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Existence and stability of fractional differential equations involving generalized Katugampola derivative

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Abstract. The present article deals with the existence and stability results for a class of fractional differential equations involving generalized Katugampola derivative. Some fixed point theorems are used to obtain the results and enlightening examples of obtained result are also given.

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1. Introduction

Fractional calculus has proven to be an useful tool in the description of various complex phenomena in the real world problems. During the theoretical development of the calculus of arbitrary order, numerous fractional integral and differential operators are emerged and/or used by timely mathematicians, see [1, 3]-[7],[10]-[18],[23]-[28]. Although the well-developed theory and many more applications of Wyel, Liouville, Riemann-Liouville, Hadamard operators, still this is a spotlight area of research in applied sciences.

U. Katugampola in [24, 25] generalized the above mentioned fractional integral and differential operators. In the same work, he obtained boundedness of generalized fractional integral in an extended Lebesgue measurable space. Further he studied existence and uniqueness of solution of initial value problem (IVP) for a class of generalized fractional differential equations (FDEs) in [26]. R. Almeida, *et al.* [7] studied these results with its Caputo counterpart. R. Almeida [6] discussed certain problems of calculus of variations dependent on Lagrange function with the same approach for first and second order. In 2015, D. Anderson *et al.* [8] studied properties of Katugampola fractional derivative with potential application in quantum mechanics as well constructed a Hamiltonian from its self adjoint operator and applied to the particle in a box model. Recently, D. S. Oliveira, *et al.* [29] proposed a generalization of Katugampola and Caputo-Katugampola fractional derivatives with the name Hilfer-Katugampola fractional derivative. This new fractional derivative interpolates the well-known fractional derivatives: Hilfer [20], Hilfer-Hadamard [23], Katugampola [25], Caputo-Katugampola [7], Riemann-Liouville [27], Hadamard [28], Caputo [27], Caputo-Hadamard [4], Liouville, Wyel as its particular cases. Following the results of [20], they further obtained existence and uniqueness of solution of nonlinear FDEs involving this generalized Katugampola derivative with initial condition [29].

The stability of functional equations was first posed by Ulam [30], thereafter, this type of stability evolved as an interesting field of research. The concept of stability of functional equations arises when the functional equation is being replaced by an inequality which acts as a perturbation of the functional equation, see the monograph [22] and the references cited therein. The considerable attention paid to recent development of stability results for FDEs can be found in [2, 3, 14, 12, 9, 20, 22, 30, 31, 32]. The present work deals with following two IVPs.
Problem I:

$$\begin{cases} \left({}^\rho D^{\alpha,\beta}_{a+}x\right)(t) &= f\big(t, x(t), ({}^\rho D^{\alpha,\beta}_{a+}x)(t)\big); \qquad t \in \Omega, \\ \left({}^\rho I^{1-\gamma}_{a+}x\right)(a) &= c_1, \qquad c_1 \in \mathbb{R}, \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{1.1}$$

where $\alpha \in (0,1), \beta \in [0,1], \rho > 0, \Omega = [a,b], f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given function and $a > 0$.
Problem II:

$$\begin{cases} \left({}^\rho D^{\alpha,\beta}_{a+}x\right)(t) &= f(t, x(t)); \qquad t \in \Omega, \\ \left({}^\rho I^{1-\gamma}_{a+}x\right)(a) &= c_2, \qquad c_2 \in \mathbb{R}, \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{1.2}$$

where $\alpha \in (0,1), \beta \in [0,1], \rho > 0, f : \Omega \times \mathbb{R} \to \mathbb{R}$ is given function. The operators ${}^\rho D^{\alpha,\beta}_{a+}$ and ${}^\rho I^{1-\gamma}_{a+}$ involved herein are the generalized Katugampola fractional derivative (of order α and type β) and Katugampola fractional integral (of order $1-\gamma$) respectively.

The rest of paper is organized as follows: Section 2 introduces some preliminary facts that we need in the sequel. Section 3 presents our main results on existence and stability of considered problems. As an application of main results, two illustrative examples are given in section 4. Concluding remarks are given in last section.

2. Preliminaries

Let $\Omega = [a,b] (0 < a < b < \infty)$. As usual C denotes the Banach space of all continuous functions $x : \Omega \to E$ with the superemum (uniform) norm

$$\|x\|_\infty = \sup_{t\in\Omega} \|x(t)\|_E$$

and $AC(\Omega)$ be the space of absolutely continuous functions from Ω into E. Denote $AC^1(\Omega)$ the space defined by

$$AC^1(\Omega) = \left\{ x : \Omega \to E | \frac{d}{dt}x(t) \in AC(\Omega) \right\}.$$

Throughout the paper, let $\delta_\rho = t^{\rho-1}\frac{d}{dt}, n = [\alpha]+1$, and mention $[\alpha]$ as integer part of α. Define the space

$$AC^n_{\delta_\rho} = \{x : \Omega \to E | \delta_\rho^{n-1} x(t) \in AC(\Omega)\}, \quad n \in \mathbb{N}.$$

