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The study of the solution of a Fredholm-Volterra integral equation by Picard operators

Maria Dobriţoiu

Abstract. In this paper we will use the Picard operators technique, in order to establish the existence and uniqueness, data dependence and Gronwall-type results for the solutions of a Fredholm-Volterra functional-integral equation. The paper ends with a result of the Ulam-Hyers stability of this integral equation.

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1. Introduction

The theory of integral equations has many applications in describing of numerous phenomena and problems from different research fields of the surrounding world, such as: mathematical physics, engineering, biology, economics and others. In what follows, we consider the following Fredholm-Volterra functional-integral equation:

$$x(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)), \tag{1.1}$$

where we denote:

$$I_{Fr}(t, s, a, b, x, K_1, h_1) = \int_a^b K_1(t, s) \cdot h_1(s, x(s), x(a), x(b))ds$$
$$I_{Vo}(t, s, a, x, K_2, h_2) = \int_a^t K_2(t, s) \cdot h_2(s, x(s), x(a))ds$$

and

$$F : [a, b] \times \mathbb{R}^3 \to \mathbb{R},\ K_1, K_2 : [a, b] \times [a, b] \to \mathbb{R},$$
$$h_1 : [a, b] \times \mathbb{R}^3 \to \mathbb{R},\ h_2 : [a, b] \times \mathbb{R}^2 \to \mathbb{R},\ g : [a, b] \times \mathbb{R} \to \mathbb{R},$$

and we will apply the Picard operators technique to prove the existence and uniqueness, data dependence, comparison and Gronwall-type results for the solution of the equation (1.1). Many authors have applied this technique to study the functional-integral equations of mixed type (see [1], [2], [6], [9], [19], [27], etc.). Also, many authors studied the functional-integral equations of Fredholm and Volterra type and we mention some of them (see [1], [3], [7], [8], [10], [11], [12], [13], [14], [16], [17], [18] [23], [24], [25], [26], [28], etc.).

In this paper we will use the notations from [22], [23] and [25] and we recall some of them.

Let (X, d) be a metric space and $A : X \to X$ an operator. We have:

$P(X) := \{Y \subset X \ / \ Y \neq \emptyset\} -$ the set of all nonempty subsets of X,

$I(A) := \{Y \in P(X) \ / \ A(Y) \subset Y\} -$ the family of the nonempty subsets of X, invariant for A,

$F_A := \{x \in X | A(x) = x\} -$ the fixed points set of A.

Also, we denote by $A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A \circ A^n,\ n \in N$ – the iterate operators of A.

Below, we present the definitions of Picard operator, c-Picard operator and weakly Picard operator.

Definition 1.1. Let (X, d) be a metric space. An operator $A : X \to X$ is called Picard operator (briefly PO) if there exists $x^* \in X$ such that:

(a) $F_A = \{x^*\}$;

(b) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.

Definition 1.2. Let (X, d) be a metric space and $c > 0$. An operator $A : X \to X$ is called c-Picard operator (briefly c-PO) if A is PO and

$$d(x, x^*) \leq c \cdot d(x, A(x)) \text{ for all } x \in X.$$

Definition 1.3. Let (X, d) be a metric space. An operator $A : X \to X$ is called weakly Picard operator (briefly WPO) if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.

If A is a WPO, then it can be considered the operator $A^\infty : X \to X$, defined by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x)$$

and we observe that $A^\infty(X) = F_A$.

In addition, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for all $x \in X$.

In the second section we study the existence and uniqueness of the solution of the integral equation (1.1).

In order to obtain the presented results of this section, we applied the Picard operators technique and the Contraction Principle.

Theorem 1.4 (Contraction Principle). *Let (X, d) be a complete metric space and $A : X \to X$ an α-contraction ($\alpha < 1$). Under these conditions we have:*

(i) $F_A = \{x^*\}$;

(ii) $x^* = \lim_{n\to\infty} A^n(x_0)$, *for all* $x_0 \in X$;

(iii) $d(x^*, A^n(x_0)) \leq \frac{\alpha^n}{1-\alpha} d(x_0, A(x_0))$.

