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Quantitative results for the convergence of the iterates of some King type operators

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Abstract. In this article we construct three q-King type operators which fix the functions e_0 and $e_2+\alpha e_1$, $\alpha > 0$. We study the rates of convergence for the iterates of these operators using the first and the second order modulus of continuity. We show that the convergence is faster in the case of q operators ($q < 1$) than in the classical case ($q = 1$).

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1. Introduction

In [4] the authors introduced the operators $B_{n,\alpha} : C[0,1] \to C[0,1]$, $n > 1$, given by

$$B_{n,\alpha}f(x) = \sum_{k=0}^{n} \binom{n}{k} (u_{n,\alpha}(x))^k (1 - u_{n,\alpha}(x))^{n-k} f\left(\frac{k}{n}\right), \tag{1.1}$$

where $\alpha \in [0, \infty)$ and

$$u_{n,\alpha}(x) = -\frac{n\alpha + 1}{2(n-1)} + \sqrt{\frac{(n\alpha + 1)^2}{4(n-1)^2} + \frac{n(\alpha x + x^2)}{n-1}}.$$

The operators $B_{n,\alpha}$ preserve the functions e_0 and $e_2+\alpha e_1$, where $e_i(x) = x^i$, $i = 0, 1, 2$. For $\alpha = 0$ the operator $B_{n,\alpha}$ reduces to the King operator (see [7]).

In this article we consider three q-operators of King type which fix the functions e_0 and $e_2 + \alpha e_1$. We study the convergence of the iterates of these operators.

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Rates of convergence are obtained by using the first and the second order modulus of smoothness, i.e.

$$\omega_1(f,\delta) = \sup\left\{|f(x+h)-f(x)| : x, x+h \in [0,1],\ 0 \le h \le \delta\right\},$$

$$\omega_2(f,\delta) = \sup\left\{|f(x+h)-2f(x)+f(x-h)| : x, x\pm h \in [0,1],\ 0 \le h \le \delta\right\},$$

where $f \in C[0,1]$ and $\delta \ge 0$. We get better results in the case of the considered q-operators ($q < 1$) than in the classical case ($q = 1$).

Other quantitative results related to the convergence of the iterates of some positive linear operators may be found in [1], [2], [3], [6], [9], [8], [12].

We remind some notations from q-calculus which we use in the construction of the operators. For $q \in (0,1)$ we have:

- q-integer:

$$[n]_q = \frac{1-q^n}{1-q},\ n \in \mathbb{N},$$

- q-factorial:

$$[n]_q! = [n]_q[n-1]_q...[1]_q,\ n = 1,2,...,\ [0]_q! = 1,$$

- q-binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

- q-integral

$$\int_0^1 f(x)d_qx = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n.$$

2. Convergence of the iterates of the positive linear operators which preserve some functions

Let $\tau : [0,1] \to [0,1]$ a continuous strictly increasing functions satisfying the conditions $\tau(0) = 0$ and $\tau(1) = 1$.

Let $P : C[0,1] \to C[0,1]$ the operator given by

$$Pf(x) = (1-\tau(x))\, f(0) + \tau(x)f(1). \tag{2.1}$$

The following theorem is a direct consequence of Theorem 3.1 from [5].

Theorem 2.1. *Let $L : C[0,1] \to C[0,1]$ a positive linear operator which preserves the functions e_0, τ and has the set of interpolation points $\{0,1\}$. If there exists $\varphi \in C[0,1]$ such that*

$$L\varphi \gtrless \varphi \text{ on } (0,1),$$

then

$$\lim_{m\to\infty} L^m f = Pf,$$

uniformly on $[0,1]$.

Theorem 2.2. [3] *Let* $\tau : [0,1] \to [0,1]$, $\tau \in C^1[0,1]$, *strictly increasing,* $\tau(0) = 0$, $\tau(1) = 1$, $\tau'(0) \neq 1$, $\tau'(1) \neq 1$ *and* $\tau(x) \neq x$, $x \in (0,1)$. *Let* $L : C[0,1] \to C[0,1]$ *be a linear positive operator which preserves* e_0 *and* τ *and let*

$$c = \sup_{0 \le x \le 1} \frac{\tau(x) - 2x\tau(x) + x^2}{|x - \tau(x)|} \tag{2.2}$$

and

$$\delta_m(x) = |L^m e_1(x) - \tau(x)|, \ x \in [0,1]. \tag{2.3}$$

If

$$0 < \delta_m(x) < 1/4, \ x \in (0,1),$$

then we have, for every $x \in [0,1]$,

$$|L^m f(x) - Pf(x)| \le \sqrt{\delta_m(x)}\omega_1(f, \sqrt{\delta_m(x)}) + \left(1 + \frac{c}{2}\right)\omega_2(f, \sqrt{\delta_m(x)})$$

and

$$|L^m f(x) - Pf(x)| \le 2\delta_m(x)\,\|f\| + \left(\frac{3}{2}\sqrt{\delta_m(x)} + \frac{7+c}{2}\right)\omega_2(f, \sqrt{\delta_m(x)}).$$

