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General inequalities related Hermite-Hadamard inequality for generalized fractional integrals

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Abstract. In this article, we first establish a new general integral identity for differentiable functions with the help of generalized fractional integral operators introduced by Raina [8] and Agarwal *et al.* [1]. As a second, by using this identity we obtain some new fractional Hermite-Hadamard type inequalities for functions whose absolute values of first derivatives are convex. Relevant connections of the results presented here with those involving Riemann-Liouville fractional integrals are also pointed out.

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1. Introduction and preliminaries

One of the most famous inequalities for convex functions is Hermite-Hadamard's inequality. This double inequality is stated as follows (see for example [3]).

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Definition 1.1. The function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$. We say that f is concave if $(-f)$ is convex.

Now, we will give some important definitions and mathematical preliminaries of fractional calculus theory which are used throughout of this paper.

Definition 1.2. [4] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du.$$

Here is $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In [5], Iqbal *et al.* proved a new identity for differentiable convex functions via Riemann-Liouville fractional integrals.

Lemma 1.3. *Let f:$[a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $f' \in L'[a,b]$, then the following identity for Riemann-Liouville fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] = \sum_{k=1}^{\infty} I_k,$$

where

$$I_1 = \int_0^{\frac{1}{2}} t^{\alpha} f'(tb + (1-t)a)\,dt, \qquad I_2 = \int_0^{\frac{1}{2}} (-t^{\alpha}) f'(ta + (1-t)b)\,dt,$$

$$I_3 = \int_{\frac{1}{2}}^{1} (t^{\alpha} - 1) f'(tb + (1-t)a)\,dt, \quad I_4 = \int_{\frac{1}{2}}^{1} (1 - t^{\alpha}) f'(ta + (1-t)b)\,dt.$$

By using the above identity, the authors obtained left-sided of Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals. Some other results related to those inequalities involving Riemann-Liouville fractional integrals can be found in the literature, for example, in [2, 7, 18, 16, 11] and the references therein.

In [8], Raina introduced a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\,\sigma(1),\,\cdots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathbb{R}) \tag{1.2}$$

where the coefficients $\sigma(k)$, $(k \in \mathbb{N} = \mathbb{N} \cup \{0\})$, is a bounded sequence of positive real numbers and $\mathbb{R}$ is the set of real numbers. With the help of (1.2), Raina [8] and Agarwal *et al.* [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} \varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[w(x-t)^{\rho}]\varphi(t)dt \ (x > a), \tag{1.3}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt \ (x<b) \tag{1.4}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits. It is easy to verify that $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x)$ and $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] < \infty. \tag{1.5}$$

In fact, for $\varphi \in L(a,b)$, we have

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)||_1 \le \mathfrak{M}(b-a)^{\lambda}||\varphi||_1 \tag{1.6}$$

and

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)||_1 \le \mathfrak{M}(b-a)^{\lambda}||\varphi||_1 \tag{1.7}$$

where

$$||\varphi||_p := \left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} of order α follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (1.3) and (1.4). Also, to see more results and generalizations for convex and some other several convex functions classes, as $Q(I), P(I), SX(h,I)$ and $r-$convex, involving generalized fractional integral operators, see [17, 14, 15, 10, 9, 13, 12, 19, 20] and references there in.

In this paper, we will prove a generalization of the identity given by Iqbal *et al.* in [5] by using generalized fractional integral operators. Then we will give some new Hermite-Hadamard type inequalities for fractional integral operators.

2. Main results

We start by giving a generalization of Lemma 1, [5]. We will use an abbreviation throughout of this study,

$$M_f(a,b;w;J) = F^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]f\left(\frac{a+b}{2}\right) - \frac{1}{2(b-a)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right]$$

that is similar to the symbol "$L_f(a,b;w;J)$" in [17].