In order to obtain several Gronwall-type and comparison results for the solution of the integral equation (1.1), in the third section we will use the Abstract Comparison Lemma, the Abstract Gronwall Lemma and the Abstract Gronwall-Comparison Lemma, which we present below.

Lemma 1.5. (*see* [25]) *Let* $(X, d, \leq)$ *be an ordered metric space and* $A : X \to X$ *an operator. If:*

(i) *A is an increasing operator;*
(ii) *the operator A is a WPO,*

then the operator A^∞ *is increasing.*

Lemma 1.6 (Abstract Comparison Lemma). (*see* [22], [23], [25]) *Let* $A, B, C : X \to X$ *be three operators defined on the ordered metric space* $(X, d, \leq)$*. If:*

(i) $A \leq B \leq C$*;*
(ii) *A, B, C are WPOs;*
(iii) *the operator B is increasing,*

then

$$x \leq y \leq z \Rightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

Remark 1.7. Let A, B, C be the operators defined in the Abstract Comparison Lemma. In addition, we suppose that B is PO, i.e. $F_B = \{x_B^*\}$. Then we have

$$A^\infty(x) \leq x_B^* \leq C^\infty(x), \text{ for all } x \in X.$$

But $A^\infty(X) = F_A$ and $C^\infty(X) = F_C$ and therefore $F_A \leq x_B^* \leq F_C$.

Lemma 1.8 (Abstract Gronwall Lemma). (*see* [22], [23], [25]) *Let* $A : X \to X$ *be an operator defined on the ordered metric space* $(X, d, \leq)$*. If:*

(i) *the operator A is PO and denote by* x_A^* *the unique fixed point of A;*
(ii) *A is an increasing operator,*

then

(a) $x \leq A(x) \Rightarrow x \leq x_A^*$*;*
(b) $x \geq A(x) \Rightarrow x \geq x_A^*$.

Lemma 1.9 (Abstract Gronwall-Comparison Lemma). (*see* [22], [23], [25]) *Let* $A_1, A_2 : X \to X$ *be two operators defined on the ordered metric space* $(X, d, \leq)$*. We assume that:*

(i) A_1 *is increasing;*
(ii) A_1 *and* A_2 *are POs;*
(iii) $A_1 \leq A_2$.

If we denote by x_2^* *the unique fixed point of* A_2*, then*

$$x \leq A_1(x) \Rightarrow x \leq x_2^*.$$

In the section 4 we prove a result of the continuous data dependence of the solution of the integral equation (1.1) using the General Data Dependence Theorem.

Theorem 1.10 (General Data Dependence Theorem). *Let (X,d) be a complete metric space, $A,B : X \to X$ two operators and suppose:*

(i) *A is c-PO with respect to the metric d and $F_A = \{x_A^*\}$;*
(ii) *there exists $x_B^* \in F_B$;*
(iii) *there exists $\eta > 0$ such that $d(A(x), B(x)) \le \eta$, for all $x \in X$.*

Under these conditions we have:

$$d(x_A^*, x_B^*) \le c \cdot \eta.$$

The last section of this paper contains a result concerning the Ulam-Hyers stability of the integral equation (1.1).

Definition 1.11. (I.A. Rus [21]) Let (X,d) be a metric space and $A : X \to X$ an operator. The equation of fixed point

$$x = A(x). \tag{1.2}$$

is Ulam-Hyers stable if there exists a real number $c_A > 0$ such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, A(y)) \le \varepsilon,$$

there exists a solution x^* of equation (1.2) such that

$$d(y^*, x^*) \le c_A \cdot \varepsilon.$$

Also, in this section we will use the Remark 2.1 from I.A. Rus [21], that you can find below.

Remark 1.12. (I.A. Rus [21], Remark 2.1) If A is a c-weakly Picard operator, then the fixed point equation (1.2) is Ulam-Hyers stable.

