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On Lupaş-Jain operators

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Abstract. In this paper, linear positive Lupaş-Jain operators are constructed and a recurrence formula for the moments is given. For the sequence of these operators; the weighted uniform approximation, also, monotonicity under convexity are obtained. Moreover, a preservation property of each Lupaş-Jain operator is presented.

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1. Introduction

In [13], Jain generalized the well known Százs-Mirakjan operators by constructing the linear positive operators given by

$$S_n^{\beta}\left(f\right)\left(x\right)=\sum_{k=0}^{\infty}\frac{nx\left(nx+k\beta\right)^{k-1}}{k!}e^{-\left(nx+k\beta\right)}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $f:[0,\infty)\to\mathbb{R},\ n\in\mathbb{N},\ x>0$ and $0\leq\beta<1$, with β may depend only on n. For some interesting works related to Jain's operators we refer to [2], [1], [5], [8], [17], [18] and references cited therein.

In [3], Agratini studied some approximation properties of the following linear positive operators

$$L_n\left(f\right)\left(x\right)=2^{-nx}\sum_{k=0}^{\infty}\frac{\left(nx\right)_k}{2^k k!}f\left(\frac{k}{n}\right) \tag{1.2}$$

for $n\in\mathbb{N},\ x\geq 0$ and some suitable $f:[0,\infty)\to\mathbb{R}$ that the operator $L_n\left(f\right)$ makes sense. These operators are special form of the well-known operators defined by Lupaş in [15] and resemble the familiar Százs-Mirakjan operators. In the paper [3], the author obtained some estimates for the order of approximation on a finite interval as well as proved a Voronovskaya type theorem. Moreover, Agratini also considered the Kantorovich extension of $L_n\left(f\right)$ for f belonging to the class of local integrable functions on $[0,\infty)$ and studied the degree of approximation [4]. Some approximation results and basic history concerning Lupaş operators can be found in [9], [10], [7].

Recently, Patel and Mishra extended the Lupaş operators given by (1.2) as

$$L_n^{\beta}(f)(x)=\sum_{k=0}^{\infty}\frac{(nx+k\beta)_k}{2^k k!}2^{-(nx+k\beta)}f\left(\frac{k}{n}\right) \tag{1.3}$$

for real valued functions f on $[0,\infty)$, where they assumed that

$$(nx+k\beta)_0=1,\ (nx+k\beta)_1=nx$$

and

$$(nx+k\beta)_k=nx\,(nx+k\beta)\,(nx+k\beta+1)\ldots(nx+k\beta+k-1)\,,\ k\geq 2$$

[19]. Here, the authors studied direct approximation results and gave Kantorovich and Durrmeyer types modifications of (1.3).

In this work, we also construct a generalization of the Lupaş operators L_n in the sense of Jain in [13]. Here, we point out that our expression is different from L_n^{β} given by (1.3) in such a way that in the construction, we take the negative subscript "-1" of the Pochhammer symbol into consideration, in which case the calculations become simpler in a remarkable degree. By using analogous Abel and Jensen combinatorial formulas for factorial powers (see, e.g., [20]), we show the monotonicity property of these operators for n under the convexity of f. We investigate that the Lupaş-Jain operator can retain the properties of the modulus of continuity function. Moreover, we study the weighted uniform approximation of functions from the polynomial weighted space given in [11].

In what follows, let α and β be real parameters such that $0<\alpha<\infty$ and $0\leq\beta<1$. Then, as in [13], Taking into account of the Lagrange inversion formula

$$\phi(z)=\phi(0)+\sum_{k=1}^{\infty}\frac{1}{k!}\left[\frac{d^{k-1}}{dz^{k-1}}(f(z))^k\phi'(z)\right]_{z=0}\left(\frac{z}{f(z)}\right)^k$$

for

$$\phi(z)=\frac{1}{(1-z)^{\alpha}}\text{ and }f(z)=\frac{1}{(1-z)^{\beta}},\ |z|<1,$$

we obtain

$$\frac{1}{(1-z)^{\alpha}}=1+\sum_{k=1}^{\infty}\frac{\alpha(\alpha+1+k\beta)_{k-1}}{k!}z^k(1-z)^{k\beta}, \tag{1.4}$$