Here we define the weighted space of continuous functions g on $\Omega^* = (a,b]$ by

$$C_{\gamma,\rho}(\Omega) = \left\{ g : \Omega^* \to \mathbb{R} | \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t) \in C(\Omega) \right\}, \quad 0 < \gamma \leq 1,$$

with the norm

$$\|g\|_{C_{\gamma,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t) \right\|_C = \max_{t\in\Omega} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t) \right|$$

and

$$C^1_{\delta_\rho,\gamma}(\Omega) = \{g \in C(\Omega) : \delta_\rho g \in C_{\gamma,\rho}(\Omega)\}$$

with the norms

$$\|g\|_{C^1_{\delta_\rho,\gamma}} = \|g\|_C + \|\delta_\rho g\|_{C_{\gamma,\rho}} \quad \text{and} \quad \|g\|_{C^1_{\delta_\rho}} = \sum_{k=0}^{1} \max_{t\in\Omega} \left|\delta_\rho^k g(t)\right|.$$

Note that $C^0_{\delta_\rho,\gamma}(\Omega) = C_{\delta_\rho,\gamma}(\Omega)$, $C_{0,\rho}(\Omega) = C(\Omega)$ and $C_{\gamma,\rho}(\Omega)$ is a complete metric space [29].

Now we introduce some preliminaries from fractional calculus. For more details, we refer the readers to [3, 24, 25, 29]:

Definition 2.1. [24] [Katugampola fractional integral] Let $\alpha \in \mathbb{R}_+, c \in \mathbb{R}$ and $g \in X^p_c(a,b)$, where $X^p_c(a,b)$ is the space of Lebesgue measurable functions. The Katugampola fractional integral of order α is defined by

$$({}^\rho I^\alpha_{a+} g)(t) = \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds, \qquad t > a, \rho > 0,$$

where $\Gamma(\cdot)$ is a Euler's gamma function.

Definition 2.2. [25] [Katugampola fractional derivative] Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\rho > 0$. The Katugampola fractional derivative ${}^\rho D^{\alpha,\beta}_{a+}$ of order α is defined by

$$\begin{aligned}({}^\rho D^\alpha_{a+} g)(t) &= \delta^n_\rho ({}^\rho I^{n-\alpha}_{a+} g)(t) \\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1} \frac{g(s)}{\Gamma(n-\alpha)} ds.\end{aligned}$$

Definition 2.3. [29] [Generalized Katugampola fractional derivative] The generalized Katugampola fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ with respect to t and is defined by

$$({}^\rho D^{\alpha,\beta}_{a\pm} g)(t) = (\pm {}^\rho I^{\beta(1-\alpha)}_{a\pm} \delta_\rho {}^\rho I^{(1-\beta)(1-\alpha)}_{a\pm} g)(t), \quad \rho > 0 \tag{2.1}$$

for the function for which right hand side expression exists.

Remark 2.4. The generalized Katugampola operator ${}^\rho D_{a+}^{\alpha,\beta}$ can be written as

$$ {}^\rho D_{a+}^{\alpha,\beta} = {}^\rho I_{a+}^{\beta(1-\alpha)} \delta_\rho {}^\rho I_{a+}^{1-\gamma} = {}^\rho I_{a+}^{\beta(1-\alpha)} {}^\rho D_{a+}^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta. $$

Lemma 2.5. [24] *[Semigroup property] If $\alpha, \beta > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$ and $\rho, c \in \mathbb{R}$ for $\rho \geq c$. Then, for $g \in X_c^p(a,b)$ the following relation hold:*

$$ ({}^\rho I_{a+}^{\alpha} {}^\rho I_{a+}^{\beta} g)(t) = ({}^\rho I_{a+}^{\alpha+\beta} g)(t). $$

Lemma 2.6. [29] *Let $t > a$, ${}^\rho I_{a+}^{\alpha}$ and ${}^\rho D_{a+}^{\alpha}$ are as in Definition 2.1 and Definition 2.2, respectively. Then the following hold:*

(i) $\left({}^\rho I_{a+}^{\alpha}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\sigma}\right)(t) = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\alpha+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\sigma+\alpha}, \quad \alpha \geq 0, \sigma > 0,$

(ii) for $\sigma = 0$, $\left({}^\rho I_{a+}^{\alpha}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\sigma}\right)(t) = \left({}^\rho I_{a+}^{\alpha} 1\right)(t) = \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}, \quad \alpha \geq 0,$

(iii) for $0 < \alpha < 1$, $\left({}^\rho D_{a+}^{\alpha}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right)(t) = 0.$

3. Main results

In this section, we present the results on existence, attractivity and Ulam stability of solutions for fractional differential equations involving generalized Katugampola fractional derivatives.

