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Choquet integral analytic inequalities

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Abstract. Based on an amazing result of Sugeno [15], we are able to transfer classic analytic integral inequalities to Choquet integral setting. We give Choquet integral inequalities of the following types: fractional-Polya, Ostrowski, fractional Ostrowski, Hermite-Hadamard, Simpson and Iyengar. We provide several examples for the involved distorted Lebesgue measure.

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1. Background

We need the following fractional calculus background:

Let $\alpha > 0$, $m = [\alpha]$ ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{1.1}$$

$a \le x \le b$. We define the subspace $C_{a+}^{\alpha}([a,b])$ of $C^m([a,b])$:

$$C_{a+}^{\alpha}([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{1.2}$$

For $f \in C_{a+}^{\alpha}([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^{\alpha} f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{1.3}$$

see [1], p. 24. Canavati first in [5] introduced the above over $[0,1]$.

Notice that $D_{a+}^{\alpha} f \in C([a,b])$.

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{1.4}$$

$x \in [a,b]$, see also [2], [9], [14]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{1.5}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{1.6}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

We need the following fractional Polya type (see [12], [13], p. 62) integral inequality without any boundary conditions.

Theorem 1.1. ([4], p. 4) *Let* $0 < \alpha < 1$, $f \in C([a,b])$. *Assume* $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ *and* $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. *Set*

$$M(f) := \max\left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \left\| D_{b-}^{\alpha} \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}. \tag{1.7}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq M(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\, 2^{\alpha}}. \tag{1.8}$$

*Inequality (*1.8*) is sharp, namely it is attained by*

$$f_*(x) = \left\{ \begin{array}{l} (x-a)^{\alpha},\ x \in \left[a, \frac{a+b}{2}\right], \\ (b-x)^{\alpha},\ x \in \left[\frac{a+b}{2}, b\right] \end{array} \right\},\ 0 < \alpha < 1. \tag{1.9}$$

The famous Ostrowski ([11]) inequality motivates this work and has as follows:

Theorem 1.2. *It holds*

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \|f'\|_{\infty}, \tag{1.10}$$

where $f \in C^1([a,b])$, $x \in [a,b]$, *and it is a sharp inequality.*

Another motivation is author's next fractional result, see [3], p. 44:

Theorem 1.3. *Let* $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$ *(*$\lceil \cdot \rceil$ *ceiling of the number),* $f \in AC^m([a,b])$ *(i.e.* $f^{(m-1)}$ *is absolutely continuous), and* $\left\| \overline{D}_{x_0-}^{\alpha} f \right\|_{\infty, [a, x_0]}$, $\left\| \overline{D}_{*x_0}^{\alpha} f \right\|_{\infty, [x_0, b]} < \infty$ *(where* $\overline{D}_{x_0-}^{\alpha} f, \overline{D}_{*x_0}^{\alpha} f$ *are the right (*[2]*) and left (*[8], *p. 50) Caputo fractional derivatives of* f *of order* α*, respectively),* $x_0 \in [a,b]$. *Assume* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right| \leq \frac{1}{(b-a)\, \Gamma(\alpha+2)}$$

$$\cdot \left\{ \left\| \overline{D}_{x_0-}^{\alpha} f \right\|_{\infty,[a,x_0]} (x_0 - a)^{\alpha+1} + \left\| \overline{D}_{*x_0}^{\alpha} f \right\|_{\infty,[x_0,b]} (b - x_0)^{\alpha+1} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha+2)} \max \left\{ \left\| \overline{D}_{x_0-}^{\alpha} f \right\|_{\infty,[a,x_0]}, \left\| \overline{D}_{*x_0}^{\alpha} f \right\|_{\infty,[x_0,b]} \right\} (b-a)^{\alpha}. \tag{1.11}$$

In the next assume that $(X, \mathcal{F})$ is a measurable space and $(\mathbb{R}^+)$ $\mathbb{R}$ is the set of all (nonnegative) real numbers.

We recall some concepts and some elementary results of capacity and the Choquet integral [6, 7].