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $\lambda > 0$. If $f' \in L[a,b]$, then the following equality for generalized fractional integral operators holds:*

$$M_f(a,b;w;J) = \frac{b-a}{2}(I_1 + I_2 + I_3 + I_4)$$

where I_1, I_2, I_3 and I_4 given in the (2.1), (2.2), (2.3) *and* (2.4), *respectively.*

Proof. Integrating by parts, we get

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}] f'(tb+(1-t)a)dt \\ &= t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}] \frac{f(tb+(1-t)a)}{b-a}\Bigg|_0^{\frac{1}{2}} \\ &- \int_0^{\frac{1}{2}} t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] \frac{f(tb+(1-t)a)}{b-a}dt \\ &= \frac{1}{b-a}\left(\frac{1}{2}\right)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] f\left(\frac{a+b}{2}\right) \\ &- \frac{1}{b-a}\int_0^{\frac{1}{2}} t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] f(tb+(1-t)a)dt. \end{aligned} \tag{2.1}$$

Analogously:

$$\begin{aligned} I_2 &= -\int_0^{\frac{1}{2}} t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}] f'(ta+(1-t)b)dt \\ &= \frac{1}{b-a}\left(\frac{1}{2}\right)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1} w\left[\left(\frac{b-a}{2}\right)^{\rho}\right] f\left(\frac{a+b}{2}\right) \\ &- \frac{1}{b-a}\int_0^{\frac{1}{2}} t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] f(ta+(1-t)b)dt \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} I_3 &= \int_{\frac{1}{2}}^1 \left[t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}] - \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]\right] f'(tb+(1-t)a)dt \\ &= t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}] \frac{f(tb+(1-t)a)}{b-a}\Bigg|_{\frac{1}{2}}^1 \\ &- \int_{\frac{1}{2}}^1 t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] \frac{f(tb+(1-t)a)}{b-a}dt \\ &- \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] \frac{f(tb+(1-t)a)}{b-a}\Bigg|_{\frac{1}{2}}^1 \\ &= \frac{1}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] f\left(\frac{a+b}{2}\right) \\ &- \frac{1}{b-a}\left(\frac{1}{2}\right)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] f\left(\frac{a+b}{2}\right) \\ &- \frac{1}{b-a}\int_{\frac{1}{2}}^1 t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] f(tb+(1-t)a)dt. \end{aligned} \tag{2.3}$$

Analogously:

$$I_4 = \int_{\frac{1}{2}}^{1} \left[\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] - t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right] f'(ta+(1-t)b)dt \tag{2.4}$$
$$= \frac{1}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] f\left(\frac{a+b}{2}\right)$$
$$- \frac{1}{b-a}\left(\frac{1}{2}\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] f\left(\frac{a+b}{2}\right)$$
$$- \frac{1}{b-a}\int_{\frac{1}{2}}^{1} t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] f(ta+(1-t)b)dt.$$

Adding the resulting equalities, we obtain

$$\begin{aligned} I_1+I_2+I_3+I_4 &= \frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] f\left(\frac{a+b}{2}\right) \\ &\quad -\frac{1}{b-a}\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] f(ta+(1-t)b)dt \\ &\quad -\frac{1}{b-a}\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] f(tb+(1-t)a)dt \\ &= \frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] f\left(\frac{a+b}{2}\right) \\ &\quad -\frac{1}{(b-a)^{\lambda+1}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a^+;w}f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w}f\right)(a)\right]. \end{aligned} \tag{2.5}$$

According to (1.3) and (1.4), changing variables with $x = tb + (1-t)a$, we get

$$\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] f(tb+(1-t)a)dt = \frac{1}{(b-a)^{\lambda}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a^+;w}f\right)(b)$$

and changing variables with $x = ta + (1-t)b$, we have

$$\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}] f(ta+(1-t)b)dt = \frac{1}{(b-a)^{\lambda}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w}f\right)(a).$$

Thus multiplying both sides of (2.5) by $\frac{(b-a)}{2}$, we get desired result. □

Remark 2.2. Taking $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$, then the above equality reduces to equality in Lemma 1, [5].

By using the above generalized new lemma, we obtain some new Hermite-Hadamard type inequalities via generalized fractional integral operators.

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for generalized fractional integral operators holds:*

$$|M_f(a,b;w;J)| \leq \frac{(b-a)}{2}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(b-a)^{\rho}]\left[|f'(a)| + |f'(b)|\right]$$

where $\rho,\lambda>0$, $w\in\mathbb{R}$ *and* $\sigma_1(k)=\sigma(k)\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k}-1}{\lambda+\rho k+1}\right)$.