Denote $BC = BC(I)$, $I = [a, \infty)$. Let $D \subset BC$ is a nonempty set and let $G : D \to D$. Consider the solutions of equation

$$ (Gx)(t) = x(t). \tag{3.1} $$

We define the attractivity of solutions for equation (3.1) as follows:

Definition 3.1. A solutions of equation (3.1) are locally attractive if there exists a ball $B(x_0, \mu)$ in the space BC such that, for arbitrary solutions $y(t)$ and $z(t)$ of equation (3.1) belonging to $B(x_0, \mu) \cap D$, we have

$$ \lim_{t\to\infty} (y(t) - z(t)) = 0. \tag{3.2} $$

Whenever limit (3.2) is uniform with respect to $B(x_0, \mu) \cap D$, solutions of equation (3.1) are said to be uniformly locally attractive.

Lemma 3.2. [14] *If $X \subset BC$. Then X is relatively compact in BC if following conditions hold:*
1. X is uniformly bounded in BC,
2. The functions belonging to X are almost equicontinuous on $\mathbb{R}_+$, i.e. equicontinuous on every compact of $\mathbb{R}_+$,
3. The functions from X are equiconvergent, i.e. given $\epsilon > 0$ there corresponds $T(\epsilon) > 0$ such that $|x(t) - \lim_{t\to\infty} x(t)| < \epsilon$ for any $t \geq T(\epsilon)$ and $x \in X$.

Now we discuss the existence and attractivity of solutions of IVP (1.1). Throughout the work, we mean $BC_{\gamma,\rho} = BC_{\gamma,\rho}(I)$ is a weighted space of all bounded and continuous functions defined by

$$BC_{\gamma,\rho} = \left\{ x : (a,\infty] \to \mathbb{R} \mid \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} x(t) \in BC \right\}$$

with the norm

$$\|x\|_{BC_{\gamma,\rho}} = \sup_{t\in\mathbb{R}_+} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} x(t) \right|.$$

Theorem 3.3. [21]*[Schauder fixed point theorem] Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E and $\Lambda : Q \to Q$ is compact, and continuous map. Then Λ has at least one fixed point in Q.*

Definition 3.4. A solution of IVP (1.1) is a measurable function $x \in BC_{\gamma,\rho}$ satisfying initial value $({}^\rho I_{a+}^{1-\gamma} x)(a^+) = c_1$ and differential equation

$$\left({}^\rho D_{a+}^{\alpha,\beta} x\right)(t) = f\left(t, x(t), ({}^\rho D_{a+}^{\alpha,\beta} x)(t)\right)$$

on I.

From ([29], Theorem 3 pp. 9), we conclude the following lemma.

Lemma 3.5. *Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1, 0 \le \beta \le 1$ and $\rho > 0$. Let $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot, x(\cdot), y(\cdot)) \in BC_{\gamma,\rho}$ for any $x, y \in BC_{\gamma,\rho}$. Then IVP (1.1) is equivalent to Volterra integral equation*

$$x(t) = \frac{c_1}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds,$$

where $g(\cdot) \in BC_{\gamma,\rho}$ such that

$$g(t) = f\left(t,\ \frac{c_1}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + ({}^\rho I_{a+}^\alpha g)(t),\ g(t)\right). \tag{3.3}$$

We use the following hypotheses in the sequel:

(H_1). Function $t \longmapsto f(t,x,y)$ is measurable on I for each $x, y \in BC_{\gamma,\rho}$, and functions $x \longmapsto f(t,x,y)$ and $y \longmapsto f(t,x,y)$ are continuous on $BC_{\gamma,\rho}$ for a.e. $t \in I$;

(H_2). There exists a continuous function $p : I \to \mathbb{R}_+$ such that

$$|f(t,x,y)| \le \frac{p(t)}{1 + |x| + |y|}, \text{ for a.e. } t \in I, \text{ and each } x, y \in \mathbb{R}.$$

Moreover, assume that

$$\lim_{t\to\infty} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho I_{a+}^\alpha p\right)(t) = 0.$$

Set

$$p^* = \sup_{t\in I} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho I_{a+}^\alpha p\right)(t).$$

Theorem 3.6. *Suppose that* (H_1) *and* (H_2) *hold. Then IVP* (1.1) *has at least one solution on* I. *Moreover, solutions of IVP* (1.1) *are locally attractive.*

Proof. For any $x \in BC_{\gamma,\rho}$, define the operator Λ such that

$$(\Lambda x)(t) = \frac{c_1}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{g(s)}{\Gamma(\alpha)}ds, \tag{3.4}$$

where $g \in BC_{\gamma,\rho}$ given by (3.3). The operator Λ is well defined and maps $BC_{\gamma,\rho}$ into $BC_{\gamma,\rho}$. Indeed, the map $\Lambda(x)$ is continuous on I for any $x \in BC_{\gamma,\rho}$, and for each $t \in I$, we have

$$\begin{aligned}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| &\le \frac{|c_1|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{|g(s)|}{\Gamma(\alpha)}ds\\ &\le \frac{|c_1|}{\Gamma(\gamma)} + \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} p(s)ds\\ &\le \frac{|c_1|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho I^\alpha_{a+}p\right)(t).\end{aligned}$$

Thus

$$\|\Lambda(x)\|_{BC_{\gamma,\rho}} \le \frac{|c_1|}{\Gamma(\gamma)} + p^* := M. \tag{3.5}$$

Hence, $\Lambda(x) \in BC_{\gamma,\rho}$. This proves that operator Λ maps $BC_{\gamma,\rho}$ into itself.