Indeed, let $\varepsilon > 0$ and y^* a solution of $d(y, A(y)) \le \varepsilon$. Since A is c-weakly Picard operator, we have that

$$d(x, A^\infty(x)) \le c \cdot d(x, A(x)), for\ all\ x \in X.$$

If we take $x := y^*$ and $x^* := A^\infty(y)$, then we have that $d(y^*, x^*) \le c_A \cdot \varepsilon$ (see [20], [21]).

2. Existence and uniqueness

In this section we present several results of existence and uniqueness for the solution of the integral equation (1.1). These results were obtained by applying the known standard techniques as in [1], [2], [5], [6] for particular integral equations.

We suppose that the following conditions are fulfilled:

(a_1) $K_1, K_2 \in C([a,b]\times[a,b])$, $h_1 \in C([a,b]\times\mathbb{R}^3)$, $h_2 \in C([a,b]\times\mathbb{R}^2)$, $g \in C([a,b]\times\mathbb{R})$;
(a_2) $F \in C([a,b] \times \mathbb{R}^3)$.

Theorem 2.1. *We assume that the conditions (a_1) and (a_2) are satisfied. In addition we assume that:*

(*i*) *there exist* $\alpha, \beta, \gamma > 0$, *such that:*

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \le \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a, b], u_i, v_i, w_i \in \mathbb{R}, i = 1, 2$;

(*ii*) *there exist* $L_1, L_2, L_3 > 0$ *such that:*

$$|h_1(s, u_1, u_2, u_3) - h_1(s, v_1, v_2, v_3)| \le L_1(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $s \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2, 3$;

$$|h_2(s, u_1, u_2) - h_2(s, v_1, v_2)| \le L_2(|u_1 - v_1| + |u_2 - v_2|),$$

for all $s \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2$;

$$|g(t, u) - g(t, v)| \le L_3|u - v|),$$

for all $t \in [a, b], u, v \in \mathbb{R}$;

(*iii*) $\alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a) < 1$,
where we denoted by M_1 *and* M_2 *respectively, two positive constants, such that* $|K_1(t, s)| \le M_1$ *and* $|K_2(t, s)| \le M_2$, *for all* $t, s \in [a, b]$.

*Under these conditions the integral equation (*1.1*) has a unique solution* $x^* \in C[a, b]$, *that can be obtained by the successive approximations method starting at any element* $x_0 \in C[a, b]$.
In addition, if x_n *is the n-th successive approximation, then we have:*

$$\|x^* - x_n\|_C \le \frac{[\alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)]^n}{1 - \alpha L_3 - (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)} \cdot \|x_0 - x_1\|_C. \tag{2.1}$$

Proof. Let $X = (C[a, b], \|\cdot\|_C)$ be a Banach space, where $\|\cdot\|_C$ is the Chebyshev's norm

$$\|x\|_C = \max_{t \in [a,b]} |x(t)|, \text{ for all } x \in C[a, b].$$

Also, we consider the operator $A : X \to X$, defined by the relation:

$$A(x)(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)) \tag{2.2}$$

for all $t \in [a, b]$.

The set of the solutions of the integral equation (1.1) coincides with the set of fixed points of the operator A. From Contraction Principle it results that the operator A must be a contraction. We have:

$$\begin{aligned}|A(x)(t) - A(y)(t)| &= |F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)) \\ &\quad - F(t, g(t, y(t)), I_{Fr}(t, s, a, b, y, K_1, h_1), I_{Vo}(t, s, a, y, K_2, h_2))|.\end{aligned}$$

From (i) and (ii) and using the Chebyshev's norm it results

$$\|A(x) - A(y)\|_{C[a,b]} \le [\alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)]\|x - y\|_{C[a,b]} \tag{2.3}$$