where

$$(a)_n=\begin{cases}a(a+1)\ldots(a+n-1) & n\in\mathbb{N}\\ 1 & n=0,\ a\neq 0,\end{cases}$$

is the well-known Pochhammer symbol, from which we have

$$(a)_{-n}=\frac{1}{(a-1)(a-2)\ldots(a-n)}=\frac{1}{(a-n)_n}=\frac{(-1)^n}{(1-a)_n}$$

for negative subscripts when $a \neq 1, 2, ..., n$ (see, e.g., p.5 of [12]). Hence, we immediately get that $(\alpha+1)_{-1} = \frac{1}{(\alpha)_1} = \frac{1}{\alpha}$. Now, we have

$$1 = \sum_{k=0}^{\infty} \frac{\alpha\,(\alpha+1+k\beta)_{k-1}}{2^k k!} 2^{-(\alpha+k\beta)} \tag{1.5}$$

for $0 < \alpha < \infty$ and $0 \leq \beta < 1$. So, denoting

$$L\,(0,\alpha,\beta) := \sum_{k=0}^{\infty} \frac{(\alpha+1+k\beta)_{k-1}}{2^k k!} 2^{-(\alpha+k\beta)} \tag{1.6}$$

it readily follows from (1.5) that

$$\alpha L\,(0,\alpha,\beta) = 1. \tag{1.7}$$

Hence, we present the following recurrence formula.

Lemma 1.1. *Let* $0 < \alpha < \infty,\ 0 \leq \beta < 1,\ r \in \mathbb{N}$ *and*

$$L\,(r,\alpha,\beta) := \sum_{k=0}^{\infty} \frac{(\alpha+1+k\beta)_{k+r-1}}{2^k k!} 2^{-(\alpha+k\beta)}. \tag{1.8}$$

Then we have

$$L\,(r,\alpha,\beta) = \sum_{k=0}^{\infty} \left(\frac{\beta+1}{2}\right)^k (\alpha+r-1+k\beta)\, L\,(r-1,\alpha+k\beta,\beta)\,.$$

Proof. Taking the fact

$$(\alpha+1+k\beta)_{k+r-1} = (\alpha+1+k\beta)_{k+r-2}\,(\alpha+r-1+k\,(\beta+1))$$

into consideration, then one finds

$$L\,(r,\alpha,\beta) = (\alpha+r-1)\, L\,(r-1,\alpha,\beta) + \frac{\beta+1}{2} L\,(r,\alpha+\beta,\beta)\,.$$

Recursive application of the last formula gives the result. □

For the calculation of moments of the operators, we can use the well-known property of the geometric series given below (see, e.g., [21]).

Remark 1.2. ([21]) Consider the geometric series

$$h_n\,(x) := \sum_{k=0}^{\infty} k^n x^k \qquad -1 < x < 1,\ n \in \mathbb{N}$$

and

$$h_0\,(x) := \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k. \tag{1.9}$$

Term-wise differentiation gives that

$$h_n'\,(x) = \sum_{k=1}^{\infty} k^{n+1} x^{k-1},$$

which satisfies the following

$$xh_n'(x) = \sum_{k=1}^{\infty} k^{n+1} x^k = h_{n+1}(x).$$

From this recurrence, one has

$$h_1(x) = \frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k x^k, \tag{1.10}$$

$$h_2(x) = \frac{x^2+x}{(1-x)^3} = \sum_{k=1}^{\infty} k^2 x^k. \tag{1.11}$$

Lemma 1.3. *For the auxiliary function $L(r,\alpha,\beta)$ defined by (1.8), one has*

$$\begin{aligned} L(1,\alpha,\beta) &= \frac{2}{1-\beta}, \\ L(2,\alpha,\beta) &= \frac{2^2(\alpha+1)}{(1-\beta)^2} + \frac{2^2\beta(\beta+1)}{(1-\beta)^3}. \end{aligned}$$

Proof. Since $0 \le \beta < 1$, then (1.9), (1.10) and (1.11), with $x = \frac{\beta+1}{2}$, give that