By Lemma 3.5, the IVP of finding the solutions of IVP (1.1) is reduced to the finding solution of the operator equation $\Lambda(x) = x$. Equation (3.5) implies that Λ transforms the ball $B_M := B(0, M) = \{x \in BC_{\gamma,\rho} : \|x\|_{BC_{\gamma,\rho}} \le M\}$ into itself.

Now we show that the operator Λ satisfies all the assumptions of Theorem 3.3. The proof is given in following steps:

Step 1: Λ is continuous.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in B_M. Then, for each $t \in I$, we have

$$\begin{aligned}&\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x_n)(t) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right|\\ &\qquad\le \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}|g_n(s) - g(s)|ds,\end{aligned} \tag{3.6}$$

where $g_n, g \in BC_{\gamma,\rho}$,

$$g_n(t) = f\left(t,\ \frac{c_1}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + ({}^\rho I^\alpha_{a+}g_n)(t),\ g_n(t)\right)$$

and g is defined by (3.3). If $t \in I$, then from (3.6), we obtain

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x_n)(t) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| \\ \leq 2\frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} p(s)ds. \tag{3.7}$$

Since $x_n \to x$ as $n \to \infty$ and $\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho I^\alpha_{a+} p\right)(t) \to 0$ as $t \to \infty$ then (3.7) implies that

$$\|\Lambda(x_n) - \Lambda(x)\|_{BC_{\gamma,\rho}} \to 0 \quad \text{as } n \to \infty.$$

Step 2: $\Lambda(B_M)$ is uniformly bounded.

This is clear since $\Lambda(B_M) \subset B_M$ and B_M is bounded.

Step 3: $\Lambda(B_M)$ is equicontinuous on every compact subset $[a, T]$ of I, $T > a$.

Let $t_1, t_2 \in [a, T], t_1 < t_2$ and $x \in B_M$. We have

$$\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\ \leq \left|\frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} g(s)ds \right. \\ \left. - \frac{\left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_1} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1} g(s)ds\right|,$$

with $g(\cdot) \in BC_{\gamma,\rho}$ given by (3.3). Thus we get

$$\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\ \leq \frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} |g(s)|ds \\ + \int_a^{t_1}\left|\left[\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} \right.\right. \\ \left.\left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\right|\frac{|g(s)|}{\Gamma(\alpha)}ds \\ \leq \frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} p(s)ds \\ + \int_a^{t_1}\left|\left[\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} \right.\right. \\ \left.\left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\right|\frac{p(s)}{\Gamma(\alpha)}ds.$$

Thus, for $p_* = \sup_{t\in[a,T]} p(t)$ and from the continuity of the function p, we obtain

$$\begin{aligned}&\left|\left(\frac{{t_2}^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2)-\left(\frac{{t_1}^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right|\\
&\qquad\le p_*\frac{\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\left(\frac{t_2^{\rho}-t_1^{\rho}}{\rho}\right)^{\alpha}\\
&\qquad+\frac{p_*}{\Gamma(\alpha)}\int_a^{t_1}\left|\left(\frac{{t_2}^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}s^{\rho-1}\left(\frac{{t_2}^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\right.\\
&\qquad\left.-\left(\frac{{t_1}^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}s^{\rho-1}\left(\frac{{t_1}^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\right|ds.\end{aligned}$$

As $t_1\to t_2$, the right hand side of the above inequation tends to zero.
Step 4: $\Lambda(B_M)$ is equiconvergent.

Let $t\in I$ and $x\in B_M$, then we have

$$\left|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right|\le\frac{|c_1|}{\Gamma(\gamma)}+\frac{\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}|g(s)|ds$$

where $g(\cdot)\in BC_{\gamma,\rho}$ is given by (3.3). Thus we get

$$\begin{aligned}\left|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right|&\le\frac{|c_1|}{\Gamma(\gamma)}+\frac{\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}p(s)ds\\
&\le\frac{|c_1|}{\Gamma(\gamma)}+\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}\left({}^{\rho}I^{\alpha}_{a+}p\right)(t).\end{aligned}$$

Since $\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}\left({}^{\rho}I^{\alpha}_{a+}p\right)(t)\to 0$ as $t\to\infty$, then, we get

$$|(\Lambda x)(t)|\le\frac{|c_1|}{\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}\Gamma(\gamma)}+\frac{\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}\left({}^{\rho}I^{\alpha}_{a+}p\right)(t)}{\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}}\to 0 \text{ as } t\to\infty.$$

Hence

$$|(\Lambda x)(t)-(\Lambda x)(+\infty)|\to 0 \text{ as } t\to\infty.$$

In view of Lemma 3.2 and immediate consequence of Steps 1 to 4, we conclude that $\Lambda: B_M\to B_M$ is continuous and compact. Theorem 3.3 implies that Λ has a fixed point x which is a solution of IVP (1.1) on I.
Step 5: Local attactivity of solutions.