Consequently, from (iii) it results that the operator A is an L_A-contraction with the coefficient

$$L_A = \alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a).$$

Now, from Contraction Principle it results that the operator A has a unique fixed point $F_A = \{x^*\}$ and consequently, the integral equation (1.1) has a unique solution $x^* \in C[a, b]$; this solution can be obtained by the successive approximations method starting at any element $x_0 \in C[a,b]$ and, if x_n is the n-th successive approximation, then the estimation (2.1) is true. The proof is complete. □

Remark 2.2. In order to obtain the Theorem 2.1, of existence and uniqueness of the solution of the integral equation (1.1) in the space $C[a,b]$, we reduced the problem of determination of the solutions of this integral equation to a fixed point problem. Under the conditions of the Theorem 2.1, the operator A, defined by (2.2), is PO.

Remark 2.3. If we consider the Banach space $X = (C[a,b], \|\cdot\|_B)$, where $\|\cdot\|_B$ is the Bielecki's norm:

$$\|x\|_B = \max_{t\in[a,b]} |x(t)|e^{-\tau(t-a)},$$

for all $x \in C[a,b]$, and $\tau > 0$ a parameter, and the operator $A : X \to X$, defined by (2.2), then we have another theorem of existence and uniqueness of the solution of the integral equation (1.1) in the space $C[a,b]$, that we present below.

Theorem 2.4. *We assume that the conditions* (a_1) *and* (a_2) *are satisfied and also, the conditions (i) and (ii) from Theorem 2.1 are fulfilled. Under these conditions the integral equation (*1.1*) has a unique solution* $x^* \in C[a,b]$.

Proof. We have

$$|A(x)(t) - A(y)(t)| \le \alpha L_3 e^{\tau(t-a)}\|x-y\|_B + 3\frac{\beta M_1 L_1}{\tau}e^{\tau(t-a)}\|x-y\|_B + 2\frac{\gamma M_2 L_2}{\tau}e^{\tau(t-a+b-t)}\|x-y\|_B$$

and therefore, using the Bielecki's norm, we obtain:

$$\|A(x) - A(y)\|_B \le [\alpha L_3 + 3\frac{\beta M_1 L_1}{\tau} + 2\frac{\gamma M_2 L_2}{\tau}e^{\tau(b-a)}]\|x-y\|_B. \tag{2.4}$$

It is clear that one can find a positive parameter τ, such that

$$\alpha L_3 + 3\frac{\beta M_1 L_1}{\tau} + 2\frac{\gamma M_2 L_2}{\tau}e^{\tau(b-a)} < 1,$$

and thus A is an L_A-contraction with

$$L_A = \alpha L_3 + 3\frac{\beta M_1 L_1}{\tau} + 2\frac{\gamma M_2 L_2}{\tau}e^{\tau(b-a)}$$

and the conclusion of theorem is obtained by applying the Contraction Principle (Theorem 1.4). □

Example 2.5. The following equation is a particular case of the integral equation (1.1), when $g(t,x(t)) = x(t)$:

$$x(t) = F(t, x(t), I_{Fr}(t,s,a,b,x,K_1,h_1), I_{Vo}(t,s,a,x,K_2,h_2)), \tag{2.5}$$

where we used the same notations for I_{Fr} and I_{Vo} as at the beginning of the first section.

Let us consider this integral equation in the following hypotheses:

(i) $F \in C([a,b]\times\mathbb{R}^3)$, $K_1, K_2 \in C([a,b]\times[a,b])$, $h_1 \in C([a,b]\times\mathbb{R}^3)$, $h_2 \in C([a,b]\times\mathbb{R}^2)$;

(ii) there exist $\alpha, \beta, \gamma > 0$, such that:

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \leq \alpha |u_1 - u_2| + \beta |v_1 - v_2| + \gamma |w_1 - w_2|,$$

for all $t \in [a, b], u_i, v_i, w_i \in \mathbb{R}, i = 1, 2;$

(iii) there exist $L_1, L_2 > 0$, such that:

$$|h_1(s, u_1, u_2, u_3) - h_1(s, v_1, v_2, v_3)| \leq L_1(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $s \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2, 3;$

$$|h_2(s, u_1, u_2) - h_1(s, v_1, v_2)| \leq L_2(|u_1 - v_1| + |u_2 - v_2|),$$

for all $s \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2;$

(iv) $\alpha + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a) < 1$,
where we denoted by M_1 and M_2 respectively, two positive constants, such that $|K_1(t, s)| \leq M_1$ and $|K_2(t, s)| \leq M_2$, for all $t, s \in [a, b]$.