If we take

$$\tau = \frac{e_2 + \alpha e_1}{1+\alpha},$$

with $\alpha \ge 0$, then the operator P from (2.1) becomes

$$Pf(x) = \frac{(1-\alpha)(1+x+\alpha)}{1+\alpha} f(0) + \frac{x(x+\alpha)}{1+\alpha} f(1), \ x \in [0,1], \ f \in C[0,1] \tag{2.4}$$

and for the constant c from (2.2) we get

$$c = \alpha + 2.$$

In the next sections we obtain estimations for $\delta_m(x)$, $x \in [0,1]$ given by (2.3) for three new operators which preserve e_0 and $e_2 + \alpha e_1$, $\alpha \ge 0$. Using Theorem 2.2 we get quantitative results for the convergence of the iterates of these operators.

3. The King modified q-Bernstein operator

The classical Berstein operator is given by

$$B_n f(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \ f \in C[0,1], \ x \in [0,1],$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

In [11] Phillips constructed the q-Bernstein operator:

$$B_{n,q} f(x) = \sum_{k=0}^{n} p_{n,k}(q;x) f\left(\frac{[k]_q}{[n]_q}\right), \ q \in (0,1], \ f \in C[0,1], \ x \in [0,1],$$

where

$$p_{n,k}(q;x) = \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x).$$

The King modified q-Bernstein operator is given by

$$K^1_{n,q,\alpha} f(x) = B_n f(u_{n,q,\alpha}(x)),\ f \in C[0,1],\ x \in [0,1],$$

where $u_{n,q,\alpha} : [0,1] \to [0,1]$, $n > 1$ are continuous and strictly increasing functions having the properties $u_{n,q,\alpha}(0) = 0$, $u_{n,q,\alpha}(1) = 1$.
From properties of the the q-Bernstein operator we have

$$\begin{aligned} K^1_{n,q,\alpha} e_0(x) &= 1, \\ K^1_{n,q,\alpha} e_1(x) &= u_{n,q,\alpha}(x), \\ K^1_{n,q,\alpha} e_2(x) &= (u_{n,q,\alpha}(x))^2 + \frac{u_{n,q,\alpha}(x)(1 - u_{n,q,\alpha}(x))}{[n]_q}. \end{aligned}$$

Imposing that the operator $K^1_{n,q,\alpha}$ preserves the function $e_2 + \alpha e_1$, $\alpha \geq 0$ we get

$$u_{n,q,\alpha}(x) = -\frac{[n]_q \alpha + 1}{2([n]_q - 1)} + \sqrt{\frac{([n]_q \alpha + 1)^2}{4([n]_q - 1)^2} + \frac{[n]_q(\alpha x + x^2)}{[n]_q - 1}}.$$

Particular cases:

- $q = 1$ – the operator constructed in [4]
- $q = 1$, $\alpha = 0$ – the King operator (see [7])
- $\alpha = 0$ – the q-Bernstein King operator (see [8])

Theorem 3.1. *The sequence of the iterates of the operator $K^1_{n,q,\alpha}$ converges uniformly to the operator P given by (2.4).*

Proof. For every $x \in [0,1]$ we have

$$\begin{aligned} K^1_{n,q,\alpha} e_1(x) - x &= u_{n,q,\alpha}(x) - x \\ &= \frac{x(x-1)}{[n]_q - 1} \cdot \frac{1}{\frac{[n]_q \alpha + 1}{2([n]_q - 1)} + \sqrt{\frac{([n]_q \alpha + 1)^2}{4([n]_q - 1)^2} + \frac{[n]_q(\alpha x + x^2)}{[n]_q - 1}} + x} \end{aligned} \tag{3.1}$$

It follows that

$$K^1_{n,q,\alpha} e_1(x) - x \leq 0,\ x \in [0,1],$$

with equality if and only if $x \in \{0,1\}$. The conclusion follows from Theorem 2.1 by taking $\varphi = e_1$. □