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\frac{\beta+1}{2}\right)^k &= \frac{2}{1-\beta}, \\ \sum_{k=1}^{\infty} k \left(\frac{\beta+1}{2}\right)^k &= \frac{2(\beta+1)}{(1-\beta)^2}, \\ \sum_{k=1}^{\infty} k^2 \left(\frac{\beta+1}{2}\right)^k &= \frac{2\left(\beta^2+4\beta+3\right)}{(1-\beta)^3}. \end{aligned}$$

Combining these results with (1.6), (1.7) and (1.8), it readily follows that

$$\begin{aligned} L(1,\alpha,\beta) &= \sum_{k=0}^{\infty} \left(\frac{\beta+1}{2}\right)^k (\alpha+k\beta) L(0,\alpha+k\beta,\beta) \\ &= \frac{2}{1-\beta}. \end{aligned} \tag{1.12}$$

Also, $L(2,\alpha,\beta)$ is obtained as

$$\begin{aligned} L(2,\alpha,\beta) &= \sum_{k=0}^{\infty} \left(\frac{\beta+1}{2}\right)^k (\alpha+1+k\beta) L(1,\alpha+k\beta,\beta) \\ &= \frac{2(\alpha+1)}{1-\beta} \sum_{k=0}^{\infty} \left(\frac{\beta+1}{2}\right)^k + \frac{2\beta}{1-\beta} \sum_{k=0}^{\infty} k \left(\frac{\beta+1}{2}\right)^k \\ &= \frac{4(\alpha+1)}{(1-\beta)^2} + \frac{4\beta(\beta+1)}{(1-\beta)^3}. \end{aligned} \tag{1.13}$$

□

2. Construction of the operators

Taking $\alpha = nx,\ n \in \mathbb{N},\ x > 0$ in (1.5), we consider the following linear positive operators

$$L_n^{\beta}(f)(x) = \sum_{k=0}^{\infty} \frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-(nx+k\beta)} f\left(\frac{k}{n}\right), \quad x \in (0,\infty) \tag{2.1}$$

and $L_n^{\beta}(f)(0) = f(0)$ for real valued bounded functions f on $[0,\infty)$, where $0 \leq \beta < 1$, depending only on n. We call the operators L_n^{β} as Lupaş-Jain. Obviously, Lupaş-Jain operators reduce to Lupaş operators in [3] when $\beta = 0$.

Lemma 2.1. *Let $e_i(t) := t^i,\ i = 0,1,2$. For the Lupaş-Jain operators, one has*

$$\begin{aligned} L_n^{\beta}(e_0)(x) &= 1, \\ L_n^{\beta}(e_1)(x) &= \frac{x}{1-\beta}, \\ L_n^{\beta}(e_2)(x) &= \frac{x^2}{(1-\beta)^2} + \frac{2x}{n(1-\beta)^3}. \end{aligned}$$

Proof. It is clear from (1.5) that $L_n^{\beta}(e_0)(x) = 1$. By taking $f = e_1$ in (2.1) and using (1.12) in the result, we easily get

$$\begin{aligned} L_n^{\beta}(e_1)(x) &= \sum_{k=1}^{\infty} \frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-(nx+k\beta)} \left(\frac{k}{n}\right) \\ &= x \sum_{k=0}^{\infty} \frac{(nx+\beta+1+k\beta)_k}{2^{k+1} k!} 2^{-(nx+\beta+k\beta)} \\ &= \frac{x}{2} L(1, nx+\beta, \beta) \\ &= \frac{x}{1-\beta}. \end{aligned}$$

By taking $f = e_2$ and using (1.12) and (1.13) we find

$$\begin{aligned} L_n^{\beta}(e_2)(x) &= \sum_{k=1}^{\infty} \frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-(nx+k\beta)} \left(\frac{k}{n}\right)^2 \\ &= \frac{x}{n} \sum_{k=0}^{\infty} \frac{(nx+\beta+1+k\beta)_k}{2^{k+1} k!} 2^{-(nx+\beta+k\beta)} (k+1) \\ &= \frac{x}{n} \left\{ \frac{1}{2^2} L(2, nx+2\beta, \beta) + \frac{1}{2} L(1, nx+\beta, \beta) \right\} \\ &= \frac{x}{n} \left\{ \frac{(nx+1+2\beta)}{(1-\beta)^2} + \frac{\beta(\beta+1)}{(1-\beta)^3} + \frac{1}{1-\beta} \right\} \\ &= \frac{x^2}{(1-\beta)^2} + \frac{2x}{n(1-\beta)^3}. \end{aligned}$$

