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Some properties of a new subclass of analytic univalent functions defined by multiplier transformation

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Abstract. The purpose of the present paper is to study the integral operator of the form

$$\int_0^z \left\{\frac{I_\mu^n f(t)}{t}\right\}^\delta dt$$

where f belongs to the subclass $C(n, \alpha, \beta, \mu)$ and δ is a real number. We obtain integral characterization for the subclass $C(n, \alpha, \beta, \mu)$ and also prove distortion, rotation and radii theorem for this class. Relevant connections of the results presented here with various known results are briefly indicated.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of functions of the form (1.1) which are also univalent in U.

A function f of S is said to be starlike of order $\alpha (0 \leq \alpha < 1)$, denoted by $f \in S^*(\alpha)$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \; z \in U,$$

and is said to be convex of order $\alpha (0 \leq \alpha < 1)$, denoted by $f \in K(\alpha)$, if and only if

$$\Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha, \;\; z \in U.$$

The classes S^* and K of starlike and convex functions, respectively, are identified by $S^*(0) \equiv S^*$ and $K(0) \equiv K$.

These classes were first studied by Robertson [17].

In 2003 Cho and Srivastava [2], (see also [1]) introduced the multiplier transformation for functions f of the form (1.1) as follows

$$I_\mu^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\mu}{1+\mu} \right)^n a_k z^k.$$

For $\mu = 1$, the operator $I_\mu^n \equiv I^n$ was studied by Uralegaddi and Somanatha [22] and for $\mu = 0$ the operator I_μ^n reduce to well-known Sălăgean operator introduced by Sălăgean [19].

Using the multiplier transformation we introduce the class $S(n, \alpha, \mu)$ of functions of the form (1.1) satisfying the following condition

$$\Re \left\{ \frac{z \left(I_\mu^n f(z) \right)'}{I_\mu^n f(z)} \right\} > \alpha, \;\; z \in U. \tag{1.2}$$

It is worthy to note that for $\mu = 0$ the class $S(n, \alpha, \mu)$ reduce to the class $S(n, \alpha)$ was first introduced by Sălăgean [19] and further studied by Kadioğlu [4].

It should be worthy to note that $S(0, \alpha, 0) = S^*(\alpha)$ and $S(1, \alpha, 0) = K(\alpha)$.

A function f of A belongs to the class $C(n, \alpha, \beta, \mu)$ if there exists a function $F \in S^*(\alpha)$ such that

$$\left| \arg \frac{I_\mu^n f(z)}{F(z)} \right| < \frac{\beta \pi}{2}, \;\; z \in U,$$

where $n \in N_0, \; 0 \leq \alpha < 1, \; 0 < \beta \leq 1, \; \mu > -1$.

By specializing the parameters in $C(n, \alpha, \beta, \mu)$ we obtain the following known subclasses of A studied earlier by various researchers.

(1) $C(0, \alpha, \beta, 0) \equiv CS^*(\alpha, \beta)$ studied by Mishra [9].
(2) $C(1, \alpha, \beta, 0) \equiv C(\alpha, \beta)$ studied by Mishra [9].
(3) $C(0, 0, \beta, 0) \equiv CS^*(\beta)$ studied by Reade [16].
(4) $C(1, 0, \beta, 0) \equiv C(\beta)$ studied by Kaplan [5].
(5) $C(0, 0, 1, 0) \equiv S^*$ studied by Roberston [17], (see also [3], [21]).
(6) $C(1, 0, 1, 0) \equiv K$ studied by Roberston [17], (see also [3], [21]).

In the present paper, we study the integral operator

$$h(z) = \int_0^z \left\{ \frac{I_\mu^n f(t)}{t} \right\}^\delta dt \tag{1.3}$$

where $n \in N_0$ and δ is a real number. For $n = 0$ and $n = 1$ this integral operator was studied by Kim [6], Merkes and Wright [8], Mishra [9], Nunokawa([10], [11]), Pfaltzgraff [13], Royster [18], Patil and Tahakare [12] and Shukla and Kumar [20], (see also [15]).

