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Strong inequalities for the iterated Boolean sums of Bernstein operators

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Dedicated to Professor Heiner Gonska on the occasion of his 70th anniversary.

Abstract. In this paper we investigate the approximation properties for the iterated Boolean sums of Bernstein operators. The approximation behaviour of those operators is presented by the so-called strong inequalities. Moreover, such strong inequalities are valid for any individual continuous function on $[0,1]$. The obtained estimate covers global direct, inverse and saturation results.

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1. Introduction

For $f \in C[0,1]$ the classical Bernstein operators is given by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

Clearly, $B_n(f,\cdot)$ is of degree at most n.

There are many papers dealt with the global approximation degree of Bernstein operators. The final estimate is obtained in [7]. Denote $||\cdot||$ the maximal norm on $[0,1]$. There exists a constant $C>0$ such that for all $f \in C[0,1]$ and all $n = 1,2,\ldots$ the following strong inequalities are true:

$$C^{-1}\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right) \le ||f-B_n(f)|| \le C\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right), \tag{1.1}$$

where

$$\varphi(x) = \sqrt{x(1-x)}$$

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and $\omega_\varphi^2(f,\cdot)$ is the second-order modulus of continuity of the function $f \in C[0,1]$ given by

$$\omega_\varphi^\ell(f,t) := \sup_{0\le\eta\le t} ||\Delta_{\eta\varphi}^\ell f||, \ \ell = 1,2,\ldots.$$

It is well-known (see e.g. [2]) that this modulus is equivalent to the K-functional $K_\varphi^\ell(f,\cdot)$:

$$K_\varphi^\ell(f,t) := \inf_{g\in C^\ell[0,1]}\{||f-g|| + t^\ell||\varphi^l g^{(\ell)}||\}.$$

Thus, the approximation behaviour of Bernstein operators can be completely characterised by (1.1). In particular the maximal approximation degree can only be $\mathcal{O}(1/n)$, i.e. the Bernstein operator is saturated with saturation degree $1/n$. There are many methods to increase the approximation degree of this operator. One of them is the so-called Boolean sum. Let P, Q be operators, $P, Q : X \longrightarrow X$ for some linear space X. Then the Boolean sum of P and Q is defined to be

$$P \oplus Q := P + Q - PQ.$$

For Bernstein operator B_n we will be concerned with iterated Boolean sums of the form $B_n \oplus B_n \oplus \cdots \oplus B_n$, and will denote such an ℓ-fold Boolean sum of the Bernstein operator by $\oplus^\ell B_n$. The easiest way to see that $\oplus^\ell B_n$ is indeed an approximation operator is to look at the error operator representation: with the identity operator I one has

$$I - \oplus^\ell B_n = (I - B_n)^\ell,$$

that can be easily verified by induction. From the last equality we obtain

$$\oplus^\ell B_n = I - (I - B_n)^\ell.$$

The right hand side of this equality represents really a linear combination of a fixed Bernstein operator. Such combination were investigated in the past. The earliest reference in regard to such an approach which we were able to located is [11] (see also [10]).

From the numerical point of view, this combination appears to be of interest, since in the case of discretely defined operators, it uses only the data required by the original operators, in the case of B_n this is just the set of numbers

$$\left\{f(0), f\left(\frac{1}{n}\right), \ldots, f\left(\frac{n-1}{n}\right), f(1)\right\}.$$

The operator $\oplus^\ell B_n$ was introduced independently in [1, 4, 8, 9] and investigated, e.g. in [3, 5] .

In 1994 Gonska and the second author of this paper (see [6]) obtain the following result for $\oplus^\ell B_n$:

Theorem 1.1. *Let $\ell \ge 1$ be fixed. Then there is constant $C > 0$ such that for any $f \in C[0,1]$ and all $n = 1,2,\ldots$*

$$||f - \oplus^\ell B_n(f)|| \le C\left\{\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + ||f||n^{-\ell}\right\}. \quad (1.2)$$

Furthermore, there holds the Steckin-type inequality

$$\omega_\varphi^{2\ell}\left(f,\frac{1}{\sqrt{n}}\right)\le\frac{C}{n^{\ell+1/2}}\sum_{k=1}^{n}k^{\ell-1/2}||f-\oplus^\ell B_k(f)||. \tag{1.3}$$