$\square$

3. Weighted approximation

In this section, we deal with the weighted uniform approximation result of the sequence of the Lupaş-Jain operators L_n^{β} by using Gadjiev's theorem in [11], for which we have the following settings:

We take $\varphi(x) = 1 + x^2$ as the suitable weight function and, for simplicity, denote $\mathbb{R}^+ := [0, \infty)$. Related to φ, we take the space

$$B_{\varphi}(\mathbb{R}^+) = \left\{ f : \mathbb{R}^+ \to \mathbb{R} \,\big|\, |f(x)| \leq M_f \varphi(x), \ x \in \mathbb{R}^+ \right\},$$

where M_f is a constant depending on f. $B_{\varphi}(\mathbb{R}^+)$ is a normed space with the norm

$$\|f\|_{\varphi} = \sup_{x \in \mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}.$$

Moreover, we denote, as usual, by $C_{\varphi}(\mathbb{R}^+)$, $C_{\varphi}^k(\mathbb{R}^+)$ the following subspaces of $B_{\varphi}(\mathbb{R}^+)$

$$\begin{aligned} C_{\varphi}(\mathbb{R}^+) &: \left\{ f \in B_{\varphi}(\mathbb{R}^+) : f \text{ is continuous} \right\}, \\ C_{\varphi}^k(\mathbb{R}^+) &= \left\{ f \in C_{\varphi}(\mathbb{R}^+) \,\Bigg|\, \lim_{x \to \infty} \frac{f(x)}{\varphi(x)} = k_f \right\}, \end{aligned}$$

respectively, where k_f is a constant depending on f. We have the following two results due to Gadjiev in [11]:

Lemma 3.1. *The linear positive operators T_n, $n \in \mathbb{N}$, act from $C_{\varphi}(\mathbb{R}^+)$ to $B_{\varphi}(\mathbb{R}^+)$ if and only if*

$$|T_n(\varphi)(x)| \leq K\varphi(x),$$

where K is a positive constant.

Theorem 3.2. *Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of linear positive operators mapping $C_{\varphi}(\mathbb{R}^+)$ into $B_{\varphi}(\mathbb{R}^+)$ and satisfying the conditions*

$$\lim_{n \to \infty} \|T_n(e_i) - e_i\|_{\varphi} = 0, \quad \textit{for } i = 0, 1, 2.$$

Then for any $f \in C_{\varphi}^k(\mathbb{R}^+)$ one has

$$\lim_{n \to \infty} \|T_n(f) - f\|_{\varphi} = 0.$$

Now, we treat weighted uniform approximation for Lupaş-Jain operators L_n^{β} acting on $C_{\varphi}(\mathbb{R}^+)$. In order to get an approximation result, as in [13], we need to make an adjustment to the parameter β by taking it as a sequence such that $\beta = \beta_n$, $0 \leq \beta_n < 1$ and $\lim_{n \to \infty} \beta_n = 0$.

Theorem 3.3. *Let $\{\beta_n\}_{n \in \mathbb{N}}$ be a sequence such that $0 \leq \beta_n < 1$ and $\lim_{n \to \infty} \beta_n = 0$. Then for each $f \in C_{\varphi}^k(\mathbb{R}^+)$ we have*

$$\lim_{n \to \infty} \left\| L_n^{\beta_n}(f) - f \right\|_{\varphi} = 0.$$

Proof. According to Lemmas 2.1 and 3.1 we get that the operators $L_n^{\beta_n}$ act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$. Now, it only remains to show the sufficient conditions of the Theorem 3.2 for $L_n^{\beta_n}$. Using Lemma 2.1 and the hypothesis on β_n, we obtain that