Theorem 3.2. *If*

$$\delta^1_{m,n,q,\alpha}(x) = \left(K^1_{n,q,\alpha}\right)^m e_1(x) - \frac{x^2 + \alpha x}{1 + \alpha},\ x \in [0,1],$$

then we have the estimation

$$\delta^1_{m,n,q,\alpha}(x) \leq \left(\frac{(\alpha+2)\left([n]_q - 1\right)}{(\alpha+2)\,[n]_q - 1} \right)^m \frac{x(1-x)}{1+\alpha} = \lambda^1_{m,n,q,\alpha}(x),\ x \in [0,1]. \tag{3.2}$$

Proof. For $x \in \{0,1\}$ we get $\delta^1_{m,n,q,\alpha}(x) = 0$ and (3.2) holds.
For $x \in (0,1)$, using (3.1) we get

$$\frac{K^1_{n,q,\alpha}e_1(x) - x}{x(1-x)} \le \frac{-1}{[n]_q\alpha + 2[n]_q - 1}.$$

We observe that

$$\frac{K^1_{n,q,\alpha}e_1(x) - x^2}{x(1-x)} = \frac{K^1_{n,q,\alpha}e_1(x) - x}{x(1-x)} + 1 \le \frac{[n]_q\alpha + 2[n]_q - 2}{[n]_q\alpha + 2[n]_q - 1}.$$

It follows that

$$\begin{aligned} K^1_{n,q,\alpha}e_1(x) - \frac{x^2+\alpha x}{1+\alpha} &= K^1_{n,q,\alpha}e_1(x) - x^2 + x^2 - \frac{x^2+\alpha x}{1+\alpha} \\ &= K^1_{n,q,\alpha}e_1(x) - x^2 - \frac{\alpha}{\alpha+1}x(1-x) \\ &\le \left(\frac{[n]_q\alpha + 2[n]_q - 2}{[n]_q\alpha + 2[n]_q - 1} - \frac{\alpha}{\alpha+1}\right)x(1-x) \\ &= \frac{(\alpha+2)\left([n]_q - 1\right)}{(\alpha+2)\,[n]_q - 1}\cdot\left(x - \frac{x^2+\alpha x}{1+\alpha}\right) \end{aligned}$$

Taking into account that the operator $K^1_{n,q,\alpha}$ preserves $(e_2+\alpha e_1)/(1+\alpha)$ we obtain

$$\left(K^1_{n,q,\alpha}\right)^m e_1(x) - \frac{x^2+\alpha x}{1+\alpha} \le \left(\frac{(\alpha+2)\left([n]_q - 1\right)}{(\alpha+2)\,[n]_q - 1}\right)^m\cdot\left(x - \frac{x^2+\alpha x}{1+\alpha}\right)$$

and the conclusion follows. □

Theorem 3.3. *We have the following estimations:*

$$\left|(K^1_{n,q,\alpha})^m f(x) - Pf(x)\right| \tag{3.3}$$
$$\le \sqrt{\lambda^1_{m,n,q,\alpha}(x)}\,\omega_1\left(f,\sqrt{\lambda^1_{m,n,q,\alpha}(x)}\right) + \left(2+\frac{\alpha}{2}\right)\omega_2\left(f,\sqrt{\lambda^1_{m,n,q,\alpha}(x)}\right),\ x\in[0,1]$$

and

$$\left|(K^1_{n,q,\alpha})^m f(x) - Pf(x)\right| \tag{3.4}$$
$$\le 2\lambda^1_{m,n,q,\alpha}(x)\,\|f\| + \left(\frac{3}{2}\sqrt{\lambda^1_{m,n,q,\alpha}(x)} + \frac{9+\alpha}{2}\right)\omega_2(f,\sqrt{\lambda^1_{m,n,q,\alpha}(x)}),\ x\in[0,1].$$

Proof. The conclusion follows using Theorem 2.2 and Theorem 3.2. □

For $\alpha = 0$ the estimations (3.3) and (3.4) were obtained in [2].
The function $h^1_{m,\alpha} : [1,\infty) \to \mathbb{R}, m \ge 1$ defined by

$$h^1_{m,\alpha}(t) = \left(\frac{(2+\alpha)(t-1)}{t(2+\alpha)-1}\right)^m$$

is strictly increasing. If $0 < q_1 < q_2 \le 1$ then $[n]_{q_1} < [n]_{q_2}$ and therefore

$$h^1_{m,\alpha}([n]_{q_1}) < h^1_{m,\alpha}([n]_{q_2}),\ m \ge 1.$$

From Theorem 3.3 it follows that the estimation $\left|(K^1_{n,q_1,\alpha})^m f(x) - Pf(x)\right|$, $x \in [0,1]$ is smaller than the estimation $\left|(K^1_{n,q_2,\alpha})^m f(x) - Pf(x)\right|$, $x \in [0,1]$. Taking $q_1 = q \in$

$(0,1)$ and $q_2 = 1$ we get that the q-operator $K^1_{n,q,\alpha}$ provides better convergence of the iterates than the classical operator $K^1_{n,1,\alpha}$.