Proof. Using Lemma 2 and the convexity of $|f'|$, we have

$$
\begin{aligned}
|M_f(a,b;w;J)| &\le \frac{b-a}{2}\left\{|I_1|+|I_2|+|I_3|+|I_4|\right\}\\
&= \frac{b-a}{2}\left\{\left|\int_0^{\frac{1}{2}} t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]f'(tb+(1-t)a)dt\right|\right.\\
&\quad+\left|\int_0^{\frac{1}{2}}\left(-t^{\lambda}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]f'(ta+(1-t)b)dt\right|\\
&\quad+\left|\int_{\frac{1}{2}}^{1}\left[t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]-\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]\right]f'(tb+(1-t)a)dt\right|\\
&\quad\left.+\left|\int_{\frac{1}{2}}^{1}\left[\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]-t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right]f'(ta+(1-t)b)dt\right|\right\}\\
&\le \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}} t^{\lambda}\left|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right|\left|f'(tb+(1-t)a)\right|dt\right.\\
&\quad+\int_0^{\frac{1}{2}} t^{\lambda}\left|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right|\left|f'(ta+(1-t)b)\right|dt\\
&\quad+\int_{\frac{1}{2}}^{1}\left|t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]-\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]\right|\left|f'(tb+(1-t)a)\right|dt\\
&\quad\left.+\int_{\frac{1}{2}}^{1}\left|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]-t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right|\left|f'(ta+(1-t)b)\right|dt\right\}\\
&\le \frac{b-a}{2}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\times\left\{\int_0^{\frac{1}{2}} t^{\lambda+\rho k}\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|\right]dt\right.\\
&\quad+\int_0^{\frac{1}{2}} t^{\lambda+\rho k}\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\\
&\quad+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|\right]dt\\
&\quad\left.+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right\}\\
&= \frac{b-a}{2}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\\
&\quad\times\left\{\left|f'(a)\right|\left[\int_0^{\frac{1}{2}} t^{\lambda+\rho k}(1-t)\,dt+\int_0^{\frac{1}{2}} t^{\lambda+\rho k+1}dt\right.\right.
\end{aligned}
$$

$$+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right](1-t)\,dt+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]tdt\Bigg]$$

$$+\left|f'(b)\right|\left[\int_{0}^{\frac{1}{2}}t^{\lambda+\rho k+1}dt+\int_{0}^{\frac{1}{2}}t^{\lambda+\rho k}(1-t)\,dt\right.$$

$$\left.\left.+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]tdt+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right](1-t)\,dt\right]\right\}$$

$$=\left(\frac{b-a}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k}-1}{\lambda+\rho k+1}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where we used the facts that

$$\begin{aligned}
\int_{0}^{\frac{1}{2}}t^{\lambda+\rho k}(1-t)\,dt &= \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2},\\
\int_{0}^{\frac{1}{2}}t^{\lambda+\rho k+1}dt &= \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2},\\
\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right](1-t)\,dt &= \frac{1}{8}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2},\\
\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]tdt &= \frac{3}{8}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}-1}{\lambda+\rho k+2}.
\end{aligned}$$

The proof is completed. □

Corollary 2.4. *If we choose* $\lambda=\alpha,\sigma(0)=1$ *and* $w=0$ *in Theorem 2.1, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}^{\alpha}_{a^{+}}f(b)+\mathcal{J}^{\alpha}_{b^{-}}f(a)\right]\right|\\
\leq\ & \frac{b-a}{4}\left(\frac{\alpha+2^{1-\alpha}-1}{\alpha+1}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].
\end{aligned}$$

Remark 2.5. The above inequality is better than one that was given in Theorem 2 of [5].

Remark 2.6. If we choose $\alpha=1$ in Corollary 1, we get the inequality in Theorem 2.2 in [6].