To prove our main results, we shall require the following definition and lemmas.

Definition 1.1. Let $P(\alpha)$ denote the class of functions of the form

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

which are regular in U and satisfy $\Re\{P(z)\} > \alpha,\ z \in U$.

Lemma 1.2. *Let*

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

be analytic in U. If $\Re\{P(z)\} > \alpha$ in U, then

$$\alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{P(re^{i\theta})\right\} d\theta < 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1), \tag{1.4}$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$, $z = re^{i\theta}$ and $0 \le r < 1$.

Proof. Since

$$\Re\{P(z)\} > \alpha.$$

It is easy to see that

$$\left(\Re\{P(z)\} - \alpha\right)\big|_{z=0} = 1 - \alpha.$$

Then by mean value theorem, we have

$$0 \le \int_{\theta_1}^{\theta_2} \left(\Re\left\{P(re^{i\theta})\right\} - \alpha\right) d\theta \le \int_{0}^{2\pi} \left(\Re\left\{P(re^{i\theta})\right\} - \alpha\right) d\theta = 2\pi\left(1-\alpha\right).$$

or, equivalently

$$0 \le \int_{\theta_1}^{\theta_2} \left(\Re\left\{P(re^{i\theta})\right\}\right) d\theta - \alpha(\theta_2 - \theta_1) \le 2\pi\left(1-\alpha\right),$$

or

$$\alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{P(re^{i\theta})\right\} d\theta < 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1).$$

□

The following lemma is a direct consequence of Lemma 1.2, and improves a result of Patil and Thakare ([12], Lemma 2.2).

Lemma 1.3. *If $f \in S^*(\alpha)$, then*

$$\alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{\frac{zf'(z)}{f(z)}\right\} d\theta < 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1), \tag{1.5}$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$, $z = re^{i\theta}$ and $0 \le r < 1$.

In the following lemma, we obtain integral characterization for the class $C(n, \alpha, \beta, \mu)$.

Lemma 1.4. *If $f \in C(n, \alpha, \beta, \mu)$, then*

$$-\beta\pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta < \beta\pi + 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1), \quad (1.6)$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$ and $0 \leq r < 1$. Conversely, let f be analytic and satisfying $I_\mu^n f(z) \neq 0$ in U, if

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta > -\beta\pi + \alpha(\theta_2 - \theta_1)$$

then $f \in C(n, \alpha, \beta, \mu)$.

Proof. $f \in C(n, \alpha, \beta, \mu)$ implies that there exists a function $F \in S^*(\alpha)$ such that

$$\left|\arg \frac{I_\mu^n f(z)}{F(z)}\right| < \frac{\beta\pi}{2}, \quad z \in U.$$

Therefore

$$-\frac{1}{2}\beta\pi < \arg I_\mu^n f(z) - \arg F(z) < \frac{1}{2}\beta\pi.$$

Let $0 \leq \theta_1 < \theta_2 \leq 2\pi$. Then with $z = re^{i\theta_2}$, we have

$$-\frac{1}{2}\beta\pi < \arg I_\mu^n f(re^{i\theta_2}) - \arg F(re^{i\theta_2}) < \frac{1}{2}\beta\pi. \quad (1.7)$$

and with $z = re^{i\theta_1}$, we have

$$-\frac{1}{2}\beta\pi < -\arg I_\mu^n f(re^{i\theta_1}) + \arg F(re^{i\theta_1}) < \frac{1}{2}\beta\pi. \quad (1.8)$$