4. The King modified q-Stancu operator

The q-Stancu operator constructed by Nowak in [10] is given by

$$P_n^{q,a} f(x) = \sum_{k=0}^{n} w_{n,k}^{q,a}(x) f\left(\frac{[k]_q}{[n]_q}\right), \; a \geq 0, \; q \in (0,1], \; f \in C[0,1], \; x \in [0,1],$$

where

$$w_{n,k}^{q,a}(x) = \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{\prod\limits_{i=0}^{k-1}(x + a[i]_q) \prod\limits_{i=0}^{n-1-k}(1 - q^i x + a[i]_q)}{\prod\limits_{i=0}^{n-1}(1 + a[i]_q)}.$$

Particular cases:

- $q = 1$ – the Stancu operator (see [13])
- $a = 0$ – the q-Bernstein operator (see [11])

We consider the King modified q-Stancu operator

$$K^2_{n,q,a,\alpha} f(x) = P_n^{q,a} f(u_{n,q,a,\alpha}(x)),$$

where $u_{n,q,a,\alpha} : [0,1] \to [0,1]$, $n > 1$ are continuous and strictly increasing functions having the properties $u_{n,q,a,\alpha}(0) = 0$, $u_{n,q,a,\alpha}(1) = 1$.
From [10] we have

$$K^2_{n,q,a,\alpha} e_0(x) = 1,$$
$$K^2_{n,q,a,\alpha} e_1(x) = u_{n,q,a,\alpha}(x),$$
$$K^2_{n,q,a,\alpha} e_2(x) = \tfrac{1}{a+1}\left(\tfrac{u_{n,q,a,\alpha}(x)(1 - u_{n,q,a,\alpha}(x))}{[n]_q} + u_{n,q,a,\alpha}(x)(x + u_{n,q,a,\alpha}(x))\right).$$

If

$$u_{n,q,a,\alpha}(x) = -\beta_{n,q,a,\alpha} + \sqrt{\beta^2_{n,q,a,\alpha} + \frac{[n]_q(1+a)(\alpha x + x^2)}{[n]_q - 1}},$$

where

$$\beta_{n,q,a,\alpha} = \frac{1 + [n]_q(a + \alpha + a\alpha)}{2([n]_q - 1)},$$

then the operator $K^2_{n,q,a,\alpha}$ fixes the functions e_0 and $e_2 + \alpha e_1$, $\alpha \geq 0$.

Theorem 4.1. *The sequence of the iterates of the operator $K^2_{n,q,a,\alpha}$ converges uniformly to the operator P given by (2.4).*

Proof. We use Theorem 2.1 with $\varphi = e_1$. Indeed, we have

$$K^2_{n,q,a,\alpha} e_1(x) - x = u_{n,q,a,\alpha}(x) - x$$

and therefore

$$K^2_{n,q,a,\alpha} e_1(x) - x = \frac{x(x-1)}{[n]_q - 1} \cdot \frac{1 + [n]_q a}{\beta_{n,q,a,\alpha} + \sqrt{\beta^2_{n,q,a,\alpha} + \frac{[n]_q(1+a)(\alpha x + x^2)}{[n]_q - 1}} + x} \leq 0, \quad (4.1)$$

for all $x \in [0,1]$, with equality only for $x \in \{0,1\}$. $\square$

Theorem 4.2. *If*

$$\delta^2_{m,n,q,a,\alpha}(x) = \left(K^2_{n,q,a,\alpha}\right)^m e_1(x) - \frac{x^2+\alpha x}{1+\alpha},\ x \in [0,1],$$

then we have the estimation

$$\delta^2_{m,n,q,a,\alpha}(x) \le \left(\frac{(\alpha+2)\left([n]_q - 1\right)}{(a\alpha+\alpha+a+2)\,[n]_q - 1}\right)^m \frac{x(1-x)}{1+\alpha} = \lambda^2_{m,n,q,a,\alpha}(x),$$

for all $x \in [0,1]$.