Let x_0 is a solution of IVP (1.1). Taking $x \in B(x_0, 2p^*)$, we have

$$\begin{aligned}\Big|\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}(\Lambda x)(t) - \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma} x_0(t)\Big| \\ &= \Big|\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}(\Lambda x)(t) - \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma} (\Lambda x_0)(t)\Big| \\ &\leq \frac{\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}\Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{\alpha-1} |f(s, g(s)) - f(s, g_0(s))| ds,\end{aligned}$$

where $g_0 \in BC_{\gamma,\rho}$ and

$$g_0(t) = f\big(t,\ \frac{c_1}{\Gamma(\gamma)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\gamma-1} + ({}^\rho I^\alpha_{a+} g_0)(t),\ g_0(t)\big).$$

Then

$$\begin{aligned}\Big|\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}(\Lambda x)(t) - \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma} x_0(t)\Big| \\ &\leq 2\frac{\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t s^{\rho-1}\Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{\alpha-1} p(s) ds.\end{aligned}$$

We obtain

$$\|(\Lambda(x) - x_0\|_{BC_{\gamma,\rho}} \leq 2p^*.$$

Hence Λ is a continuous function such that $\Lambda(B(x_0, 2p^*)) \subset B(x_0, 2p^*)$.
Moreover, if x is a solution of IVP (1.1), then

$$\begin{aligned}|x(t) - x_0(t)| &= |(\Lambda x)(t) - (\Lambda x_0)(t)| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t s^{\rho-1}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} |g(s) - g_0(s)| ds \\ &\leq 2({}^\rho I^\alpha_{a+} p)(t).\end{aligned}$$

Thus

$$|x(t) - x_0(t)| \leq 2\frac{\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}({}^\rho I^\alpha_{a+} p)(t)}{\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}}. \tag{3.8}$$

With the fact that $\lim\limits_{t\to\infty}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}({}^\rho I^\alpha_{a+} p)(t) = 0$ and inequation (3.8), we obtain

$$\lim_{t\to\infty} |x(t) - x_0(t)| = 0.$$

Consequently, all solutions of IVP (1.1) are locally attractive. □

Now onwards in this section, we deal with existence of solutions and Ulam stability for IVP (1.2).

Lemma 3.7. *Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1, 0 \leq \beta \leq 1$ and $\rho > 0$. If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot, x(\cdot)) \in C_{\gamma,\rho}(\Omega)$ for any $x \in C_{\gamma,\rho}(\Omega)$. Then IVP (1.2) is equivalent to the Volterra integral equation*

$$x(t) = \frac{c_2}{\Gamma(\gamma)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\gamma-1} + ({}^\rho I^\alpha_{a+} f(\cdot, x(\cdot)))(t).$$

Let $\epsilon > 0, \Phi : \Omega \to [0, \infty)$ be a continuous function and consider the following inequalities:

$$|\big({}^\rho D^{\alpha,\beta}_{a+} x\big)(t) - f(t, x(t))| \leq \epsilon; \qquad t \in \Omega, \tag{3.9}$$

$$|\big({}^\rho D^{\alpha,\beta}_{a+} x\big)(t) - f(t, x(t))| \leq \Phi(t); \qquad t \in \Omega, \tag{3.10}$$

$$|\big({}^\rho D^{\alpha,\beta}_{a+} x\big)(t) - f(t, x(t))| \leq \epsilon\Phi(t); \qquad t \in \Omega. \tag{3.11}$$

Definition 3.8. [1] IVP (1.2) is Ulam-Hyers stable if there exists a real number $\psi > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C_{\gamma,\rho}$ of inequality (3.9) there exists a solution $\bar{x} \in C_{\gamma,\rho}$ of IVP (1.2) with

$$|x(t) - \bar{x}(t)| \leq \epsilon\psi; \qquad t \in \Omega.$$

Definition 3.9. [1] IVP (1.2) is generalized Ulam-Hyers stable if there exists $\Psi : C([0, \infty), [0, \infty))$ with $\Psi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $x \in C_{\gamma,\rho}$ of inequality (3.9) there exists a solution $\bar{x} \in C_{\gamma,\rho}$ of IVP (1.2) with

$$|x(t) - \bar{x}(t)| \leq \Psi(\epsilon); \qquad t \in \Omega.$$

Definition 3.10. [1] IVP (1.2) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $\psi_\phi > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C_{\gamma,\rho}$ of inequality (3.11) there exists a $\bar{x} \in C_{\gamma,\rho}$ of IVP (1.2) with

