

Sufficient conditions of boundedness of \mathbf{L} -index and analog of Hayman's Theorem for analytic functions in a ball

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Abstract. We generalize some criteria of boundedness of \mathbf{L} -index in joint variables for analytic in an unit ball functions. Our propositions give an estimate maximum modulus of the analytic function on a skeleton in polydisc with the larger radii by maximum modulus on a skeleton in the polydisc with the lesser radii. An analog of Hayman's Theorem for the functions is obtained. Also we established a connection between class of analytic in ball functions of bounded l_j -index in every direction $\mathbf{1}_j$, $j \in \{1, \dots, n\}$ and class of analytic in ball of functions of bounded \mathbf{L} -index in joint variables, where $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $l_j : \mathbb{B}^n \rightarrow \mathbb{R}_+$ is continuous function, $\mathbf{1}_j = (0, \dots, 0, \underbrace{1}_{j\text{-th place}}, 0, \dots, 0) \in \mathbb{R}_+^n$, $z \in \mathbb{C}^n$.

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1. Introduction

Recently, there was introduced a concept of analytic function in a ball in \mathbb{C}^n of bounded \mathbf{L} -index in joint variables [8]. We also obtained criterion of boundedness of \mathbf{L} -index in joint variables which describes a local behavior of partial derivatives on a skeleton in the polydisc and established other important properties of analytic functions in a ball of bounded \mathbf{L} -index in joint variables. Those investigations used an idea of exhaustion of a ball in \mathbb{C}^n by polydiscs.

The presented paper is a continuation of our investigations from [8]. We set the goal to prove new analogues of criteria of boundedness of \mathbf{L} -index in joint variables for analytic in a ball functions. Particular, we prove an estimate of maximum modulus on a greater polydisc by maximum modulus on a lesser polydisc (Theorems 3.1, 3.2) and obtain an analog of Hayman's Theorem for analytic functions in a ball of bounded

L-index in joint variables (Theorems 4.1 and 4.2). For entire functions similar propositions were obtained by A. I. Bandura, M. T. Bordulyak, O. B. Skaskiv [4, 5] in a case $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $z \in \mathbb{C}^n$. Also A. I. Bandura, N.V. Petrechko, O. B. Skaskiv [6, 7] deduced same results for analytic in a polydisc functions. Hayman’s Theorem and its generalizations for different classes of analytic functions [1, 3, 5, 7, 12, 15, 20, 21] are very important in theory of functions of bounded index. The criterion is helpful [1, 9] to investigate boundedness of index of entire solutions of ordinary or partial differential equations.

Note that the corresponding theorems for entire functions of bounded l -index and of bounded L -index in direction were also applied to investigate infinite products (see bibliography in [21, 1]). Thus, those generalizations for analytic in a ball functions are necessary to study **L**-index in joint variables of analytic solutions of PDE’s, its systems and multidimensional counterparts of Blaschke products. At the end of the paper, we present a scheme of application of Hayman’s Theorem to study properties of analytic solutions in the unit ball.

2. Main definitions and notations

We need some standard notations. Denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$, $\mathbf{1}_j = (0, \dots, 0, \underbrace{1}_{j\text{-th place}}, 0, \dots, 0) \in \mathbb{R}_+^n$,

$$R = (r_1, \dots, r_n) \in \mathbb{R}_+^n, z = (z_1, \dots, z_n) \in \mathbb{C}^n, |z| = \sqrt{\sum_{j=1}^n |z_j|^2}.$$

For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$ we will use formal notations without violation of the existence of these expressions

$$AB = (a_1 b_1, \dots, a_n b_n), A/B = (a_1/b_1, \dots, a_n/b_n), A^B = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}, \|A\| = a_1 + \dots + a_n,$$

and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, \dots, n\}$; the relation $A \leq B$ is defined similarly. For $K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ denote $K! = k_1! \cdot \dots \cdot k_n!$. The polydisc $\{z \in \mathbb{C}^n : |z_j - z_j^0| < r_j, j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : |z_j - z_j^0| = r_j, j = 1, \dots, n\}$ is denoted by $\mathbb{T}^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : |z_j - z_j^0| \leq r_j, j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n[z^0, R]$. The open ball $\{z \in \mathbb{C}^n : |z - z^0| < r\}$ is denoted by $\mathbb{B}^n(z^0, r)$, its boundary is a sphere $\mathbb{S}^n(z^0, r) = \{z \in \mathbb{C}^n : |z - z^0| = r\}$, the closed ball $\{z \in \mathbb{C}^n : |z - z^0| \leq r\}$ is denoted by $\mathbb{B}^n[z^0, r]$, $\mathbb{B}^n = \mathbb{B}^n(\mathbf{0}, 1)$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : |z| < 1\}$.