Then the integral equation (1.1) has a unique solution $x^* \in C[a, b]$, that can be obtained by the successive approximations method starting at any element $x_0 \in C[a, b]$. Moreover, if x_n is the n-th successive approximation, then we have:

$$\|x^* - x_n\|_C \leq \frac{[\alpha + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)]^n}{1 - \alpha - (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)} \cdot \|x_0 - x_1\|_C. \tag{2.6}$$

In order to prove this result, we applied the Theorem 2.1 in particular case of

$$g(t, x(t)) = x(t).$$

Remark 2.6. A similar result can be obtained for the solution of integral equation

$$x(t) = F(t, x(a), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)), \tag{2.7}$$

by applying the Theorem 2.1 in particular case of $g(t, x(t)) = x(a)$.

Remark 2.7. In the paper [9] has been studied the existence and uniqueness of the solution of nonlinear Fredholm-Volterra functional-integral equation:

$$x(t) = F(t, x(a), \int_a^b K_1(t, s, x(g_1(s)))ds, \int_a^t K_2(t, s, x(g_2(s)))ds). \tag{2.8}$$

3. Comparison results and Gronwall lemmas

We present below a comparison result and two Gronwall-type lemmas for the solution of the integral equation (1.1). These results have been obtained by using the Picard operators technique and applying the Abstract Comparison Lemma, the Abstract Gronwall Lemma and the Abstract Gronwall-Comparison Lemma as in [4], [5], [15] for particular operatorial equations.

In order to obtain a comparison result, we consider the integral equations:

$$x(t) = F_i(t, g(t, x(t)), I^i_{Fr}(t, s, a, b, x, K_1, h^i_1), I^i_{Vo}(t, s, a, x, K_2, h^i_2)), \tag{3.1}$$

where we denoted:

$$I^i_{Fr}(t,s,a,b,x,K_1,h^i_1) = \int_a^b K_1(t,s)\cdot h^i_1(s,x(s),x(a),x(b))ds$$
$$I^i_{Vo}(t,s,a,x,K_2,h^i_2) = \int_a^t K_2(t,s)\cdot h^i_2(s,x(s),x(a))ds$$

where

$$F_i \in C([a,b]\times\mathbb{R}^3),\ g\in C([a,b]\times\mathbb{R}),$$
$$K_1,K_2\in C([a,b]\times[a,b],\mathbb{R}_+),\ h^i_1\in C([a,b]\times\mathbb{R}^3),$$
$$h^i_2\in C([a,b]\times\mathbb{R}^2),\ i=1,2,3.$$

We have:

Theorem 3.1. *Suppose that:*

(*i*) *the functions* $F_i, g, K_1, K_2, h^i_1, h^i_2$, $i=1,2,3$ *satisfy the conditions of Theorem 2.1, and let* x^*_i *be the unique solution of the integral equation (*3.1*) corresponding to* F_i, h^i_1, h^i_2, $i=1,2,3$;
(*ii*) *the functions* $F_2(t,\cdot,\cdot,\cdot), h^2_1(t,\cdot,\cdot,\cdot), h^2_2(t,\cdot,\cdot)$ *are increasing;*
(*iii*) $F_1\le F_2\le F_3$, $h^1_1\le h^2_1\le h^3_1$ *and* $h^1_2\le h^2_2\le h^3_2$.