$$\lim_{n\to\infty} \left\| L_n^{\beta_n}(e_0) - e_0 \right\|_\varphi = 0$$

and that

$$\left\| L_n^{\beta_n}(e_1) - e_1 \right\|_\varphi \leq \frac{\beta_n}{1-\beta_n},$$

which gives

$$\lim_{n\to\infty} \left\| L_n^{\beta_n}(e_1) - e_1 \right\|_\varphi = 0.$$

Finally, since $2x \leq 1 + x^2$, we get

$$\begin{aligned}
\left\| L_n^{\beta_n}(e_2) - e_2 \right\|_\varphi &= \sup_{x\in\mathbb{R}^+} \frac{\left| L_n^{\beta_n}(e_2) - e_2 \right|}{1+x^2} \\
&= \sup_{x\in\mathbb{R}^+} \left| \frac{1}{1+x^2} \left(\frac{x^2}{(1-\beta_n)^2} + \frac{2x}{n(1-\beta_n)^3} - x^2 \right) \right| \\
&= \sup_{x\in\mathbb{R}^+} \left| \frac{x^2}{1+x^2} \frac{2\beta_n - \beta_n^2}{(1-\beta_n)^2} + \frac{2x}{1+x^2} \frac{1}{n(1-\beta_n)^3} \right| \\
&\leq \frac{2\beta_n - \beta_n^2}{(1-\beta_n)^2} + \frac{1}{n(1-\beta_n)^3},
\end{aligned}$$

which clearly gives that

$$\lim_{n\to\infty} \left\| L_n^{\beta_n}(e_2) - e_2 \right\|_\varphi = 0.$$

This completes the proof. □

4. The monotonicity of the sequence of Lupaş-Jain operators

Recall that *a continuous function f is said to be convex in $D \subseteq \mathbb{R}$, if*

$$f\left(\sum_{i=1}^n \alpha_i t_i\right) \leq \sum_{i=1}^n \alpha_i f(t_i)$$

for every $t_1, t_2, ..., t_n \in D$ and for every nonnegative numbers $\alpha_1, \alpha_2, ..., \alpha_n$ such that $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$.

For the proof of the main result of this section, we need the corresponding definition of the well-known Jensen and Abel combinatorial formulas for factorial powers. Below, we reproduce these formulas from the work of Stancu and Occorsio (pp.175-176 of [20]) for the increment -1, respectively.

$$\begin{aligned}
&(u+v)(u+v+1+m\beta)_{m-1} \\
&= \sum_{k=0}^m \binom{m}{k} u(u+1+k\beta)_{k-1} v(v+1+(m-k)\beta)_{m-k-1} \qquad (4.1)
\end{aligned}$$

and

$$(u+v+m\beta)_m = \sum_{k=0}^{m} \binom{m}{k} (u+k\beta)_k \, v \, (v+1+(m-k)\beta)_{m-k-1} \,. \tag{4.2}$$

Note that the monotonicity of Szász-Mirakjan operators of convex function was proved by Cheney and Sharma [6]. On the other hand, the same result for the Lupaş operators was obtained by Erençin et al. [7]. Now, we present the monotonicity of each Lupaş-Jain operator $L_n^{\beta}(f)$ for n, when f is a convex function.

Theorem 4.1. *Let f be a convex function defined on $[0,\infty)$. Then, for all n, $L_n^{\beta}(f)$ is non-increasing in n.*