Definition 1.4. A set function $\mu : \mathcal{F} \to \mathbb{R}^+$ is called a non-additive measure (or capacity) if it satisfies

(1) $\mu(\varnothing) = 0$;

(2) $\mu(A) \leq \mu(B)$ for any $A \subseteq B$ and $A, B \in \mathcal{F}$.

The non-additive measure μ is called concave if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \tag{1.12}$$

for all $A, B \in \mathcal{F}$. In the literature the concave non-additive measure is known as submodular or 2-alternating non-additive measure. If the above inequality is reverse, μ is called convex. Similarly, convexity is called supermodularity or 2-monotonicity, too.

First note that the Lebesgue measure λ for an interval $[a, b]$ is defined by $\lambda([a, b]) = b - a$, and that given a distortion function m, which is increasing (or non-decreasing) and such that $m(0) = 0$, the measure $\mu(A) = m(\lambda(A))$ is a distorted Lebesgue measure. We denote a Lebesgue measure with distortion m by $\mu = \mu_m$. It is known that μ_m is concave (convex) if m is a concave (convex) function.
The family of all the nonnegative, measurable function $f : (X, \mathcal{F}) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ is denoted as L_∞^+, where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field of $\mathbb{R}^+$. The concept of the integral with respect to a non-additive measure was introduced by Choquet [6].

Definition 1.5. Let $f \in L_\infty^+$. The Choquet integral of f with respect to non-additive measure μ on $A \in \mathcal{F}$ is defined by

$$(C) \int_A f d\mu := \int_0^\infty \mu(\{x : f(x) \geq t\} \cap A) \, dt, \tag{1.13}$$

where the integral on the right-hand side is a Riemann integral.

Instead of $(C) \int_X f d\mu$, we shall write $(C) \int f d\mu$. If $(C) \int f d\mu < \infty$, we say that f is Choquet integrable and we write

$$L_C^1(\mu) = \left\{ f : (C) \int f d\mu < \infty \right\}.$$

The next lemma summarizes the basic properties of Choquet integrals [7].

Lemma 1.6. *Assume that* $f, g \in L_C^1(\mu)$.

(1) $(C) \int 1_A d\mu = \mu(A)$*,* $A \in \mathcal{F}$*.*

(2) (Positive homogeneity) For all $\lambda \in \mathbb{R}^{+}$*, we have*

$$(C)\int \lambda f d\mu = \lambda \cdot (C)\int f d\mu.$$

(3) (Translation invariance) For all $c \in \mathbb{R}$*, we have*

$$(C)\int (f+c)\, d\mu = (C)\int f d\mu + c.$$

(4) (Monotonicity in the integrand) If $f \leq g$*, then we have*

$$(C)\int f d\mu \leq (C)\int g d\mu.$$

(Monotonicity in the set function) If $\mu \leq \nu$*, then we have*

$$(C)\int f d\mu \leq (C)\int f d\nu.$$

(5) (Subadditivity) If μ *is concave, then*

$$(C)\int (f+g)\, d\mu \leq (C)\int f d\mu + (C)\int g d\mu.$$

(Superadditivity) If μ *is convex, then*

$$(C)\int (f+g)\, d\mu \geq (C)\int f d\mu + (C)\int g d\mu.$$

(6) (Comonotonic additivity) If f *and* g *are comonotonic, then*

$$(C)\int (f+g)\, d\mu = (C)\int f d\mu + (C)\int g d\mu,$$

where we say that f *and* g *are comonotonic, if for any* $x, x' \in X$*, then*

$$(f(x) - f(x'))\,(g(x) - g(x')) \geq 0.$$

We next mention the amazing result from [15], which permits us to compute the Choquet integral when the non-additive measure is a distorted Lebesgue measure.

Theorem 1.7. *Let* f *be a nonnegative and measurable function on* $\mathbb{R}^{+}$ *and* $\mu = \mu_m$ *be a distorted Lebesgue measure. Assume that* $m(x)$ *and* $f(x)$ *are both continuous and* $m(x)$ *is differentiable. When* f *is an increasing (non-decreasing) function on* $\mathbb{R}^{+}$*, the Choquet integral of* f *with respect to* μ_m *on* $[0,t]$ *is represented as*

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(t-x)\, f(x)\, dx, \tag{1.14}$$

however, when f *is a decreasing (non-increasing) function on* $\mathbb{R}^{+}$*, the Choquet integral of* f *is*

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(x)\, f(x)\, dx. \tag{1.15}$$

2. Main results

From now on we assume that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone continuous function, and $\mu = \mu_m$ i.e. $\mu(A) = m(\lambda(A))$, denotes a distorted Lebesgue measure where m is such that $m(0) = 0$, m is increasing (non-decreasing) and continuously differentiable.