Then

$$x^*_1\le x^*_2\le x^*_3.$$

Proof. We consider the Banach space $X=(C[a,b],\|\cdot\|_C)$ and the operators $A_i : X\to X$, defined by the relation (2.2) corresponding to functions $F_i, g, K_1, K_2, h^i_1, h^i_2$, $i=1,2,3$:

$$A_i(x)(t) = F_i(t, g(t,x(t)), I^i_{Fr}(t,s,a,b,x,K_1,h^i_1), I^i_{Vo}(t,s,a,x,K_2,h^i_2)).$$

From condition (i) it results that the operators $A_i : X\to X$, $i=1,2,3$ are PO's and therefore each of these operators has a unique fixed point, $F_{A_i}=\{x^*_i\}$.
From condition (ii) we deduce that the operator A_2 is increasing and from condition (iii) we obtain that $A_1\le A_2\le A_3$.
Now, applying the Abstract Comparison Lemma (Lemma 1.6), it results that

$$x_1\le x_2\le x_3 \implies A^\infty_1(x_1)\le A^\infty_2(x_2)\le A^\infty_3(x_3),$$

but A_1,A_2,A_3 are PO's and then by Remark 1.7, the conclusion of this theorem follows, i.e. $x^*_1\le x^*_2\le x^*_3$. The proof is complete. □

For the solution of the integral equation (1.1) we present below, the following two Gronwall-type lemmas.

Theorem 3.2. *We suppose that:*

(*i*) $F\in C([a,b]\times\mathbb{R}^3)$, $K_1,K_2\in C([a,b]\times[a,b],\mathbb{R}_+)$, $h_1\in C([a,b]\times\mathbb{R}^3)$, $h_2\in C([a,b]\times\mathbb{R}^2)$, $g\in C([a,b]\times\mathbb{R})$;
(*ii*) F, K_1, K_2, h_1, h_2, g *satisfy the conditions (i)-(iii) of Theorem 2.1, and denote by* $x^*\in C[a,b]$ *the unique solution of the integral equation (*1.1*);*
(*iii*) $h_1(s,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}, h_2(s,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}$ *are increasing functions for all* $s\in[a,b]$;

(*iv*) *$F(t,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}$ is increasing function for all $t\in[a,b]$.*

Under these conditions, the following statements are true:

(*a*) *if x is a lower-solution of integral equation (1.1) then $x\le x^*$;*
(*b*) *if x is a upper-solution of integral equation (1.1) then $x\ge x^*$.*

Proof. We consider the operator $A: X\to X$, defined by (2.2). From conditions (i) and (ii) it results that this operator is PO and denote by x^* the unique fixed point of A. From the assumptions (i), (iii) and (iv) it results that the operator A is increasing.

Now, the conditions of the Abstract Gronwall Lemma (Lemma 1.8), being satisfied, it results that the conclusions of this theorem:

– if x is a lower-solution of the integral equation (1.1), i.e. $x\le A(x)$, then $x\le x^*$;
– if x is a upper-solution of the integral equation (1.1), i.e. $x\ge A(x)$, then $x\ge x^*$,

are true. The proof is complete. □

To obtain an effective Gronwall-type lemma, it can use the Abstract Gronwall-Comparison Lemma (Lemma 1.9), and we obtain a result that we present below.

Theorem 3.3. *We consider the integral equation (1.1) corresponding to F_i, g, K_1, K_2, h_1^i, h_2^i, for $i=1,2$. We assume that:*

(*i*) $F_i\in C([a,b]\times\mathbb{R}^3)$, $K_1,K_2\in C([a,b]\times[a,b],\mathbb{R}_+)$, $h_1^i\in C([a,b]\times\mathbb{R}^3)$, $h_2^i\in C([a,b]\times\mathbb{R}^2)$, $g\in C([a,b]\times\mathbb{R})$, $i=1,2$;
(*ii*) *$F_i,g,K_1,K_2,h_1^i,h_2^i$ satisfy the conditions (i)-(iii) of Theorem 2.1, for $i=1,2$;*
(*iii*) *$h_1^1(s,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}, h_2^1(s,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}$ are increasing functions for all $s\in[a,b]$;*
(*iv*) *$F_1(t,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}, g(t,\cdot):\mathbb{R}\to\mathbb{R}$ are increasing functions for all $t\in[a,b]$.*
(*v*) *$F_1\le F_2, h_1^1\le h_1^2$ and $h_2^1\le h_2^2$.*