Proof. For $x=0$, the result is obvious. So, for $x>0$, we can write

$$2^x = \sum_{k=0}^{\infty} \frac{x(x+1+k\beta)_{k-1}}{2^k k!} 2^{-k\beta}$$

by (1.5) with $\alpha = x$. Using this formula we can write

$$\begin{aligned}
& L_n^{\beta}(f)(x) - L_{n+1}^{\beta}(f)(x) \\
= \; & 2^x \sum_{k=0}^{\infty} \frac{nx(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-[(n+1)x+k\beta]} f\left(\frac{k}{n}\right) \\
& - \sum_{k=0}^{\infty} \frac{(n+1)x((n+1)x+1+k\beta)_{k-1}}{2^k k!} 2^{-[(n+1)x+k\beta]} f\left(\frac{k}{n+1}\right) \\
= \; & \sum_{l=0}^{\infty} \frac{x(x+1+l\beta)_{l-1}}{2^l l!} 2^{-l\beta} \sum_{k=0}^{\infty} \frac{nx(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-[(n+1)x+k\beta]} f\left(\frac{k}{n}\right) \\
& - \sum_{k=0}^{\infty} \frac{(n+1)x((n+1)x+1+k\beta)_{k-1}}{2^k k!} 2^{-[(n+1)x+k\beta]} f\left(\frac{k}{n+1}\right) \\
= \; & \sum_{l=0}^{\infty} \frac{x(x+1+l\beta)_{l-1}}{2^l l!} 2^{-l\beta} \\
& \times \sum_{k=l}^{\infty} \frac{nx(nx+1+(k-l)\beta)_{k-l-1}}{2^{k-l}(k-l)!} 2^{-[(n+1)x+(k-l)\beta]} f\left(\frac{k-l}{n}\right) \\
& - \sum_{k=0}^{\infty} \frac{(n+1)x((n+1)x+1+k\beta)_{k-1}}{2^k k!} 2^{-[(n+1)x+k\beta]} f\left(\frac{k}{n+1}\right).
\end{aligned}$$

Changing the order of the above summations, we obtain that

$$\begin{aligned}
& L_n^{\beta}(f)(x) - L_{n+1}^{\beta}(f)(x) \\
= & \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{x(x+1+l\beta)_{l-1}}{l!} \frac{nx(nx+1+(k-l)\beta)_{k-l-1}}{2^k (k-l)!} 2^{-[(n+1)x+k\beta]} f\left(\frac{k-l}{n}\right) \\
& - \sum_{k=0}^{\infty} \frac{(n+1)x((n+1)x+1+k\beta)_{k-1}}{2^k k!} 2^{-[(n+1)x+k\beta]} f\left(\frac{k}{n+1}\right)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \left\{ \sum_{l=0}^{k} \frac{nx\left(nx+1+l\beta\right)_{l-1}}{l!} \frac{x\left(x+1+\left(k-l\right)\beta\right)_{k-l-1}}{2^k\left(k-l\right)!} f\left(\frac{l}{n}\right) \right.$$
$$\left. - \frac{\left(n+1\right)x\left(\left(n+1\right)x+1+k\beta\right)_{k-1}}{2^k k!} f\left(\frac{k}{n+1}\right) \right\} 2^{-[(n+1)x+k\beta]} \tag{4.3}$$

Now, denote

$$\alpha_l := \binom{k}{l} \frac{nx\left(nx+1+l\beta\right)_{l-1} x\left(x+1+\left(k-l\right)\beta\right)_{k-l-1}}{\left(n+1\right)x\left(\left(n+1\right)x+1+k\beta\right)_{k-1}} > 0$$

and

$$t_l := \frac{l}{n}.$$

Taking $u = nx,\ v = x$ and $m = k$ in (4.1) one has

$$\begin{aligned} & \left(n+1\right)x\left(\left(n+1\right)x+1+k\beta\right)_{k-1} \\ = \ & \sum_{l=0}^{k} \binom{k}{l} nx\left(nx+1+l\beta\right)_{l-1} x\left(x+1+\left(k-l\right)\beta\right)_{k-l-1}, \end{aligned}$$

which clearly gives that

$$\sum_{l=0}^{k} \alpha_l = 1.$$

On the other hand, taking $u = nx+\beta+1,\ v = x$ and $m = k-1$ in (4.2), it follows that

$$\begin{aligned} & \left(\left(n+1\right)x+1+k\beta\right)_{k-1} \\ = \ & \left(nx+\beta+1+x+\left(k-1\right)\beta\right)_{k-1} \\ = \ & \sum_{l=0}^{k-1} \binom{k-1}{l} \left(nx+\beta+1+l\beta\right)_l x\left(x+1+\left(k-1-l\right)\beta\right)_{k-l-2}. \end{aligned}$$