By Theorem 1.7 and mean value theorem for integrals we get:

i) If f is an increasing (non-decreasing) function on $\mathbb{R}^+$, we have

$$(C)\int_{[0,t]} f d\mu_m \overset{(1.14)}{=} \int_0^t m'(t-x) f(x)\,dx$$
$$= m'(t-\xi)\int_0^t f(x)\,dx, \text{ where } \xi \in (0,t). \tag{2.1}$$

ii) If f is a decreasing (non-increasing) function on $\mathbb{R}^+$, we have

$$(C)\int_{[0,t]} f d\mu_m \overset{(1.15)}{=} \int_0^t m'(x) f(x)\,dx = m'(\xi)\int_0^t f(x)\,dx, \tag{2.2}$$

where $\xi \in (0,t)$.

We denote by

$$\gamma(t,\xi) := \begin{cases} m'(t-\xi), \text{ when } f \text{ is increasing (non-decreasing)} \\ m'(\xi), \text{ when } f \text{ is decreasing (non-increasing),} \end{cases} \tag{2.3}$$

for some $\xi \in (0,t)$ per case.

We give the following Choquet-fractional-Polya inequality:

Theorem 2.1. *Let $0 < \alpha < 1$, $f = f|_{[0,t]}$, $t \in \mathbb{R}^+$, all considered as above in this section. Assume further that $f \in C^{\alpha}_{0+}\left(\left[0,\frac{t}{2}\right]\right)$ and $f \in C^{\alpha}_{t-}\left(\left[\frac{t}{2},t\right]\right)$. Set*

$$M^*(f)(t) := \max\left\{\left\|D^{\alpha}_{0+}f\right\|_{\infty,\left[0,\frac{t}{2}\right]}, \left\|D^{\alpha}_{t-}f\right\|_{\infty,\left[\frac{t}{2},t\right]}\right\}. \tag{2.4}$$

Then

$$(C)\int_{[0,t]} f d\mu_m \le \gamma(t,\xi) M^*(f)(t)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)2^{\alpha}}. \tag{2.5}$$

Proof. By Theorem 1.1 and earlier comments. □

Usual Polya inequality with ordinary derivative requires boundary conditions making a Choquet-Polya inequality impossible.

We give applications:

Remark 2.2. i) If $m(t) = \frac{t}{1+t}$, $t \in \mathbb{R}^+$, then $m(0) = 0$, $m(t) \ge 0$, $m'(t) = \frac{1}{(1+t)^2} > 0$, and m is increasing. Then $\gamma(t,\xi) \le 1$.

ii) If $m(t) = 1 - e^{-t} \ge 0$, $t \in \mathbb{R}^+$, then $m(0) = 0$, $m'(t) = e^{-t} > 0$, and m is increasing. Then $\gamma(t,\xi) \le 1$.

iii) If $m(t) = e^t - 1 \ge 0$, $t \in \mathbb{R}^+$, $m(0) = 0$, $m'(t) = e^t > 0$, and m is increasing. Then $\gamma(t,\xi) \le e^t$.

iv) If $m(t) = \sin t$, for $t \in \left[0,\frac{\pi}{2}\right]$, we get $m(0) = 0$, $m'(t) = \cos t \ge 0$, and m is increasing. Then $\gamma(t,\xi) \le 1$.