Theorem 2.7. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|^{q}$ *is convex on* $[a,b]$ *for some fixed* $q>1$, *then the following inequality for generalized fractional integral operators holds:*

$$\left|M_{f}(a,b;w;J)\right|\leq\frac{(b-a)\,\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(b-a)^{\rho}]}{2}$$
$$\times\left\{\left(\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}\right\}$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$,

$$\phi = \int_{\frac{1}{2}}^{1} \left(1 - t^{\lambda+\rho k}\right)^{p} dt$$

and

$$\sigma_2(k) = \sigma(k) \left[\left(\frac{\left(\frac{1}{2}\right)^{(\lambda+\rho k)p+1}}{(\lambda+\rho k)p+1} \right)^{\frac{1}{p}} + \phi^{\frac{1}{p}} \right].$$

Proof. By using Lemma 2 and properties of modulus, we have

$$|M_f(a,b;w;J)| \le \frac{b-a}{2} \left[|I_1| + |I_2| + |I_3| + |I_4| \right]. \tag{2.6}$$

Then by using Hölder integral inequality and convexity of $|f'|^q$, we have

$$\begin{aligned} |I_1| &\le \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \qquad (2.7) \\ &\quad \times \left(\int_0^{\frac{1}{2}} \left(t^{\lambda+\rho k}\right)^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left[t|f'(b)|^q + (1-t)|f'(a)|^q \right] dt \right)^{\frac{1}{q}} \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \left(\frac{\left(\frac{1}{2}\right)^{(\lambda+\rho k)p+1}}{(\lambda+\rho k)p+1} \right)^{\frac{1}{p}} \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} |I_2| &\le \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \qquad (2.8) \\ &\quad \times \left(\int_0^{\frac{1}{2}} \left(t^{\lambda+\rho k}\right)^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left[t|f'(a)|^q + (1-t)|f'(b)|^q \right] dt \right)^{\frac{1}{q}} \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \left(\frac{\left(\frac{1}{2}\right)^{(\lambda+\rho k)p+1}}{(\lambda+\rho k)p+1} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} |I_3| &\le \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \qquad (2.9) \\ &\quad \times \left(\int_{\frac{1}{2}}^{1} \left(1-t^{\lambda+\rho k}\right)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left[t|f'(b)|^q + (1-t)|f'(a)|^q \right] dt \right)^{\frac{1}{q}} \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \phi^{\frac{1}{p}} \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} |I_4| &\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \qquad (2.10) \\ &\quad \times \left(\int_{\frac{1}{2}}^{1} \left(1-t^{\lambda+\rho k}\right)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left[t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q \right] dt \right)^{\frac{1}{q}} \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \phi^{\frac{1}{p}} \left(\frac{\left|f'(b)\right|^q + 3\left|f'(a)\right|^q}{4} \right)^{\frac{1}{q}} \end{aligned}$$

where $\phi = \int_{\frac{1}{2}}^{1} \left(1-t^{\lambda+\rho k}\right)^p dt$.

If we use the inequalities (2.7), (2.8), (2.9) and (2.10) in the inequality (2.6), we get the desired result. So, the proof is completed. □

Corollary 2.8. *If we choose $\lambda = \alpha, \sigma(0) = 1$ and $w = 0$ in Theorem 2.2, we have*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[\mathcal{J}_{a^+}^{\alpha} f(b) + \mathcal{J}_{b^-}^{\alpha} f(a) \right] \right| \\ \leq\ & \frac{b-a}{2} \left\{ \left(\frac{\left(\frac{1}{2}\right)^{\alpha p+1}}{\alpha p+1} \right)^{\frac{1}{p}} + \Omega^{\frac{1}{p}} \right\} \\ &\times \left\{ \left(\frac{3\left|f'(a)\right|^q + \left|f'(b)\right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q + 3\left|f'(b)\right|^q}{4} \right)^{\frac{1}{q}} \right\} \\ \leq\ & \frac{b-a}{2} \left\{ \left(\frac{\left(\frac{1}{2}\right)^{\alpha p+1}}{\alpha p+1} \right)^{\frac{1}{p}} + \Omega^{\frac{1}{p}} \right\} \left(\frac{3^{\frac{1}{q}}+1}{4^{\frac{1}{q}}} \right) \left[\left|f'(a)\right| + \left|f'(b)\right| \right] \end{aligned}$$

where we used the fact that

$$\sum_{i=1}^{n} (a_i + b_i)^r \leq \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r \qquad (2.11)$$

for $0 \leq r < 1$, $a_1, a_2, a_{3,}..., a_n \geq 0$ *and* $b_1, b_2, b_3, ..., b_n \geq 0$. *Also,*

$$\Omega = \int_{\frac{1}{2}}^{1} (1-t^{\alpha})^p \, dt.$$

The following result is obtained by using the well-known power-mean integral inequality.