Taking into account of the above fact, it follows that

$$\begin{aligned} \sum_{l=0}^{k} \alpha_l t_l &= \frac{\sum_{l=1}^{k} \binom{k}{l} nx\left(nx+1+l\beta\right)_{l-1} x\left(x+1+\left(k-l\right)\beta\right)_{k-l-1} \left(\frac{l}{n}\right)}{\left(n+1\right)x\left(\left(n+1\right)x+1+k\beta\right)_{k-1}} \\ &= \frac{k\sum_{l=0}^{k-1} \binom{k-1}{l} nx\left(nx+\beta+1+l\beta\right)_l x\left(x+1+\left(k-1-l\right)\beta\right)_{k-l-2}}{n\left(n+1\right)x\left(\left(n+1\right)x+1+k\beta\right)_{k-1}} \\ &= \frac{k}{n+1} \frac{\sum_{l=0}^{k-1} \binom{k-1}{l} \left(nx+\beta+1+l\beta\right)_l x\left(x+1+\left(k-1-l\right)\beta\right)_{k-l-2}}{\left(\left(n+1\right)x+1+k\beta\right)_{k-1}} \\ &= \frac{k}{n+1}. \end{aligned}$$

Hence, making use of the convexity of f, (4.3) gives that

$$L_n^{\beta}(f)(x) \geq L_{n+1}^{\beta}(f)(x)$$

for all $n \in \mathbb{N}$, which completes the proof. $\square$

5. A preservation property

We recall the following definition for the subsequent result.

Definition 5.1. A continuous, and non-negative function ω defined on $[0,\infty)$ is called a function of modulus of continuity, if each of the following conditions is satisfied:

i) $\omega(u+v) \leq \omega(u) + \omega(v)$ for $u, v \in [0,\infty)$, i.e., ω is subadditive,

ii) $\omega(u) \geq \omega(v)$ for $u \geq v$, i.e., ω is non-decreasing,

iii) $\lim_{u\to 0^+} \omega(u) = \omega(0) = 0$ ([16]).

In [14], Li noticed a new preservation property that the Bernstein polynomials $B_n,\ n \in \mathbb{N}$ satisfy. Li proved that *if $\omega(x)$ is a modulus of continuity function, then for each $n \in \mathbb{N}$, $B_n(\omega; x)$ is also a modulus of continuity function.* The same result for the Lupaş operators was obtained in [7]. Below, we show that this result is satisfied by the Lupaş-Jain operators as well.

Theorem 5.2. *Let ω be a modulus of continuity function. Then, for all n, $L_n^{\beta}(\omega)$ is also a modulus of continuity function.*

Proof. Let $x, y \in [0,\infty)$ and $x \leq y$. Then from the definition of L_n^{β}, we have

$$L_n^{\beta}(\omega)(y) = \sum_{k=0}^{\infty} \frac{ny\,(ny+1+k\beta)_{k-1}}{2^k k!} 2^{-(ny+k\beta)} \omega\left(\frac{k}{n}\right).$$

Taking nx and $n(y-x)$ in place of u and v, respectively in (4.1), we obtain

$$\begin{aligned} & ny\,(ny+1+m\beta)_{m-1} \qquad (5.1) \\ = & \sum_{i=0}^{k} \binom{k}{i} nx\,(nx+1+i\beta)_{i-1}\, n(y-x)\,(n(y-x)+1+(k-i)\beta)_{k-i-1} \end{aligned}$$

which implies

$$\begin{aligned} & L_n^{\beta}(\omega)(y) \\ = & \sum_{k=0}^{\infty}\sum_{i=0}^{k} \omega\left(\frac{k}{n}\right)\binom{k}{i} \frac{nx\,(nx+1+i\beta)_{i-1}}{2^k k!} 2^{-(ny+k\beta)} \\ & \times n(y-x)\,(n(y-x)+1+(k-i)\beta)_{k-i-1}. \end{aligned}$$

Interchanging the order of the above summations gives that

$$\begin{aligned} & L_n^{\beta}(\omega)(y) \\ = & \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \omega\left(\frac{k}{n}\right) \frac{1}{i!\,(k-i)!} nx\,(nx+1+i\beta)_{i-1} \frac{2^{-(ny+k\beta)}}{2^k} \qquad (5.2) \\ & n(y-x)\,(n(y-x)+1+(k-i)\beta)_{k-i-1}. \end{aligned}$$