For $K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ and the partial derivatives of an analytic in \mathbb{B}^n function $F(z) = F(z_1, \dots, z_n)$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \dots + k_n} F}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z) : \mathbb{B}^n \rightarrow \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{B}^n) : l_j(z) > \beta / (1 - |z|), j \in \{1, \dots, n\}, \tag{2.1}$$

where $\beta > \sqrt{n}$ is a some constant. For a polydisc A.I. Bandura, N.V. Petrechko and O.B. Skaskiv [6, 7] imposed the restriction $(\forall z \in \mathbb{D}^n(\mathbf{0}, \mathbf{1})) : l_j(z) > \beta/(1 - |z_j|)$, $j \in \{1, \dots, n\}$. A similar condition is used in one-dimensional case by S.N. Strochyk, M.M. Sheremeta, V.O. Kushnir [22, 14, 21].

Note that if $R \in \mathbb{R}_+^n$, $|R| \leq \beta$, $z^0 \in \mathbb{B}^n$ and $z \in \mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]$ then $z \in \mathbb{B}^n$ (see Remark 1 in [8]).

An analytic function $F: \mathbb{B}^n \rightarrow \mathbb{C}$ is said to be of *bounded \mathbf{L} -index (in joint variables)*, if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{B}^n$ and for all $J \in \mathbb{Z}_+^n$

$$\frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} \leq \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : K \in \mathbb{Z}_+^n, \|K\| \leq n_0 \right\}. \tag{2.2}$$

The least such integer n_0 is called the *\mathbf{L} -index in joint variables of the function F* and is denoted by $N(F, \mathbf{L}, \mathbb{B}^n)$ (see [8]). Entire and analytic in polydisc functions of bounded \mathbf{L} -index in joint variables are considered in [4, 5, 6, 7, 10, 13, 19, 18, 16, 17].

By $Q(\mathbb{B}^n)$ we denote the class of functions \mathbf{L} , which satisfy (2.1) and the following condition

$$(\forall R \in \mathbb{R}_+^n, |R| \leq \beta, j \in \{1, \dots, n\}) : 0 < \lambda_{1,j}(R) \leq \lambda_{2,j}(R) < \infty, \tag{2.3}$$

where $\lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{B}^n} \inf \{l_j(z)/l_j(z^0) : z \in \mathbb{D}^n [z^0, R/\mathbf{L}(z^0)]\}$,

$$\lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{B}^n} \sup \{l_j(z)/l_j(z^0) : z \in \mathbb{D}^n [z^0, R/\mathbf{L}(z^0)]\}.$$

$$\Lambda_1(R) = (\lambda_{1,1}(R), \dots, \lambda_{1,n}(R)), \quad \Lambda_2(R) = (\lambda_{2,1}(R), \dots, \lambda_{2,n}(R)).$$

We need the following results.

Theorem 2.1 ([8]). *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. An analytic in \mathbb{B}^n function F has bounded \mathbf{L} -index in joint variables if and only if for each $R \in \mathbb{R}_+^n$, $|R| \leq \beta$, there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for every $z^0 \in \mathbb{B}^n$ there exists $K^0 \in \mathbb{Z}_+^n$, $\|K^0\| \leq n_0$, and*

$$\max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : \|K\| \leq n_0, z \in \mathbb{D}^n [z^0, R/\mathbf{L}(z^0)] \right\} \leq p_0 \frac{|F^{(K^0)}(z^0)|}{K^0! \mathbf{L}^{K^0}(z^0)}. \tag{2.4}$$

Denote $\tilde{\mathbf{L}}(z) = (\tilde{l}_1(z), \dots, \tilde{l}_n(z))$. The notation $\mathbf{L} \asymp \tilde{\mathbf{L}}$ means that there exist

$$\Theta_1 = (\theta_{1,j}, \dots, \theta_{1,n}) \in \mathbb{R}_+^n, \quad \Theta_2 = (\theta_{2,j}, \dots, \theta_{2,n}) \in \mathbb{R}_+^n$$

such that $\forall z \in \mathbb{B}^n \theta_{1,j} \tilde{l}_j(z) \leq l_j(z) \leq \theta_{2,j} \tilde{l}_j(z)$ for each $j \in \{1, \dots, n\}$.

Theorem 2.2 ([8]). *Let $\mathbf{L} \in Q(\mathbb{B}^n)$, $\mathbf{L} \asymp \tilde{\mathbf{L}}$, $\beta|\Theta_1| > \sqrt{n}$. An analytic in \mathbb{B}^n function F has bounded \mathbf{L} -index in joint variables if and only if F has bounded \mathbf{L} -index in joint variables.*

3. Local behaviour of maximum modulus of analytic in ball function

For an analytic in \mathbb{B}^n function F we put

$$M(r, z^0, F) = \max\{|F(z)| : z \in \mathbb{T}^n(z^0, r)\},$$

where $z^0 \in \mathbb{B}^n$, $r \in \mathbb{R}_+^n$. Then $M(R, z^0, F) = \max\{|F(z)|: z \in \mathbb{D}^n[z^0, R]\}$, because the maximum modulus for an analytic function in a closed polydisc is attained on its skeleton.