If x is a solution of integral inequality

$$x(t)\le F_1(t,g(t,x(t)),I^1_{Fr}(t,s,a,b,x,K_1,h_1^1),I^1_{Vo}(t,s,a,x,K_2,h_2^1)),\tag{3.2}$$

where

$$I^1_{Fr}(t,s,a,b,x,K_1,h_1^1)=\int_a^b K_1(t,s)\cdot h_1^1(s,x(s),x(a),x(b))ds$$
$$I^1_{Vo}(t,s,a,x,K_2,h_2^1)=\int_a^t K_2(t,s)\cdot h_2^1(s,x(s),x(a))ds,$$

then $x\le x_2^*$, where x_2^* is the unique solution of integral equation (1.1) corresponding to $F_2,g,K_1,K_2,h_1^2,h_2^2$:

$$x(t)=F_2(t,g(t,x(t)),I^2_{Fr}(t,s,a,b,x,K_1,h_1^2),I^2_{Vo}(t,s,a,x,K_2,h_2^2)),$$

where

$$I^2_{Fr}(t,s,a,b,x,K_1,h_1^2)=\int_a^b K_1(t,s)\cdot h_1^2(s,x(s),x(a),x(b))ds$$
$$I^2_{Vo}(t,s,a,x,K_2,h_2^2)=\int_a^t K_2(t,s)\cdot h_2^2(s,x(s),x(a))ds.$$

Proof. We consider the operator A_1, A_2 defined by (2.2), corresponding to F_1, g, K_1, K_2, h_1^1, h_2^1 and F_2, g, K_1, K_2, h_1^2, h_2^2.

From Theorem 2.1 we have that A_1 and A_2 are POs, and we denote by x_i^* the unique fixed point of operator $A_i, i = 1, 2$.

From condition (ii) it results that A_1 is increasing and from condition (iii) we obtain that $A_1 \leq A_2$.

If x is a solution of (3.2), then $x \leq A_1(x)$.

Now, we apply the Abstract Gronwall-Comparison Lemma (Lemma 1.9), and we obtain the conclusion of the theorem. The proof is complete. □

4. Data dependence

In order to study the data dependence of the solution of the integral equation (1.1) we consider the following perturbed integral equation:

$$x(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, k_1), I_{Vo}(t, s, a, x, K_2, k_2)), \tag{4.1}$$

where

$$I_{Fr}(t, s, a, b, x, K_1, k_1) = \int_a^b K_1(t, s) \cdot k_1(s, x(s), x(a), x(b))ds$$
$$I_{Vo}(t, s, a, x, K_2, k_2) = \int_a^t K_2(t, s) \cdot k_2(s, x(s), x(a))ds$$

and

$$F : [a, b] \times \mathbb{R}^3 \to \mathbb{R}, \ K_1, K_2 : [a, b] \times [a, b] \to \mathbb{R},$$
$$k_1 : [a, b] \times \mathbb{R}^3 \to \mathbb{R}, \ k_2 : [a, b] \times \mathbb{R}^2 \to \mathbb{R}, \ g : [a, b] \times \mathbb{R} \to \mathbb{R}.$$

We have the following data dependence theorem of the solution of the integral equation (1.1):

Theorem 4.1. *Suppose that:*

(i) F, K_1, K_2, h_1, h_2, g *satisfy the conditions of Theorem 2.1 and we denote by* $x^* \in C[a, b]$ *the unique solution of integral equation (1.1);*
(ii) $k_1 \in C([a, b] \times \mathbb{R}^3), k_2 \in C([a, b] \times \mathbb{R}^2)$*;*
(iii) there exists $\eta_1, \eta_2 > 0$ *such that*

$$|h_1(s, u, v, w) - k_1(s, u, v, w)| \leq \eta_1, \textit{for all } s \in [a, b], u, v, w \in \mathbb{R}, \textit{ and}$$
$$|h_2(s, u, v) - k_2(s, u, v)| \leq \eta_2, \textit{for all } s \in [a, b], u, v \in \mathbb{R}.$$