Combining (1.7) and (1.8), we obtain

$$\begin{aligned} -\beta\pi + \arg F(re^{i\theta_2}) - \arg F(re^{i\theta_1}) &< \arg I_\mu^n f(re^{i\theta_2}) - \arg I_\mu^n f(re^{i\theta_1}) \\ &< \beta\pi + \arg F(re^{i\theta_2}) - \arg F(re^{i\theta_1}), \end{aligned}$$

or

$$-\beta\pi + \int_{\theta_1}^{\theta_2} d\arg F(re^{i\theta}) < \int_{\theta_1}^{\theta_2} d\arg I_\mu^n f(re^{i\theta}) < \beta\pi + \int_{\theta_1}^{\theta_2} d\arg F(re^{i\theta}),$$

or

$$\begin{aligned} -\beta\pi + \int_{\theta_1}^{\theta_2} \Re\left\{\frac{zF'(z)}{F(z)}\right\} d\theta &< \int_{\theta_1}^{\theta_2} \Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta \\ &< \beta\pi + \int_{\theta_1}^{\theta_2} \Re\left\{\frac{zF'(z)}{F(z)}\right\} d\theta. \quad (1.9) \end{aligned}$$

But $F \in S^*(\alpha)$, then using Lemma 1.3 in (1.9), we have

$$-\beta\pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta < \beta\pi + 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1)$$

and this completes the proof of direct part of the lemma.

To prove the converse part, we follow the techniques of Kaplan [5] and Patil and Thakare [12] and can obtain the desired result.

□

Remark 1.5. If we put $n = 1$, $\mu = 0$ in Lemma 1.4, we obtain the following result
If $f \in C(\alpha, \beta)$, then

$$-\beta\pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta < \beta\pi + 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1), \quad (1.10)$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$, $z = re^{i\theta}$ and $0 \le r < 1$. Conversely, let f be analytic and satisfying $f'(z) \neq 0$ in U, if

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\beta\pi + \alpha(\theta_2 - \theta_1) \quad (1.11)$$

then $f \in C(\alpha, \beta)$.

2. Main results

Theorem 2.1. *If $f \in C(n, \alpha, \beta, \mu)$, then $h \in C(\eta, \gamma)$, provided*

$$\frac{-\gamma}{\beta + 2(1-\alpha)} \le \delta \le \frac{\gamma + 2(1-\eta)}{\beta + 2(1-\alpha)}. \quad (2.1)$$

The result is sharp when (i) $\gamma = 0$ (ii) $\eta = 0, \gamma = 1$.

Proof. From relation (1.3) we have

$$h'(z) = \left\{\frac{I_\mu^n f(z)}{z}\right\}^\delta.$$

Applying logarithmic differentiation and then taking real parts of both sides, we obtain

$$Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} = \delta Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} + (1-\delta).$$

For $\delta > 0$, using Lemma 1.4, we get

$$\begin{aligned}\int_{\theta_1}^{\theta_2} Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} d\theta &= \delta \int_{\theta_1}^{\theta_2} Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta + (1-\delta)(\theta_2 - \theta_1) \\ &> \delta[-\beta\pi + \alpha(\theta_2 - \theta_1)] + (1-\delta)(\theta_2 - \theta_1) \\ &= -\beta\delta\pi + [1 - (1-\alpha)\delta](\theta_2 - \theta_1).\end{aligned}$$

To prove that $h \in C(\eta, \gamma)$, we have to show that the right hand side of the above inequality is not less than $-\gamma\pi + \eta(\theta_2 - \theta_1)$, provided

$$0 \le \delta \le \frac{\gamma + 2(1-\eta)}{\beta + 2(1-\alpha)}. \quad (2.2)$$

Similarly, for $\delta < 0$, using Lemma 1.4, we get

$$\int_{\theta_1}^{\theta_2} \Re\left\{1+\frac{zh''(z)}{h'(z)}\right\} d\theta > \delta\left[\beta\pi + 2(1-\alpha) + \alpha(\theta_2-\theta_1)\right] + (1-\delta)(\theta_2-\theta_1).$$