We continue with the Choquet-Ostrowski type inequalities:

Theorem 2.3. *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone continuous function, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and is twice continuously differentiable on $\mathbb{R}^+$. Here $0 \le x_0 \le t \in \mathbb{R}^+$. Then*

1)

$$\left| \frac{1}{t} (C) \int_{[0,t]} f d\mu_m - m'(t - x_0) f(x_0) \right|$$
$$\le \left(\frac{1}{4} + \frac{\left(x_0 - \frac{t}{2}\right)^2}{t^2} \right) t \left\| \left(m'(t - \cdot) f\right)' \right\|_{\infty,[0,t]}, \tag{2.6}$$

if f is an increasing function on $\mathbb{R}^+$,
and

2)

$$\left| \frac{1}{t} (C) \int_{[0,t]} f d\mu_m - m'(x_0) f(x_0) \right|$$
$$\le \left(\frac{1}{4} + \frac{\left(x_0 - \frac{t}{2}\right)^2}{t^2} \right) t \left\| \left(m' f\right)' \right\|_{\infty,[0,t]}, \tag{2.7}$$

if f is a decreasing function on $\mathbb{R}^+$.

Proof. By (1.10) we have that ($x_0 \in [0, t]$)

$$\left| \frac{1}{t} (C) \int_{[0,t]} f d\mu_m - m'(t - x_0) f(x_0) \right|$$
$$\overset{(1.14)}{=} \left| \frac{1}{t} \int_0^t m'(t - x) f(x) dx - m'(t - x_0) f(x_0) \right|$$
$$\le \left(\frac{1}{4} + \frac{\left(x_0 - \frac{t}{2}\right)^2}{t^2} \right) t \left\| \left(m'(t - \cdot) f\right)' \right\|_{\infty,[0,t]}, \tag{2.8}$$

when f is an increasing function on $\mathbb{R}^+$.

Also we have that

$$\left| \frac{1}{t} (C) \int_{[0,t]} f d\mu_m - m'(x_0) f(x_0) \right|$$
$$\overset{(1.15)}{=} \left| \frac{1}{t} \int_0^t m'(x) f(x) dx - m'(x_0) f(x_0) \right|$$
$$\overset{(1.10)}{\le} \left(\frac{1}{4} + \frac{\left(x_0 - \frac{t}{2}\right)^2}{t^2} \right) t \left\| \left(m' f\right)' \right\|_{\infty,[0,t]}, \tag{2.9}$$

when f is a decreasing function on $\mathbb{R}^+$. □

We make

Remark 2.4. (continuing from Remark 2.2) Assuming m is twice continuously differentiable is quite natural. Indeed:

i) If $m(t) = \frac{t}{1+t}$, $t \in \mathbb{R}^+$, then $m'(t) = (1+t)^{-2}$, $m''(t) = -2(1+t)^{-3}$, $m^{(3)}(t) = 6(1+t)^{-4}$, $m^{(4)}(t) = -24(1+t)^{-5}$, etc, all higher order derivatives exist and are continuous.

ii) If $m(t) = 1 - e^{-t}$, $t \in \mathbb{R}^+$, then $m'(t) = e^{-t}$, $m''(t) = -e^{-t}$, $m^{(3)}(t) = e^{-t}$, $m^{(4)}(t) = -e^{-t}$, etc, all higher order derivatives exist and are continuous.

iii) If $m(t) = e^t - 1$, $t \in \mathbb{R}^+$, then $m^{(i)}(t) = e^t$, $i = 1, 2, ...$, all derivatives exist and are continuous.

iv) If $m(t) = \sin t$, $t \in \left[0, \frac{\pi}{2}\right]$, then $m'(t) = \cos t$, $m''(t) = -\sin t$, $m^{(3)}(t) = -\cos t$, $m^{(4)}(t) = \sin t$, etc, all derivatives exist and are continuous.

We continue with fractional Choquet-Ostrowski type inequalities.

Theorem 2.5. *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function, μ_m is a distorted Lebesgue measure and $0 \le x_0 \le t \in \mathbb{R}^+$.*
Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $(m'(t-\cdot) f) \in AC^m([0,t])$, and $\left\| \overline{D}^{\alpha}_{x_0-} (m'(t-\cdot) f) \right\|_{\infty,[0,x_0]}$, $\left\| \overline{D}^{\alpha}_{*x_0} (m'(t-\cdot) f) \right\|_{\infty,[x_0,t]} < \infty$. Assume $(m'(t-\cdot) f)^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. Then