Taking $k-i=l$, (5.2) reduces to

$$\begin{aligned} & L_n^{\beta}(\omega)(y) \\ = \;& \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{i+l}{n}\right)nx\left(nx+1+i\beta\right)_{i-1}\frac{2^{-(ny+(i+l)\beta)}}{2^{i+l}i!l!} \\ & \times n\left(y-x\right)\left(n\left(y-x\right)+1+l\beta\right)_{l-1}. \end{aligned} \tag{5.3}$$

On the other hand, $L_n^{\beta}(\omega)(x)$ can be written as

$$\begin{aligned} L_n^{\beta}(\omega)(x) &= \sum_{i=0}^{\infty}\omega\left(\frac{i}{n}\right)nx\left(nx+1+i\beta\right)_{i-1}\frac{2^{-(nx+i\beta)}}{2^{i}i!} \\ &= \sum_{i=0}^{\infty}\omega\left(\frac{i}{n}\right)nx\left(nx+1+i\beta\right)_{i-1}\frac{2^{-(ny+i\beta)}2^{n(y-x)}}{2^{i}i!}. \end{aligned} \tag{5.4}$$

Since

$$2^{n(y-x)}=\sum_{l=0}^{\infty}n\left(y-x\right)\left(n\left(y-x\right)+1+l\beta\right)_{l-1}\frac{2^{-l\beta}}{2^{l}l!}$$

then, one may write

$$\begin{aligned} L_n^{\beta}(\omega)(x) &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{i}{n}\right)nx\left(nx+1+i\beta\right)_{i-1}\frac{2^{-(ny+(i+l)\beta)}}{2^{i+l}i!l!} \\ & \quad\times n\left(y-x\right)\left(n\left(y-x\right)+1+l\beta\right)_{l-1}. \end{aligned} \tag{5.5}$$

Subtracting (5.5) from (5.3)

$$\begin{aligned} & L_n^{\beta}(\omega)(y)-L_n^{\beta}(\omega)(x) \\ = \;& \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\left[\omega\left(\frac{i+l}{n}\right)-\omega\left(\frac{i}{n}\right)\right]nx\left(nx+1+i\beta\right)_{i-1}\frac{2^{-(ny+(i+l)\beta)}}{2^{i+l}i!l!} \\ & \times n\left(y-x\right)\left(n\left(y-x\right)+1+l\beta\right)_{l-1} \end{aligned} \tag{5.6}$$

and using the hypothesis that ω is a modulus of continuity function, one obtains

$$\begin{aligned} \leq\;& \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)nx\left(nx+1+i\beta\right)_{i-1}\frac{2^{-(ny+(i+l)\beta)}}{2^{i+l}i!l!} \\ & \times n\left(y-x\right)\left(n\left(y-x\right)+1+l\beta\right)_{l-1} \\ =\;& \sum_{i=0}^{\infty}nx\left(nx+1+i\beta\right)_{i-1}\frac{2^{-i\beta}}{2^{i}i!} \\ & \times\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)n\left(y-x\right)\left(n\left(y-x\right)+1+l\beta\right)_{l-1}\frac{2^{-(ny+l\beta)}}{2^{l}l!} \\ =\;& \sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)n\left(y-x\right)\left(n\left(y-x\right)+1+l\beta\right)_{l-1}\frac{2^{-(n(y-x)+l\beta)}}{2^{l}l!} \\ =\;& L_n^{\beta}(\omega)\left(y-x\right)). \end{aligned} \tag{5.7}$$

This shows that $L_n^{\beta}(\omega)$ satisfies the subadditivity property. Since ω is non-decreasing, then (5.6) provides that $L_n^{\beta}(\omega)(y) \geq L_n^{\beta}(\omega)(x)$ when $y \geq x$, namely, $L_n^{\beta}(\omega)$ is non-decreasing. From the definition of L_n^{β} it is obvious that $\lim_{x\to 0} L_n^{\beta}(\omega; x) = L_n^{\beta}(\omega; 0) = \omega(0) = 0$. Therefore, $L_n^{\beta}(\omega)$ is a function of modulus of continuity. □

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