ *and* (H_3) *hold. If* Λ *is nonempty, then*
(i) Λ *is bounded from above, that* $\lambda^* < +\infty$.
(ii) $\lambda^* \in \Lambda$.

Proof. Suppose to the contrary that there exists an increasing sequence $\{\lambda_n\}_1^{+\infty} \subset \Lambda$ such that $\lim_{n\to+\infty} \lambda_n = +\infty$. Set $x_{\lambda_n} \in K/\{\theta\}$ is a fixed point of T_{λ_n} that is ,

$$T_{\lambda_n} u_{\lambda_n} = u_{\lambda_n}.$$

There are two cases to be considered.
Case 1. $\{u_{\lambda_n}\}_1^{+\infty}$ is bounded, that is there exists a constant $M > 0$ such that

$$\left\|u_{\lambda_n}\right\| \leq M,\ for\ n = 1, 2, \ldots.$$

Hence, from $(H_1), (H_2),$ *and* (H_3) and Lemma 2.3, we have

$$\begin{aligned} M \geq \left\|u_{\lambda_n}\right\| &\geq \left(T_{\lambda_n} u_{\lambda_n}\right)(t) \\ &\geq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f(s, 0)\, ds \\ &= \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right) F_a \to +\infty, \end{aligned}$$

which is a contradiction.
Case 2. $\{u_{\lambda_n}\}_1^{+\infty}$ is unbounded, that is there exists subsequence of $\{u_{\lambda_n}\}_1^{+\infty}$ still denoted by $\{u_{\lambda_n}\}_1^{+\infty}$ such that $\lim_{n\to+\infty} \left\|u_{\lambda_n}\right\| = +\infty$.
When (H_3), take

$$L > \frac{N(N-2)\left(b^{N-2}-a^{N-2}\right)}{\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\lambda_1}$$

there exists $N_1 > 0$ such that $f(t,u) \geq Lu,\ for\ u \geq N_1,\ t \in [a,b]$. Choose n_1 such that $\left\|u_{\lambda_{n_1}}\right\| > NN_1$.
Thus, for $t \in [a,b]$, we have

$$f\left(t, \frac{1}{N}\left\|u_{\lambda_{n_1}}\right\|\right) \geq \frac{1}{N} L \left\|u_{\lambda_{n_1}}\right\|.$$

Moreover, from (H_1) and the definition of K, we have

$$\left\|x_{\lambda_{n_1}}\right\| \geq \left(T_{\lambda_{n_1}} u_{\lambda_{n_1}}\right)(t)$$

$$\geq \frac{\lambda_{n_1}}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s, u_{\lambda_{n_1}}(s)\right) ds$$

$$\geq \frac{\lambda_{n_1}}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s, \frac{1}{6}\left\|u_{\lambda_{n_1}}(s)\right\|\right) ds$$

$$= \frac{\lambda_{n_1} L\left(1-\left(\frac{a}{b}\right)^{N-2}\right)}{N(N-2)\left(b^{N-2}-a^{N-2}\right)}\left\|u_{\lambda_{n_1}}\right\| > \left\|u_{\lambda_{n_1}}\right\|,$$

which is a contradiction.
Consequently, we find that Λ is bounded from above.

(ii) From the definition of λ^*, there exists a nondecreasing sequence $\{\lambda_n\}_1^{+\infty}$ such that $\lim_{n\to+\infty} \lambda_n = \lambda^*$. Let $\{u_{\lambda_n}\}_1^{+\infty} \in K \setminus \{\theta\}$ be a fixed point of T_{λ_n}. Arguing similarly as above in Case 2, we can show that $\{u_{\lambda_n}\}_1^{+\infty}$ is bounded subset in K, that is there exists a constant $M > 0$. Hence from $(H_1), (H_2),\ and\ (H_3)$, we have

$$\left\|u_{\lambda_n}\right\| = \left(T_{\lambda_n} u_{\lambda_n}\right)\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)$$

$$\leq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f\left(s, u_{\lambda_n}(s)\right) ds$$

$$\leq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f\left(s, u_{\lambda_n}(b)\right) ds$$

$$\leq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f\left(s, v_{\lambda_n}(b)\right) ds$$

$$= \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f(s,0)\, ds$$

$$= \frac{\lambda_n F_a}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right) \to \frac{\lambda_* F_a\left(\left(\frac{b}{a}\right)^{N-2}\right)}{(N-2)\left(b^{N-2}-a^{N-2}\right)} = M,$$

as $n \to \infty$.

Therefore

$$\|u_{\lambda_n}\| \leq M, \ n = 1, 2, ...$$

which shows that $\{u_{\lambda_n}\}_1^{+\infty}$ is uniformly bounded.
From the proof of Theorem 3.3 we know that $\{u_{\lambda_n}\}_1^{+\infty}$ is equicontinuous subset in K and by an application of the Arzela-Ascoli theorem we conclude that $\{u_{\lambda_n}\}_1^{+\infty}$ is a relatively compact set in K. So, there exists a subsequence $\left\{u_{\lambda_{n_i}}\right\} \subset \{u_{\lambda_n}\}$ converging to $u^* \in K$. Note that

$$\left(u_{\lambda_{n_i}}\right)(t) = \lambda_{n_i} \int_0^1 s^{N-1} G(t,s) f\left(s, u_{\lambda_{n_i}}(s)\right) ds.$$