$$\begin{aligned} &\left| \frac{1}{t} (C) \int_{[0,t]} f d\mu_m - m'(t-x_0) f(x_0) \right| \\ &\le \frac{1}{t\Gamma(\alpha+2)} \left\{ \left\| \overline{D}^{\alpha}_{x_0-} (m'(t-\cdot) f) \right\|_{\infty,[0,x_0]} x_0^{\alpha+1} \right. \\ &\left. + \left\| \overline{D}^{\alpha}_{*x_0} (m'(t-\cdot) f) \right\|_{\infty,[x_0,t]} (t-x_0)^{\alpha+1} \right\} \qquad (2.10) \\ &\le \frac{t^\alpha}{\Gamma(\alpha+2)} \max \left\{ \left\| \overline{D}^{\alpha}_{x_0-} (m'(t-\cdot) f) \right\|_{\infty,[0,x_0]}, \left\| \overline{D}^{\alpha}_{*x_0} (m'(t-\cdot) f) \right\|_{\infty,[x_0,t]} \right\}. \end{aligned}$$

Proof. By Theorem 1.3. ☐

Theorem 2.6. *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing continuous function, μ_m is a distorted Lebesgue measure and $0 \le x_0 \le t \in \mathbb{R}^+$. Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $(m'f) \in AC^m([0,t])$, and $\left\| \overline{D}^{\alpha}_{x_0-} (m'f) \right\|_{\infty,[0,x_0]}$, $\left\| \overline{D}^{\alpha}_{*x_0} (m'f) \right\|_{\infty,[x_0,t]} < \infty$. Assume $(m'f)^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. Then*

$$\begin{aligned} &\left| \frac{1}{t} (C) \int_{[0,t]} f d\mu_m - m'(x_0) f(x_0) \right| \\ &\le \frac{1}{t\Gamma(\alpha+2)} \left\{ \left\| \overline{D}^{\alpha}_{x_0-} (m'f) \right\|_{\infty,[0,x_0]} x_0^{\alpha+1} + \left\| \overline{D}^{\alpha}_{*x_0} (m'f) \right\|_{\infty,[x_0,t]} (t-x_0)^{\alpha+1} \right\} \\ &\le \frac{t^\alpha}{\Gamma(\alpha+2)} \max \left\{ \left\| \overline{D}^{\alpha}_{x_0-} (m'f) \right\|_{\infty,[0,x_0]}, \left\| \overline{D}^{\alpha}_{*x_0} (m'f) \right\|_{\infty,[x_0,t]} \right\}. \qquad (2.11) \end{aligned}$$

Proof. By Theorem 1.3. ☐

We need the well-known Hermite-Hadamard inequality:

Theorem 2.7. *Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function, $[a,b] \subset \mathbb{R}$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{2.12}$$

We give the following Choquet-Hermite-Hadamard inequalities:

Theorem 2.8. *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone continuous convex function, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and continuously differentiable on $\mathbb{R}^+$. Here $[a,b] \subseteq \mathbb{R}^+$. Then*

i) If f is decreasing, we have that

$$m'(\xi) f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}(C)\int_{[a,b]} f(x)\,d\mu_m(x) \leq m'(\xi)\frac{f(a)+f(b)}{2}, \tag{2.13}$$

for some $\xi \in (0, b-a)$.

ii) If f is increasing, we have that

$$m'(b-a-\psi) f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}(C)\int_{[a,b]} f(x)\,d\mu_m(x)$$
$$\leq m'(b-a-\psi)\frac{f(a)+f(b)}{2}, \tag{2.14}$$

for some $\psi \in (0, b-a)$.

Proof. Let f be a convex function from $[a,b] \subset \mathbb{R}^+$ into $\mathbb{R}^+$. Let $t_1, t_2 \in [0, b-a]$, these are of the form $t_1 = x - a$, $t_2 = y - a$, where $x, y \in [a,b]$.

Consider ($\lambda \in (0,1)$)

$$\begin{aligned} f(a + \lambda t_1 + (1-\lambda)t_2) &= f(a + \lambda(x-a) + (1-\lambda)(y-a)) \\ &= f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \\ &= \lambda f(a + x - a) + (1-\lambda) f(a + y - a) \\ &= \lambda f(a + t_1) + (1-\lambda) f(a + t_2), \end{aligned}$$

proving that $f(a + \cdot)$ is convex over $[0, b-a]$.