To show that $h \in C(\eta,\gamma)$, we have to prove that the right-hand side of the above inequality is not less than $-\gamma\pi + \eta(\theta_2-\theta_1)$, provided

$$\frac{-\gamma}{\beta+2(1-\alpha)} \leq \delta \leq 0. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain (2.1).
Thus the proof of Theorem 2.1 is established.
To show the sharpness, let us take the function $f(z)$ defined by the relation

$$I_\mu^n f(z) = \frac{z}{(1-z)^{2(1-\alpha)+\beta}}, \tag{2.4}$$

then it is easy to see that this function belongs to $C(n,\alpha,\beta,\mu)$ with respect to the function $\frac{z}{(1-z)^{2(1-\alpha)}}$ belonging to $S^*(\alpha)$. Then

$$h(z) = \int_0^z \frac{dt}{(1-t)^{[2(1-\alpha)+\beta]\delta}} \tag{2.5}$$

and from condition (1.11) this functions belongs to $C(0,1)$ if and only if

$$\frac{-1}{2(1-\alpha)+\beta} \leq \delta \leq \frac{3}{2(1-\alpha)+\beta}.$$

Again for $\gamma = 0$, from (2.5) we have

$$1+\frac{zh''(z)}{h'(z)} = \frac{1+\left[1-2\left(1-\frac{\{2(1-\alpha)+\beta\}\delta}{2}\right)\right]z}{1-z}$$

and $\Re\left\{1+\frac{zh''(z)}{h'(z)}\right\} > \eta$ if and only if

$$1-\frac{\{2(1-\alpha)+\beta\}\,\delta}{2} \geq \eta \Rightarrow 0 \leq \delta \leq \frac{2(1-\eta)}{\beta+2(1-\alpha)}. \qquad \square$$

Remark 2.2. The undermentioned results are particular cases of Theorem 2.1.

(i) If we put $n = 0$ and $n = 1$ with $\mu = 0$ in Theorem 2.1 we obtain the corresponding results of Mishra [9].
(ii) If we put $n = 1, \beta = 0, \gamma = 0$ with $\mu = 0$ we obtain a result of Patil and Thakare [12].
(iii) If we put $n = 1, \beta = 0, \eta = 0$ with $\mu = 0$ we obtain a result of Patil and Thakare [12].
(iv) If we put $n = 1, \alpha = 0, \eta = 0$ with $\mu = 0$ we obtain a result of Patil and Thakare [12].
(v) If we put $n = 0, \beta = 0, \eta = 0$ we obtain a result of Patil and Thakare [12].
(vi) If we put $n = 1, \alpha = 0, \beta = 0, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Nunokawa [11] as well as that of Merkes and Wright [8].
(vii) If we put $n = 0, \alpha = 0, \beta = 0, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Nunokawa [11] as well as that of Merkes and Wright [8].

(viii) If we put $n = 1, \alpha = 0, \beta = 1, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Nunokawa [11] as well as that of Merkes and Wright [8].

(ix) If we put $n = 0, \alpha = 0, \eta = 0$ with $\mu = 0$ we obtain a result of Shukla and Kumar [20].

(x) If we put $n = 0, \alpha = 0, \beta = 1, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Kim [6].

(xi) If we put $n = 0, \alpha = 1/2, \beta = 0, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Nunokawa [11] as well as that of Merkes and Wright [8].

Theorem 2.3. *Let $f \in C(n, \alpha, \beta, \mu)$. Then for $|z| = r$*

$$\frac{r(1-r)^{\beta}}{(1+r)^{\beta+2(1-\alpha)}} \leq |I_{\mu}^{n} f(z)| \leq \frac{r(1+r)^{\beta}}{(1-r)^{\beta+2(1-\alpha)}}$$

The result is sharp.