By taking the limit we have

$$u^*(t) = (T_{\lambda^*} u^*)(t) \geq \frac{\lambda_1 F_a}{(N-2)\left(b^{N-2} - a^{N-2}\right)} \left(1 - \left(\frac{a}{t}\right)^{N-2}\right),$$

that is $\lambda^* \in \Lambda$. The proof is complete. □

Theorem 3.6. *Suppose that* $(H_1)-(H_3)$ *holds. Then there exists* $\lambda^* \geq \lambda_* > 0$ *such that BVP* (2.1) *has at least two, one and no positive solutions for* $0 < \lambda \leq \lambda_*,\ \lambda_* < \lambda \leq \lambda^*$ *and* $\lambda > \lambda^*$ *respectively.*

Proof. From $(H_1), (H_2)$ and (H_3) we have $(0, \lambda_*] \subset \Lambda$. So $\lambda^* \geq \lambda_* > 0$.
From Lemma 3.2 and 3.5, we have $(0, \lambda^*] = \Lambda$. Therefore, from the definition of λ^* we only to prove that T_λ has at least two fixed points in $K \setminus \{\theta\}$ for $\lambda \in (0, \lambda_*]$.
Now, given $\lambda \in (0, \lambda_*]$. Theorem 3.3 means that T_λ has at least one fixed point $u_{\lambda,1} \in K \setminus \{\theta\}$ which satisfies $\|u_{\lambda,1}\| \leq \left(\frac{b}{a}\right)^{N-2} - 1$.
Let

$$K_1 = \left\{ x \in K \mid \|u\| < \left(\frac{b}{a}\right)^{N-2} - 1 \right\}.$$

For $t \in [a,b]$, so for $u \in K$ with $\|u\| = \left(\frac{b}{a}\right)^{N-2} - 1$, i.e $u \in \partial K_1$, we have

$$\|u\| = \|T_\lambda u\| = (T_\lambda u)\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)$$

$$\leq \frac{\lambda}{(N-2)\left(b^{N-2} - a^{N-2}\right)} \left(\left(\frac{b}{a}\right)^{N-2} - 1\right) \int_a^b s^{N-1} f(s, u(s))\, ds$$

$$\leq \frac{\lambda_*}{(N-2)\left(b^{N-2} - a^{N-2}\right)} \left(\left(\frac{b}{a}\right)^{N-2} - 1\right) \int_a^b s^{N-1} f\left(s, \left(\frac{b}{a}\right)^{N-2} - 1\right) ds$$

$$< \frac{\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)}{N-2} < \|u\|. \tag{3.3}$$

When (H_3), take

$$L > \frac{N\,(N-2)\left(b^{N-2}-a^{N-2}\right)}{\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\lambda_1}$$

there exists $N_1 > 0$ such that $f\,(t,u) \geq Lu,\ for\ u \geq N_1,\ t \in [a,b]$.
Set $K_2 = \{u\ :\ \|u\| < NN_1\}$. Then $\overline{K_1} \subset K_2$. If $u \in \partial K_2$, we have

$$\|u\| = \|T_\lambda u\| = (T_\lambda u)\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\,(s,u\,(s))\,ds$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s,\frac{1}{N}\,\|u\|\right)ds$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s,\frac{1}{N}\,\|u\,(s)\|\right)ds$$

$$\geq \frac{\lambda L\left(1-\left(\frac{a}{b}\right)^{N-2}\right)}{N\,(N-2)\left(b^{N-2}-a^{N-2}\right)}\,\|u\| > \|u\|\,.$$

Consequently, Applying Theorem 3.1 that T_λ has a fixed point $u_{\lambda,2} \in \overline{K_2} \setminus K_1$. Equation (3.3) implies that T_λ has no fixed point in ∂K_1. In conclusion, for $\lambda \in (0,\lambda_*]$, T_λ has at least two fixed points $u_{\lambda,1}$ and $u_{\lambda,2}$ in K. The proof is complete. □

We present an example to illustrate the applicability of the results shown before.

Example 3.7. Consider in $\mathbb{R}^3$ the elliptic boundary value problem

$$\begin{cases} -\triangle u\,(x) = \lambda\,(|x| + u + \ln\,(1+u))\,, & x \in \Omega, \\ u\,(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

To the system (3.4) we associate the the second order boundary value problem

$$\begin{cases} -u''\,(t) - \dfrac{2}{t}\,u(t) = \lambda\,(t + u + \ln\,(1+u))\,,\ t \in (a,b)\,, \\ u\,(a) = u\,(b) = 0, \end{cases}$$

By direct computation, we have

$$F_\infty = 2,\ F_0 = \frac{1}{4},\ F_1 = \frac{1}{2} + \frac{2}{3}\,(1+\ln\,(2))\ \text{ and }\ \lambda_* = \frac{48-9\pi}{6+8\,(1+\ln\,(2))}.$$

So, the assumptions $(H_1)\,,\,(H_2)$ and (H_3) are satisfied, it follows from Theorem 3.4 there exists $\lambda^* = 3 \geq \lambda_*$ such that boundary value problem (3.4) has at least one positive solution for $0 < \lambda \leq 3$ and has no positive solution for $\lambda > \lambda^*$.

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