Under these conditions, if $y^* \in C[a, b]$ *is a solution of the integral equation (4.1), then we have:*

$$\|x^* - y^*\|_C \leq \frac{(M_1\eta_1 + M_2\eta_2)(b - a)}{1 - \alpha L_3 - (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)}. \tag{4.2}$$

Proof. We consider the operator from the proof of Theorem 2.1, $A : C[a,b] \to C[a,b]$, attached to integral equation (1.1) and defined by the relation (2.2):

$$A(x)(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)),$$

for all $t \in [a,b]$.

From condition (i) it results that the operator A is a L_A-contraction with the coefficient

$$L_A = \alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a)$$

(Theorem 2.1) and therefore, A is c-PO with $c = \frac{1}{1-L_A}$.

Also, we attach to the integral equation (4.1) the operator $B : C[a,b] \to C[a,b]$, defined by the relation:

$$B(x)(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, k_1), I_{Vo}(t, s, a, x, K_2, k_2)) \tag{4.3}$$

for all $t \in [a,b]$.

From conditions (i) and (ii) it results that the operator B is correctly defined.

The set of the solutions of the perturbed integral equation (4.1) in the space $C[a,b]$ coincides with the fixed points set of the operator B defined by the relation (4.3).

We have:

$$\begin{aligned}|A(x)(t) - B(x)(t)| = &|F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)) \\ &- F(t, g(t, y(t)), I_{Fr}(t, s, a, b, y, K_1, k_1), I_{Vo}(t, s, a, y, K_2, k_2))|\end{aligned}$$

and from condition (iii) it results that

$$|A(x)(t) - B(x)(t)| \leq (M_1\eta_1 + M_2\eta_2)(b-a), \ \textit{for all } t \in [a,b].$$

Now, using the Chebyshev's norm, we obtain:

$$\|A(x) - B(x)\|_C \leq (M_1\eta_1 + M_2\eta_2)(b-a) \tag{4.4}$$

and applying the General Data Dependence Theorem (Theorem 1.10), with

$$c = \frac{1}{1-L_A} \text{ and } \eta = (M_1\eta_1 + M_2\eta_2)(b-a),$$

it results the estimation (4.2). The proof is complete. □

5. Ulam-Hyers stability

Theorem 5.1. *Under the conditions of Theorem 2.1, the integral equation (1.1) is Ulam-Hyers stable, i.e. for $\varepsilon > 0$ and $y^* \in C[a,b]$ a solution of the inequation*

$$|y(t) - F(t, g(t, y(t)), I_{Fr}(t, s, a, b, y, K_1, h_1), I_{Vo}(t, s, a, y, K_2, h_2))| \leq \varepsilon$$

for all $t \in [a,b]$, there exists a solution of the integral equation (1.1), $x^* \in C([a,b]$, such that

$$|y^*(t) - x^*(t)| \leq \frac{1}{1-L_A}\varepsilon, \ \textit{for all } t \in [a,b],$$

where

$$L_A = \alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a).$$

Proof. We consider the operator A, defined by the relation (2.2). Under the conditions of Theorem 2.1, it results that the operator A is a contraction and therefore, A is c-PO with the constant $c = \frac{1}{1-L_A}$,

$$L_A = \alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a).$$

Now, the conclusion of this theorem is obtained as an application of the Remark 1.12 (I.A.Rus [21], Remark 2.1) anf the proof is complete. □

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Maria Dobriţoiu
University of Petroşani
Faculty of Sciences
20, Universităţii Street
302006 Petroşani, Romania
e-mail: mariadobritoiu@yahoo.com