Proof. By definition $f \in C(n, \alpha, \beta, \mu)$ if and only if there exists a function $P \in P(0)$ and $F(z) \in S^{*}(\alpha)$ such that

$$\frac{I_{\mu}^{n} f(z)}{F(z)} = [P(z)]^{\beta}.$$

Therefore

$$\left|I_{\mu}^{n} f(z)\right| = |P(z)|^{\beta} |F(z)|.$$

Now using the well-known inequalities (see [3])

$$\frac{1-r}{1+r} \leq |P(z)| \leq \frac{1+r}{1-r}$$

and

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leq |F(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}},$$

we obtain the required inequalities.
Sharpness follows if we take $f(z)$ connected by the relation

$$I_{\mu}^{n} f(z) = \frac{z(1+z)^{\beta}}{(1-z)^{\beta+2(1-\alpha)}}$$

and

$$F(z) = \frac{z}{(1-z)^{2(1-\alpha)}}.$$

□

Theorem 2.4. *If $f \in C(n, \alpha, \beta, \mu)$, then*

$$\left|\arg \frac{I_{\mu}^{n} f(z)}{z}\right| \leq \beta \sin^{-1} \frac{2r}{1+r^{2}} + 2(1-\alpha) \sin^{-1} r.$$

The result is sharp.

Proof. If $f \in C(n, \alpha, \beta, \mu)$, then

$$\frac{I_{\mu}^{n} f(z)}{F(z)} = [P(z)]^{\beta},$$

for some $P(z) \in P(0)$ and $F(z) \in S^{*}(\alpha)$.

Thus

$$\left|\arg \frac{I_\mu^n f(z)}{z}\right| \leq \alpha\left|\arg P(z)\right| + \left|\arg \frac{F(z)}{z}\right|. \tag{2.6}$$

Now using the well-known results

$$|\arg P(z)| \leq \sin^{-1} \frac{2r}{1+r^2} \tag{2.7}$$

and a result of Pinchuk [14]

$$\left|\arg \frac{F(z)}{z}\right| \leq 2(1-\alpha)\sin^{-1} r, \tag{2.8}$$

Using (2.7) and (2.8) in (2.6) we get the required result.
Sharpness follows if we take $f(z)$ to be the same as in Theorem 2.3. $\square$

Theorem 2.5. *If $f \in C(n,\alpha,\beta,\mu)$, then $f \in S(n)$ for $|z| < r_0$, where*

$$r_0 = \frac{(1+\beta-\alpha) - \sqrt{\alpha^2 - 2\beta\alpha + \beta(2+\beta)}}{1-2\alpha}, \ \textit{when } \alpha \neq 1/2$$

and

$$r_0 = \frac{1}{1+2\beta}, \ \textit{when } \alpha = 1/2.$$

The result is sharp.

Proof. $f \in C(n,\alpha,\beta,\mu)$, if and only if there there exists a function $P \in P(0)$ and $F(z) \in S^*(\alpha)$ such that

$$\frac{I_\mu^n f(z)}{F(z)} = [P(z)]^\beta.$$

$$I_\mu^n f(z) = [P(z)]^\beta F(z). \tag{2.9}$$

Logarithmic differentiation of (2.9) yields

$$\frac{z(I_\mu^n f(z))'}{I_\mu^n f(z)} = \beta\frac{zP'(z)}{P(z)} + \frac{zF'(z)}{F(z)}.$$

Now by a result of MacGregor [7], we know that

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{2r}{1-r^2}.$$

Therefore

$$\begin{aligned}\Re\left\{\frac{z(I_\mu^n f(z))'}{I_\mu^n f(z)}\right\} &\geq \Re\left\{\frac{zF'(z)}{F(z)}\right\} - \beta\left|\frac{zP'(z)}{P(z)}\right| \\ &\geq \frac{1-(1-2\alpha)r}{1+r} - \beta\left(\frac{2r}{1-r^2}\right) \\ &= \frac{(1-2\alpha)r^2 - 2(1+\beta-\alpha)r + 1}{1-r^2}.\end{aligned}$$

The right hand side of the above inequality is not less than or equal to zero provided $|z| = r < r_0$, where r_0 is as given in the statement of theorem. Sharpness follows if we take $f(z)$ to be the same as in Theorem 2.3. $\square$

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