The o-saturation class is described as follows:

$$||f-\oplus^\ell B_n(f)||=o\left(\frac{1}{n^\ell}\right)\quad\Longleftrightarrow\quad f \text{ is a linear function.}$$

It follows immediately from (1.2) and (1.3) that for all $0<\alpha\le 2l$

$$||f-\oplus^\ell B_n(f)||=\mathcal{O}(n^{-\alpha/2})\quad\Longleftrightarrow\quad\omega_\varphi^{2\ell}(f,t)=\mathcal{O}(t^\alpha).$$

Thus, Theorem 1.1 covers global direct, inverse and saturation results for the Boolean sum of Bernstein operator B_n. In this paper we will show that like (1.1) we have also the strong inequalities for $\oplus^\ell B_n$ in some weak form. To this end, denote $E_n(f)$ to be the best approximation constant of f via algebraic polynomials p_n of degree n, i.e.

$$E_n(f):=\min_{p_n}||f-p_n||.$$

We have

Theorem 1.2. *Let $\ell\ge 1$ be fixed. Then there are constants $C>0$ and $A\ge 1$ such that for any $f\in C[0,1]$ and all $n=1,2,\dots$*

$$\begin{aligned}C^{-1}\left\{\omega_\varphi^{2\ell}\left(f,\frac{1}{\sqrt{n}}\right)+E_1(f)n^{-\ell}\right\}&\le\max_{n\le k\le An}(||f-\oplus^\ell B_k(f)||+E_1(f)k^{-\ell}) \qquad (1.4)\\&\le\max_{k\ge n}(||f-\oplus^\ell B_k(f)||+E_1(f)k^{-\ell})\\&\le C\left\{\omega_\varphi^{2\ell}\left(f,\frac{1}{\sqrt{n}}\right)+E_1(f)n^{-\ell}\right\}.\end{aligned}$$

Moreover, if f is not an algebraic polynomial of degree less than 2ℓ, then for some constants $D,A>0$ and all $n=1,2,\dots$ there holds

$$\begin{aligned}D^{-1}\omega_\varphi^{2\ell}\left(f,\frac{1}{\sqrt{n}}\right)&\le\max_{n\le k\le An}||f-\oplus^\ell B_k(f)|| \qquad (1.5)\\&\le\max_{k\ge n}||f-\oplus^\ell B_k(f)||\le D\omega_\varphi^{2\ell}\left(f,\frac{1}{\sqrt{n}}\right).\end{aligned}$$

We prove this result in the next section.

2. Proof of Theorem 1.2

Proof of Theorem 1.2. First we note that $\oplus^l B_n$ is invariant for linear functions. Hence we conclude from (1.2)

$$||f-\oplus^\ell B_n(f)||\le C\left\{\omega_\varphi^{2\ell}\left(f,\frac{1}{\sqrt{n}}\right)+E_1(f)n^{-\ell}\right\}. \tag{2.1}$$

Let $0 < \delta_1 < \delta_2 < 1/2$. We obtain from (1.3) for $i = 1, 2$

$$\begin{aligned} \omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell} &\le \frac{C}{n^{\ell+1/2}} \sum_{k=1}^{n} k^{\ell-1/2}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le Cn^{-\delta_i-\ell} \max_{1\le k\le n} k^{\ell+\delta_i}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}). \end{aligned}$$

Noticing $\omega_\varphi^{2\ell}(f, t_1)/t_1^{2\ell} \le C\omega_\varphi^{2\ell}(f, t_2)/t_2^{2\ell}$ for $0 \le t_2 \le t_1$, we conclude from (2.1) for $1 \le k \le n$

$$\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell} \le C\frac{n^\ell}{k^\ell}\left(\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + n^{-\ell}E_1(f)\right).$$

It follows from the last two estimates that for $i = 1, 2$

$$\begin{aligned} \omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell} &\le Cn^{-\delta_i-\ell} \max_{1\le k\le n} k^{\ell+\delta_i}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le C_1\left(\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell}\right). \end{aligned}$$

Consequently, for some constant $C > 0$

$$\begin{aligned} &\frac{1}{n^{\delta_1+\ell}} \max_{1\le k\le n} k^{\ell+\delta_1}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le \frac{C}{n^{\delta_2+\ell}} \max_{1\le k\le n} k^{\ell+\delta_2}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}). \end{aligned}$$