Also it holds

$$(C)\int_{[a,b]} f(x)\,d\mu_m(x) = (C)\int_{[0,b-a]} f(a+x)\,d\mu_m(x). \tag{2.15}$$

Clearly, if f is increasing over $[a,b]$, then $f(a+\cdot)$ is increasing on $[0,b-a]$, and vice verca. And if f is decreasing over $[a,b]$, then $f(a+\cdot)$ is decreasing on $[0,b-a]$, and vice verca.

i) If f is decreasing, then

$$(C)\int_{[0,b-a]} f(a+x)\,d\mu_m(x) \overset{(1.15)}{=} \int_0^{b-a} m'(x) f(a+x)\,dx$$
$$= m'(\xi)\int_0^{b-a} f(a+x)\,dx, \quad \text{for some } \xi \in (0, b-a). \tag{2.16}$$

By (2.12) we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_0^{b-a} f(a+x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{2.17}$$

and then

$$f\left(\frac{a+b}{2}\right) m'(\xi) \leq \frac{m'(\xi)}{b-a}\int_0^{b-a} f(a+x)\,dx \leq \left(\frac{f(a)+f(b)}{2}\right) m'(\xi). \tag{2.18}$$

That is we proved (by (2.15), (2.16))

$$f\left(\frac{a+b}{2}\right) m'(\xi) \leq \frac{(C)\int_{[a,b]} f(x)\,d\mu_m(x)}{b-a} \leq \left(\frac{f(a)+f(b)}{2}\right) m'(\xi), \tag{2.19}$$

for some $\xi \in (0, b-a)$.

ii) If f is increasing, then

$$(C)\int_{[0,b-a]} f(a+x)\,d\mu_m(x) \overset{(1.14)}{=} \int_0^{b-a} m'(b-a-x) f(a+x)\,dx$$

$$= m'(b-a-\psi)\int_0^{b-a} f(a+x)\,dx, \quad \text{for some } \psi \in (0, b-a). \tag{2.20}$$

Again by (2.12) we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_0^{b-a} f(a+x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{2.21}$$

and

$$f\left(\frac{a+b}{2}\right) m'(b-a-\psi) \leq \frac{m'(b-a-\psi)}{b-a}\int_0^{b-a} f(a+x)\,dx$$

$$\leq \left(\frac{f(a)+f(b)}{2}\right) m'(b-a-\psi). \tag{2.22}$$

That is we proved (by (2.15), (2.20))

$$f\left(\frac{a+b}{2}\right) m'(b-a-\psi) \leq \frac{(C)\int_{[a,b]} f(x)\,d\mu_m(x)}{b-a}$$

$$\leq \left(\frac{f(a)+f(b)}{2}\right) m'(b-a-\psi), \tag{2.23}$$

for some $\psi \in (0, b-a)$. ☐

We need the well-known Simpson inequality:

Theorem 2.9. *If $f : [a,b] \to \mathbb{R}$ is four times continuously differentiable on (a,b) and*

$$\left\| f^{(4)} \right\|_\infty = \sup_{x\in(a,b)} \left| f^{(4)}(x) \right| < \infty,$$

then the Simpson inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4. \tag{2.24}$$

We give the corresponding Choquet-Simpson inequalities:

Theorem 2.10. *Here $f:\mathbb{R}^+\to\mathbb{R}^+$ is a monotone function which is four times continuously differentiable on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0)=0$, m is increasing and five times continuously differentiable on $\mathbb{R}^+$, $t\in\mathbb{R}^+$. Then*

i) if f is increasing, we have that

$$\left|\frac{1}{t}(C)\int_{[0,t]} f d\mu_m - \frac{1}{3}\left[\frac{m'(t)f(0)+m'(0)f(t)}{2} + 2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\leq \frac{1}{2880}\left\|\left(m'(t-\cdot)f\right)^{(4)}\right\|_{\infty,[0,t]} t^4, \tag{2.25}$$

and

ii) if f is decreasing, we have that

$$\left|\frac{1}{t}(C)\int_{[0,t]} f d\mu_m - \frac{1}{3}\left[\frac{m'(0)f(0)+m'(t)f(t)}{2} + 2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\leq \frac{1}{2880}\left\|\left(m'f\right)^{(4)}\right\|_{\infty,[0,t]} t^4. \tag{2.26}$$