Proof. From (4.1) it follows that

$$\frac{\left(K^2_{n,q,a,\alpha}\right)^m e_1(x) - x}{x(1-x)} \le -\frac{1+[n]_q a}{2\left([n]_q - 1\right)\left(1+\beta_{n,q,a,\alpha}\right)},\ x \in (0,1),$$

Using the same steps as in Theorem 3.2 we get the conclusion. $\square$

Theorem 4.3. *We have the following estimations:*

$$\left|(K^2_{n,q,a,\alpha})^m f(x) - Pf(x)\right| \le \sqrt{\lambda^2_{m,n,q,a,\alpha}(x)}\,\omega_1\left(f, \sqrt{\lambda^2_{m,n,q,a,\alpha}(x)}\right)$$

$$+\left(2+\frac{\alpha}{2}\right)\omega_2\left(f, \sqrt{\lambda^2_{m,n,q,a,\alpha}(x)}\right),\ x \in [0,1]$$

and

$$\left|(K^2_{n,q,a,\alpha})^m f(x) - Pf(x)\right| \le 2\lambda^2_{m,n,q,a,\alpha}(x)\,\|f\|$$

$$+\left(\frac{3}{2}\sqrt{\lambda^2_{m,n,q,a,\alpha}(x)} + \frac{9+\alpha}{2}\right)\omega_2(f, \sqrt{\lambda^2_{m,n,q,a,\alpha}(x)}),\ x \in [0,1].$$

Proof. The conclusion follows from Theorem 2.2 and Theorem 4.2. $\square$

The function $h^2_{m,a,\alpha} : [1,\infty) \to \mathbb{R}$, $m \ge 1$ defined by

$$h^2_{m,a,\alpha}(t) = \left(\frac{(2+\alpha)(t-1)}{t(2+\alpha+a+a\alpha)-1}\right)^m$$

is strictly increasing. If $0 < q_1 < q_2 \le 1$ then

$$h^2_{m,a,\alpha}([n]_{q_1}) < h^2_{m,a,\alpha}([n]_{q_2}),\ m \ge 1.$$

From Theorem 4.3 it follows that the estimation $\left|(K^2_{n,q_1,a,\alpha})^m f(x) - Pf(x)\right|$, $x \in [0,1]$ is smaller than the estimation $\left|(K^2_{n,q_2,a,\alpha})^m f(x) - Pf(x)\right|$, $x \in [0,1]$. In particular, taking $q_1 = q \in (0,1)$ and $q_2 = 1$ we get that the q-operator $K^2_{n,q,a,\alpha}$ has a better rate of convergence for the iterates than the operator $K^2_{n,1,a,\alpha}$.

5. The King modified q-genuine Bernstein-Durrmeyer operator

The q-genuine Bernstein-Durrmeyer operator introduced in [9] is given by

$$U_{n,q}f(x) = p_{n,0}(q;x)f(0) + p_{n,n}(q;x)f(1) \tag{5.1}$$

$$+[n-1]_q \sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;x) \int_0^1 p_{n-2,k-1}(q;qt)f(t)d_qt,$$

for every $f \in C[0,1]$ and every $x \in [0,1]$. For $q = 1$ we get the classical genuine Bernstein-Durrmeyer operator.
We consider the King modification of the q-genuine Bernstein-Durrmeyer operator

$$K^3_{n,q,\alpha}f(x) = U_{n,q}f(u_{n,q,\alpha}(x)),$$

where $u_{n,q,\alpha} : [0,1] \to [0,1]$, $n > 1$ are continuous and strictly increasing functions having the properties $u_{n,q,\alpha}(0) = 0$, $u_{n,q,\alpha}(1) = 1$.
From [9] we have

$$K^3_{n,q,\alpha}e_0(x) = 1,$$
$$K^3_{n,q,\alpha}e_1(x) = u_{n,q,\alpha}(x),$$
$$K^3_{n,q,\alpha}e_2(x) = (u_{n,q,\alpha}(x))^2 + \frac{[2]_q u_{n,q,\alpha}(x)(1-u_{n,q,\alpha}(x))}{[n+1]_q}.$$

If

$$u_{n,q,\alpha}(x) = -\frac{[n+1]_q\alpha + [2]_q}{2([n+1]_q - [2]_q)} + \sqrt{\frac{([n+1]_q\alpha + [2]_q)^2}{4([n+1]_q - [2]_q)^2} + \frac{[n+1]_q(\alpha x + x^2)}{[n+1]_q - [2]_q}}, \tag{5.2}$$

then the operator $K^3_{n,q,\alpha}$ preserves the functions e_0 and $e_2 + \alpha e_1$, $\alpha > 0$.