Let the maximum on the right hand side be reached at n_0. So we have

$$\begin{aligned} &n^{-\delta_1-\ell}n_0^{\ell+\delta_1}(\|f - \oplus^\ell B_{n_0}(f)\| + E_1(f)n_0^{-\ell}) \\ &\le Cn^{-\delta_2-\ell}n_0^{\ell+\delta_2}(\|f - \oplus^\ell B_{n_0}(f)\| + E_1(f)n_0^{-\ell}). \end{aligned}$$

In other words, for some $c' > 0$ there holds $c'n \le n_0$. Therefore,

$$\begin{aligned} \omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell} &\le Cn^{-\delta_2-\ell}n_0^{\ell+\delta_2}(\|f - \oplus^\ell B_{n_0}(f)\| + E_1(f)n_0^{-\ell}) \\ &\le C \max_{c'n\le k\le n}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le C \max_{k\ge c'n}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}). \end{aligned}$$

Or for some constant $A > 0$

$$\begin{aligned} \omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell} &\le C \max_{n\le k\le An}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le C \max_{k\ge n}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}). \end{aligned}$$

Clearly, by (2.1)

$$\max_{k\ge n}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \le C\left(\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell}\right).$$

The first assertion (1.4) is proved.

It remains to show the assertion (1.5). To this end we note that f is not an algebraic polynomial of degree less than 2ℓ. Hence, $\omega_\varphi^{2\ell}(f,1) \neq 0$. On the other hand, as we mention at the beginning of this paper $\omega_\varphi^{2\ell}(f,\cdot)$ is equivalent to the $K-$functional $K_\varphi^{2\ell}(f,\cdot)$. Thus, for $0 \le t \le 1$ we have with some constant $C > 0$ the inequality

$$\omega_\varphi^{2\ell}(f,1)/1^{2\ell} \le C\omega_\varphi^{2\ell}(f,t)/t^{2\ell}.$$

But $E_1(f) \le C\omega_\varphi^{2\ell}(f,1)$. Therefore,

$$E_1(f)\frac{1}{n^\ell} \le C\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right).$$

Thus, (2.1) can be written as

$$||f - \oplus^\ell B_n(f)|| \le C\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right).$$

Combining this estimate with (1.3) and using the same approach as above we obtain (1.5). □

References

[1] Agrawal, P.N., Kasana, H.S., *On the iterative combinations of Bernstein polynomials*, Demonstratio Math., **17**(1984), no. 3, 777-783.

[2] Ditzian, Z., Totik, V., *Moduli of Smoothness*, New York, Springer-Verlag, 1987.

[3] Dzamalov, M.S., *On a theorem of E.V. Vornovskoi*, (Russian), in: Operators and their Applications, Approximation of Functions, Equalities, Leningrad, 1985, 22-27.

[4] Felbecker, G., *Linearkombinationen von iterierten Bernsteinoperatoren*, Manuscripta Math., **29**(1979), 229-246.

[5] Gawronski, W., Stadtmüller, U., *Linear combinations of iterated generalized Bernstein functions with an application to density estimation*, Acta Sci. Math. (Szeged), **47**(1984), 205-221.

[6] Gonska, H.H., Zhou, X., *Approximation theorem for the iterated Boolean sums of Bernstein operators*, J. of Computational and Applied Mathematics, **53**(1994), 21-31.

[7] Knoop, K.B., Zhou, X., *The lower estimate for linear positive operators (II)*, Results in Mathematics, **25**(1994), 315-330.

[8] Mastroianni, G., Occorsio, M.R., *Una generalizzazione dell'operatore di Bernstein*, Rend. Accad. Sci. Fis. Mat. Napoli, **44**(1977), 151-169.

[9] Micchelli, C., *The saturation class and iterates of the Bernstein polynomials*, J. Approx. Theory, **8**(1973), 1-18.

[10] Natanson, G.I., *Application class and the method of I.P. Natanson and I. Yu. Kharrik in the algebraic case*, (Russian), in: Operator Theory and Function Theory, No. 1 Leningrad Univ., Leningrad, 1983, 166-170.

[11] Natanson, I.P., *On the approximation of multiply differentiable periodic functions by means of singular integrals*, (Russian), Dokl. Akad. Nauk SSSR, **82**(1952), 337-339.

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