Proof. i) If f is increasing, then

$$\left|\frac{1}{t}(C)\int_{[0,t]} f d\mu_m - \frac{1}{3}\left[\frac{m'(t)f(0)+m'(0)f(t)}{2} + 2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\overset{(1.14)}{=} \left|\frac{1}{t}\int_0^t m'(t-x)f(x)\,dx - \frac{1}{3}\left[\frac{m'(t)f(0)+m'(0)f(t)}{2} + 2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\overset{(2.24)}{\leq} \frac{1}{2880}\left\|\left(m'(t-\cdot)f\right)^{(4)}\right\|_{\infty,[0,t]} t^4. \tag{2.27}$$

ii) If f is decreasing, then

$$\left|\frac{1}{t}(C)\int_{[0,t]} f d\mu_m - \frac{1}{3}\left[\frac{m'(0)f(0)+m'(t)f(t)}{2} + 2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\overset{(1.15)}{=} \left|\frac{1}{t}\int_0^t m'(x)f(x)\,dx - \frac{1}{3}\left[\frac{m'(0)f(0)+m'(t)f(t)}{2} + 2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\overset{(2.24)}{\leq} \frac{1}{2880}\left\|\left(m'f\right)^{(4)}\right\|_{\infty,[0,t]} t^4. \tag{2.28}$$

$\square$

We need the famous Iyengar inequality [10] coming next:

Theorem 2.11. *Let f be a differentiable function on $[a,b] \subset \mathbb{R}$ and $|f'(x)| \leq M_1$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M_1 (b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M_1}. \quad (2.29)$$

We present the corresponding Choquet-Iyengar inequalities:

Theorem 2.12. *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0)=0$, m is increasing and twice continuously differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. Then*

i) if f is increasing and $\left|(m'(t-\cdot)f)'(x)\right| \leq M_2$, $\forall\, x \in [0,t]$, $M_2 > 0$, we have that

$$\left| (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}(m'(t)f(0)+m'(0)f(t)) \right|$$
$$\leq \frac{M_2 t^2}{4} - \frac{(m'(0)f(t)-m'(t)f(0))^2}{4M_2}. \quad (2.30)$$

ii) if f is decreasing and $\left|(m'f)'(x)\right| \leq M_3$, $\forall\, x \in [0,t]$, $M_3 > 0$, we have that

$$\left| (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}(m'(0)f(0)+m'(t)f(t)) \right|$$
$$\leq \frac{M_3 t^2}{4} - \frac{(m'(t)f(t)-m'(0)f(0))^2}{4M_3}. \quad (2.31)$$

Proof. i) If f is increasing and $\left|(m'(t-\cdot)f)'(x)\right| \leq M_2$, $\forall\, x \in [0,t]$, then

$$\left| (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}(m'(t)f(0)+m'(0)f(t)) \right|$$
$$\overset{(1.14)}{=} \left| \int_0^t m'(t-x)f(x)\,dx - \frac{t}{2}(m'(t)f(0)+m'(0)f(t)) \right|$$
$$\overset{(2.29)}{\leq} \frac{M_2 t^2}{4} - \frac{(m'(0)f(t)-m'(t)f(0))^2}{4M_2}. \quad (2.32)$$

ii) If f is decreasing and $\left|(m'f)'(x)\right| \leq M_3$, $\forall\, x \in [0,t]$, then

$$\left| (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}(m'(0)f(0)+m'(t)f(t)) \right|$$
$$\overset{(1.15)}{=} \left| \int_0^t m'(x)f(x)\,dx - \frac{t}{2}(m'(0)f(0)+m'(t)f(t)) \right|$$
$$\overset{(2.29)}{\leq} \frac{M_3 t^2}{4} - \frac{(m'(t)f(t)-m'(0)f(0))^2}{4M_3}. \qquad \square$$

Note 2.13. One can transfer many analytic integral classic inequalities to Choquet integral setting but we choose to stop here.

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