Theorem 5.1. *The sequence of the iterates of the operator $K^3_{n,q,\alpha}$ converges uniformly to the operator P given by (2.4).*

Proof. We have

$$\begin{aligned} K^3_{n,q,\alpha}e_1(x) - x &= u_{n,q,\alpha}(x) - x \\ &= \frac{x(x-1)}{[n+1]_q - [2]_q} \cdot \frac{[2]_q}{\gamma_{n,q,\alpha} + \sqrt{\gamma^2_{n,q,\alpha} + \frac{[n+1]_q(\alpha x + x^2)}{[n+1]_q - [2]_q}} + x}, \end{aligned}$$

where

$$\gamma_{n,q,\alpha} = \frac{[n+1]_q\alpha + [2]_q}{2([n+1]_q - [2]_q)}.$$

It follows that

$$K^3_{n,q,\alpha}e_1(x) - x \le 0,\ x \in [0,1],$$

with equality only for $x \in \{0,1\}$. Using Theorem 2.1 with $\varphi = e_1$ we get the conclusion. □

Theorem 5.2. *If*

$$\delta^3_{m,n,q,\alpha}(x) = (K^3_{n,q,\alpha})^m e_1(x) - \frac{x^2+\alpha x}{1+\alpha},\ x \in [0,1],$$

then we get

$$\delta^3_{m,n,q,\alpha}(x) \leq \left(\frac{(\alpha+2)\left([n+1]_q - [2]_q\right)}{(\alpha+2)\,[n+1]_q - [2]_q} \right)^m \frac{x(1-x)}{1+\alpha} = \lambda^3_{m,n,q,\alpha}(x),$$

for all $x \in [0,1]$.

Proof. We get the conclusion using the same steps as in Theorem 3.2 and taking into account the inequality

$$\frac{\left(K^3_{n,q,\alpha}\right)^m e_1(x) - x}{x(1-x)} \leq -\frac{[2]_q}{(\alpha+1)[n+1]_q - [2]_q},\ x \in (0,1). \qquad \square$$

Theorem 5.3. *We have the following estimations:*

$$\left|(K^3_{n,q,\alpha})^m f(x) - Pf(x)\right| \leq \tag{5.3}$$

$$\sqrt{\lambda^3_{m,n,q,\alpha}(x)}\omega_1\left(f, \sqrt{\lambda^2_{m,n,q}(x)}\right) + \left(2+\frac{\alpha}{2}\right)\omega_2\left(f, \sqrt{\lambda^3_{m,n,q,\alpha}(x)}\right),\ x \in [0,1]$$

and

$$\left|(K^3_{n,q,\alpha})^m f(x) - Pf(x)\right| \leq \tag{5.4}$$

$$2\lambda^2_{m,n,q,\alpha}(x)\,\|f\| + \left(\frac{3}{2}\sqrt{\lambda^3_{m,n,q,\alpha}(x)} + \frac{9+\alpha}{2}\right)\omega_2(f, \sqrt{\lambda^3_{m,n,q,\alpha}(x)}),\ x \in [0,1].$$

Proof. The conclusion follows from Theorem 2.2 and Theorem 5.2. $\square$

For $\alpha = 0$ the estimations (5.3) and (5.4) were obtained in [3].

The function $h^3_{m,n,\alpha} : (0,1] \to \mathbb{R}$, $m \geq 1$ defined by

$$h^3_{m,n,\alpha}(q) = \left(\frac{(\alpha+2)(q^2 - q^{n+1})}{(\alpha+2)(1-q^{n+1}) - 1 + q^2} \right)^m$$

is strictly increasing. From Theorem 5.3 it follows that if $0 < q_1 < q_2 \leq 1$ then the estimation $\left|(K^3_{n,q_1,\alpha})^m f(x) - Pf(x)\right|$, $x \in [0,1]$ is smaller than the estimation $\left|(K^3_{n,q_2,\alpha})^m f(x) - Pf(x)\right|$, $x \in [0,1]$. Taking $q_1 = q \in (0,1)$ and $q_2 = 1$ we get that the convergence of the iterates of the q-operator $K^3_{n,q,\alpha}$ is better than that of the operator $K^3_{n,1,\alpha}$.

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