Theorem 2.9. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$ for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following* *inequality for generalized fractional integral operators holds:*

$$\left|M_f(a,b;w;J)\right| \leq \frac{b-a}{2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(b-a)^{\rho}\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \tag{2.12}$$

$$\times\left\{\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}\mu_1+\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}\mu_2\right\}$$

$\rho,\lambda>0$, $w\in\mathbb{R}$ *and where*

$$\mu_1=\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)^{\frac{1}{q}}+\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)^{\frac{1}{q}}$$

and

$$\mu_2=\left(\frac{3}{8}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)^{\frac{1}{q}}+\left(\frac{1}{8}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)^{\frac{1}{q}}.$$

Proof. By using Lemma 2 and properties of modulus, we have

$$\left|M_f(a,b;w;J)\right| \leq \frac{b-a}{2}\left\{|I_1|+|I_2|+|I_3|+|I_4|\right\}$$

Then by using the power mean-integral inequality for $p>1$, we have

$$\begin{aligned}|I_1| &\leq \sum_{k=0}^{\infty}\frac{\sigma(k)\,|w|^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \\ &\quad\times\left(\int_0^{\frac{1}{2}}t^{\lambda+\rho k}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^{\lambda+\rho k}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\end{aligned} \tag{2.13}$$

and by using convexity of $|f'|^{\frac{p}{p-1}}$ in (2.13), we have

$$\begin{aligned}\int_0^{\frac{1}{2}}t^{\lambda+\rho k}\left|f'(tb+(1-t)a)\right|^q dt &= \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\left|f'(b)\right|^q \\ &\quad+\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)\left|f'(a)\right|^q.\end{aligned}$$

If we use last equality in inequality of (2.13), then we get the following inequality as

$$|I_1| \leq \sum_{k=0}^{\infty}\frac{\sigma(k)\,|w|^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}$$

$$\times\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}\left\{\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\left|f'(b)\right|^q+\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)\left|f'(a)\right|^q\right\}.$$

As similar to computation of $|I_1|$, we can get $|I_2|$, $|I_3|$ and $|I_4|$ as following:

$$|I_2| \leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}$$

$$\times\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}\left\{\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}|f'(a)|^q+\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)|f'(b)|^q\right\}^{\frac{1}{q}},$$

$$|I_3| \leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\frac{3}{8}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)|f'(b)|^q\right.$$

$$\left.+\left(\frac{1}{8}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)|f'(a)|^q\right\}^{\frac{1}{q}}$$

and

$$|I_4| \leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\frac{3}{8}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)|f'(a)|^q\right.$$

$$\left.+\left(\frac{1}{8}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)|f'(b)|^q\right\}^{\frac{1}{q}}.$$

Then by using the fact (2.11) in the inequalities of $|I_1|$, $|I_2|$, $|I_3|$ and $|I_4|$ and by using necessary arrangement we get the desired result in (2.12). □

Corollary 2.10. *If we choose $\lambda=\alpha, \sigma(0)=1$ and $w=0$ in Theorem 2.3, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha} f(b)+\mathcal{J}_{b-}^{\alpha} f(a)\right]\right|$$

$$\leq \frac{b-a}{2}\left\{\left(\frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}\eta_1+\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\alpha+1}-1}{\alpha+1}\right)^{1-\frac{1}{q}}\eta_2\right\}\left[|f'(a)|+|f'(b)|\right]$$

where

$$\eta_1=\left(\frac{\left(\frac{1}{2}\right)^{\alpha+2}}{\alpha+2}\right)^{\frac{1}{q}}+\left(\frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1}-\frac{\left(\frac{1}{2}\right)^{\alpha+2}}{\alpha+2}\right)^{\frac{1}{q}}$$

and

$$\eta_2=\left(\frac{3}{8}+\frac{1-\left(\frac{1}{2}\right)^{\alpha+2}}{\alpha+2}\right)^{\frac{1}{q}}+\left(\frac{1}{8}+\frac{\left(\frac{1}{2}\right)^{\alpha+1}-1}{\alpha+1}+\frac{1-\left(\frac{1}{2}\right)^{\alpha+2}}{\alpha+2}\right)^{\frac{1}{q}}.$$

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