$$|x(t) - \bar{x}(t)| \leq \epsilon\psi_\phi\Phi(t); \qquad t \in \Omega.$$

Definition 3.11. [1] IVP (1.2) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $\psi_\phi > 0$ such that for each solution $x \in C_{\gamma,\rho}$ of inequality (3.10) there exists a $\bar{x} \in C_{\gamma,\rho}$ of IVP (1.2) with

$$|x(t) - \bar{x}(t)| \leq \psi_\phi\Phi(t); \qquad t \in \Omega.$$

Remark 3.12. It is clear that

- **(i).** If the IVP (1.2) is Ulam-Hyers stable then, for a real number $\psi > 0$ as a continuous function defined in Definition 3.9, it is generalized Ulam-Hyers stable.
- **(ii).** If the IVP (1.2) is Ulam-Hyers-Rassias stable then it is generalized Ulam-Hyers-Rassias stable.
- **(iii).** If the IVP (1.2) is Ulam-Hyers-Rassias stable with respect to Φ then, for $\Phi(\cdot) = 1$, it is Ulam-Hyers stable.

Definition 3.13. A solution of IVP (1.2) is a measurable function $x \in C_{\gamma,\rho}$ that satisfies initial condition $\big({}^\rho I^{1-\gamma}_{a+} x\big)(a) = c_2$ and differential equation $\big({}^\rho D^{\alpha,\beta}_{a+} x\big)(t) = f(t, x(t))$ on Ω.

Consider the following hypotheses:

(H_3). Function $t \longmapsto f(t,x)$ is measurable on Ω for each $x \in C_{\gamma,\rho}$ and function $x \longmapsto f(t,x)$ is continuous on $C_{\gamma,\rho}$ for a.e. $t \in \Omega$,

(H_4). There exists a continuous function $p : \Omega \to [0,\infty)$ such that

$$|f(t,x)| \leq \frac{p(t)}{1+|x|}|x|, \text{ for a.e. } t \in \Omega, \text{ and each } x \in \mathbb{R}.$$

Set $p^* = \sup_{t\in\Omega} p(t)$. Now we shall give the existence theorem in the following:

Theorem 3.14. *Assume that* (H_3) *and* (H_4) *hold. Then IVP* (1.2) *has at least one solution defined on* Ω.

Proof. Consider the operator $\Lambda : C_{\gamma,\rho} \to C_{\gamma,\rho}$ such that

$$(\Lambda x)(t) = \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}ds. \tag{3.12}$$

Clearly, the fixed points of this operator equation $(\Lambda x)(t) = x(t)$ are solutions of IVP (1.2). For any $x \in C_{\gamma,\rho}$ and each $t \in \Omega$, we have

$$\begin{aligned}
\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)\right| &\leq \frac{|c_2|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{|f(s,x)|}{\Gamma(\alpha)}ds \\
&\leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}p(s)ds \\
&\leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}ds \\
&\leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha} \\
&\leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha+1-\gamma}.
\end{aligned}$$

Thus

$$\|\Lambda x\|_C \leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha+1-\gamma} := N. \tag{3.13}$$

Thus Λ transforms the ball $B_N = B(0,N) = \{z \in C_{\gamma,\rho} : \|z\|_C \leq N\}$ into itself. We shall show that the operator $\Lambda : B_N \to B_N$ satisfies all the conditions of Theorem 3.16. The proof is given in following several steps.

Step 1: $\Lambda : B_N \to B_N$ is continuous.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in B_N. Then, for each $t \in I$, we have

$$\begin{aligned}
&\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x_n)(t) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| \\
&\qquad \leq \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}|f(s,x_n(s)) - f(s,x(s))|ds.
\end{aligned} \tag{3.14}$$

Since $x_n \to x$ as $n \to \infty$ and f is continuous, then by Lebesgue dominated convergence theorem, inequality (3.14) implies $\|\Lambda(x_n) - \Lambda(x)\|_C \to 0$ as $n \to \infty$.
Step 2: $\Lambda(B_N)$ is uniformly bounded.

Since $\Lambda(B_N) \subset B_N$ and B_N is bounded. Hence, $\Lambda(B_N)$ is uniformly bounded.
Step 3: $\Lambda(B_N)$ is equicontinuous.

Let $t_1, t_2 \in \Omega, t_1 < t_2$ and $x \in B_N$. We have

$$\begin{aligned}
&\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\
&\qquad \le \left|\frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s,x(s))ds \right.\\
&\qquad \left. - \frac{\left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_1} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s,x(s))ds\right| \\
&\qquad \le \frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} |f(s,x(s))|ds \\
&\qquad + \int_a^{t_1}\left|\left[\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right.\right. \\
&\qquad \left.\left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\right|\frac{|f(s,x(s))|}{\Gamma(\alpha)}ds \\
&\qquad \le \frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} p(s)ds \\
&\qquad + \int_a^{t_1}\left|\left[\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right.\right. \\
&\qquad \left.\left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\right|\frac{p(s)}{\Gamma(\alpha)}ds.
\end{aligned}$$

Thus, for $p_* = \sup_{t\in\Omega} p(t)$ and from the continuity of the function p, we obtain

$$\begin{aligned}
&\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\
&\qquad \le \frac{p_*}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}\left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{\alpha} \\
&\qquad + \frac{p_*}{\Gamma(\alpha)}\int_a^{t_1}\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right. \\
&\qquad \left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right|ds.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with Arzela-Ascoli Theorem, we can conclude that Λ is continuous and compact. By applying the Schauder fixed point theorem, we conclude that Λ has a fixed point x which is a solution of IVP (1.2). $\square$

Theorem 3.15. *Assume that* (H_3), (H_4) *and the following hypotheses hold:*

(H_5). *There exists* $\lambda_\phi > 0$ *such that for each* $t \in \Omega$, *we have*

$$({}^\rho I_{a+}^{\alpha} \Phi(t)) \leq \lambda_\phi \Phi(t);$$

(H_6). *There exists* $q \in C(\Omega, [0, \infty))$ *such that for each* $t \in \Omega$,

$$p(t) \leq q(t)\Phi(t).$$

Then, IVP (1.2) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Consider the operator $\Lambda : C_{\gamma,\rho} \to C_{\gamma,\rho}$ defined in (3.12). Let x be a solution of inequality (3.10), and let us assume that $\bar{x}$ is a solution of IVP (1.2). Thus

$$\bar{x}(t) = \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{f(s, \bar{x}(s))}{\Gamma(\alpha)} ds.$$

From inequality (3.10), for each $t \in \Omega$, we have

$$\left| x(t) - \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{f(s, x(s))}{\Gamma(\alpha)} ds \right| \leq \Phi(t).$$

Set $q^* = \sup_{t\in\Omega} q(t)$. From the hypotheses (H5) and (H6), for each $t \in \Omega$, we get

$$\begin{aligned}
|x(t) - \bar{x}(t)| &\leq \left|x(t) - \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{f(s, x(s))}{\Gamma(\alpha)} ds\right| \\
&\quad + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{|f(s, x(s)) - f(s, \bar{x}(s))|}{\Gamma(\alpha)} ds \\
&\leq \Phi(t) + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{2q^*\Phi(s)}{\Gamma(\alpha)} ds \\
&\leq \Phi(t) + 2q^*({}^\rho I_{a+}^{\alpha} \Phi)(t) \\
&\leq \Phi(t) + 2q^*\lambda_\phi \Phi(t) \\
&= [1 + 2q^*\lambda_\phi]\Phi(t).
\end{aligned}$$

Thus

$$|x(t) - \bar{x}(t)| \leq \psi_\phi \Phi(t).$$

Hence, IVP (1.2) is generalized Ulam-Hyers-Rassias stable. $\square$

Define the metric

$$d(x, y) = \sup_{t\in\Omega} \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}|x(t) - y(t)|}{\Phi(t)}$$

in the space $C_{\gamma,\rho}(\Omega)$. Following fixed point theorem is used in our further result.

Theorem 3.16. [19] *Let $\Theta : C_{\gamma,\rho} \to C_{\gamma,\rho}$ be a strictly contractive operator with a Lipschitz constant $L < 1$. There exists a nonnegative integer k such that*

$$d(\Theta^{k+1}x, \Theta^k x) < \infty$$

for some $x \in C_{\gamma,\rho}$, then the following propositions hold true:
(A1) The sequence $\{\Theta^k x\}_{n\in\mathbb{N}}$ converges to a fixed point x^* of Θ;
(A2) x^* is a unique fixed point of Θ in $X = \{y \in C_{\gamma,\rho}(\Omega) : d(\Theta^k x, y) < \infty\}$;
(A3) If $y \in X$, then $d(y, x^*) \leq \frac{1}{1-L} d(y, \Theta x)$.

Theorem 3.17. *Assume that (H_5) and the following hypothesis hold:*

(H_7)**.** *There exists $\phi \in C(\Omega, [0,\infty))$ such that for each $t \in \Omega$, and all $x, \bar{x} \in \mathbb{R}$, we have*

$$|f(t,x) - f(t,\bar{x})| \leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \phi(t)\Phi(t)|x - \bar{x}|.$$

If

$$L = \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma} \phi^* \lambda_\phi < 1,$$

where $\phi^* = \sup_{t\in\Omega} \phi(t)$, then there exists a unique solution x_0 of IVP (1.2)*, and IVP* (1.2) *is generalized Ulam-Hyers-Rassias stable. Furthermore, we have*

$$|x(t) - \bar{x}(t)| \leq \frac{\Phi(t)}{1-L}.$$

Proof. Let $\Lambda : C_{\gamma,\rho} \to C_{\gamma,\rho}$ be the operator defined in (3.12). Apply Theorem 3.16, we have

$$\begin{aligned}
|(\Lambda x)(t) - (\Lambda \bar{x})(t)| &\leq \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{|f(s,x(s)) - f(s,\bar{x}(s))|}{\Gamma(\alpha)} ds \\
&\leq \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \phi(s) \\
&\quad \times \Phi(s) \frac{\left|\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma} x(s) - \left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma} \bar{x}(s)\right|}{\Gamma(\alpha)} ds \\
&\leq \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \phi^*(s)\Phi(s) \frac{\|x - \bar{x}\|_C}{\Gamma(\alpha)} ds \\
&\leq \phi^* ({}^\rho I^\alpha_{a+})\Phi(t)\|x - \bar{x}\|_C \\
&\leq \phi^* \lambda_\phi \Phi(t) \|x - \bar{x}\|_C.
\end{aligned}$$

Thus

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda \bar{x})\right| \leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma} \phi^* \lambda_\phi \Phi(t)\|x - \bar{x}\|_C.$$

Hence

$$d(\Lambda(x), \Lambda(\bar{x})) = \sup_{t\in\Omega} \frac{\|(\Lambda x)(t) - (\Lambda \bar{x})(t)\|_C}{\Phi(t)} \leq L\|x - \bar{x}\|_C$$

from which we conclude the theorem. $\square$

4. Examples

In this section we present some examples to illustrate our main results.

Example 4.1. Consider the following IVP involving generalized Katugampola fractional derivative:

$$\begin{cases} \left({}^\rho D_{a+}^{\frac{1}{2},\frac{1}{2}} x\right)(t) & = f(t,x,y); \qquad t \in [a,b], \\ \left({}^\rho I_{a+}^{\frac{1}{4}} x\right)(a) & = (1-a), \end{cases} \tag{4.1}$$

where $\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \rho > 0, \gamma = \frac{3}{4}, 0 < a < b \leq e$, and

$$\begin{cases} f(t,x,y) & = \frac{\theta (t-a)^{-\frac{1}{4}} \sin(t-a)}{64(1+\sqrt{t-a})(1+|x|+|y|)}; \quad t \in (a,b],\ x,y \in \mathbb{R}, \\ f(a,x,y) & = 0; \qquad x,y \in \mathbb{R}. \end{cases}$$

Clearly, function f is continuous for each $x, y \in \mathbb{R}$ and (H_2) is satisfied with

$$\begin{cases} p(t) & = \frac{\theta (t-a)^{-\frac{1}{4}} |\sin(t-a)|}{64(1+\sqrt{t-a})}; \quad 0 < \theta \leq 1,\ t \in (a,+\infty), \\ p(a) & = 0. \end{cases}$$

Thus, all the conditions of Theorem 3.6 are satisfied. Hence, IVP (4.1) has at least one solution defined on $[a, +\infty)$.

Also, we have

$$\begin{aligned} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} ({}^\rho I_{a+}^{\frac{1}{2}} p)(t) &= \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\frac{1}{4}} \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\frac{1}{2}} \frac{p(s)}{\Gamma(\frac{1}{2})} ds \\ &\leq \frac{1}{8} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\frac{1}{4}} \to 0 \quad \text{as } t \to +\infty. \end{aligned}$$

This implies that solutions of IVP (4.1) are locally asymptotically stable.

Example 4.2. Consider the following IVP involving generalized Katugampola derivative:

$$\begin{cases} \left({}^\rho D_{a+}^{\frac{1}{2},\frac{1}{2}} x\right)(t) & = f(t,x); \qquad t \in [a,b], \\ \left({}^\rho I_{a+}^{\frac{1}{4}} x\right)(a) & = (1-a), \end{cases} \tag{4.2}$$

where $\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \rho > 0, \gamma = \frac{3}{4}, 0 < a < b \leq e$, and

$$\begin{cases} f(t,x) & = \frac{\theta (t-a)^{-\frac{1}{4}} \sin(t-a)}{64(1+\sqrt{t-a})(1+|x|)}; \quad t \in (a,b],\ x \in \mathbb{R}, \\ f(a,x) & = 0; \qquad x \in \mathbb{R}. \end{cases}$$

Clearly, function f is continuous for all $x \in \mathbb{R}$ and (H_4) is satisfied with

$$\begin{cases} p(t) & = \frac{\theta (t-a)^{-\frac{1}{4}} |\sin(t-a)|}{64(1+\sqrt{t-a})}; \quad 0 < \theta \leq 1,\ t \in (a,b],\ x \in \mathbb{R}, \\ p(a) & = 0. \end{cases}$$

Hence, Theorem 3.14 implies that IVP (4.2) has at least one solution defined on $[a, b]$. Also, one can see that (H_5) is satisfied with

$$\Phi(t) = e^3, \quad \text{and} \quad \lambda_\phi = \frac{1}{\Gamma(\frac{3}{2})}.$$

Consequently, Theorem 3.15 implies that IVP (4.2) is generalized Ulam-Hyers-Rassias stable.

5. Concluding remarks

In this article, two IVPs involving generalized Katugampola fractional derivative are considered. The existence and local attractivity of solution is obtained for first IVP while Ulam-type stability of second IVP is obtained by using fixed point theorems. Both the results are supported with suitable illustrative examples.

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