The following proposition uses an idea about the possibility of replacing universal quantifier by existential quantifier in sufficient conditions of index boundedness [2]. To prove an analog of Hayman’s Theorem we need this theorem which has an independent interest.

Theorem 3.1. *Let $\mathbf{L} \in Q^n$, $F : \mathbb{B}^n \rightarrow \mathbb{C}$ be analytic function. If there exist $R', R'' \in \mathbb{R}_+^n$, $R' < R''$, $|R''| < \beta$ and $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{C}^n$*

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \leq p_1 M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) \tag{3.1}$$

then F is of bounded \mathbf{L} -index in joint variables.

Proof. At first, we assume that $\mathbf{0} < R' < \mathbf{1} < R''$.

Let $z^0 \in \mathbb{B}^n$ be an arbitrary point. We expand a function F in power series

$$F(z) = \sum_{K \geq \mathbf{0}} b_K (z - z^0)^K = \sum_{k_1, \dots, k_n \geq 0} b_{k_1, \dots, k_n} (z_1 - z_1^0)^{k_1} \dots (z_n - z_n^0)^{k_n}, \tag{3.2}$$

where $b_K = b_{k_1, \dots, k_n} = \frac{F^{(K)}(z^0)}{K!}$.

Let $\mu(R, z^0, F) = \max\{|b_K|R^K : K \geq \mathbf{0}\}$ be a maximal term of power series (3.2) and

$$\nu(R) = \nu(R, z^0, F) = (\nu_1^0(R), \dots, \nu_n^0(R))$$

be a set of indices such that

$$\mu(R, z^0, F) = |b_{\nu(R)}| R^{\nu(R)},$$

$$\|\nu(R)\| = \sum_{j=1}^n \nu_j(R) = \max\{\|K\| : K \geq \mathbf{0}, |b_K|R^K = \mu(R, z^0, F)\}.$$

In view of inequality (3.8) we obtain for any $|R| < 1 - |z^0|$,

$$\mu(R, z^0, F) \leq M(R, z^0, F).$$

Then for given R' and R'' with $0 < |R'| < 1 < |R''| < \beta$ we conclude

$$\begin{aligned} M(R'R, z^0, F) &\leq \sum_{k \geq \mathbf{0}} |b_k|(R'R)^k \leq \sum_{k \geq \mathbf{0}} \mu(R, z^0, F)(R')^k \\ &= \mu(R, z^0, F) \sum_{k \geq \mathbf{0}} (R')^k = \prod_{j=1}^n \frac{1}{1 - r'_j} \mu(R, z^0, F). \end{aligned}$$

Besides,

$$\begin{aligned} \ln \mu(R, z^0, F) &= \ln\{|b_{\nu(R)}|R^{\nu(R)}\} = \ln\left\{|b_{\nu(R)}|(RR'')^{\nu(R)} \frac{1}{(R'')^{\nu(R)}}\right\} \\ &= \ln\{|b_{\nu(R)}|(RR'')^{\nu(R)}\} + \ln\left\{\frac{1}{(R'')^{\nu(R)}}\right\} \\ &\leq \ln \mu(R''R, z^0, F) - \|\nu(R)\| \ln \min_{1 \leq j \leq n} r''_j. \end{aligned}$$

This implies that

$$\begin{aligned} \|\nu(R)\| &\leq \frac{1}{\ln \min_{1 \leq j \leq n} r''_j} (\ln \mu(R''R, z^0, F) - \ln \mu(R, z^0, F)) \\ &\leq \frac{1}{\ln \min_{1 \leq j \leq n} r''_j} \left(\ln M(R''R, z^0, F) - \ln \left(\prod_{j=1}^n (1 - r'_j) M(R'R, z^0, F) \right) \right) \\ &\leq \frac{1}{\ln \min_{1 \leq j \leq n} r''_j} (\ln M(R''R, z^0, F) - \ln M(R'R, z^0, F)) - \frac{\sum_{j=1}^n \ln(1 - r'_j)}{\min_{1 \leq j \leq n} r''_j} \\ &= \frac{1}{\min_{1 \leq j \leq n} r''_j} \ln \frac{M(R''R, z^0, F)}{M(R'R, z^0, F)} - \frac{\sum_{j=1}^n \ln(1 - r_j)}{\min_{1 \leq j \leq n} r''_j}. \end{aligned} \tag{3.3}$$

Put $R = \frac{1}{\mathbf{L}(z^0)}$. Now let $N(F, z^0, \mathbf{L})$ be the **L**-index of the function F in joint variables at point z^0 i. e. it is the least integer for which inequality (2.2) holds at point z^0 . Clearly that

$$N(F, z^0, \mathbf{L}) \leq \nu\left(\frac{1}{\mathbf{L}(z^0)}, z^0, F\right) = \nu(R, z^0, F). \tag{3.4}$$

But

$$M(R''/\mathbf{L}(z^0), z^0, F) \leq p_1(R', R'')M(R'/\mathbf{L}(z^0), z^0, F). \tag{3.5}$$

Therefore, from (3.3), (3.4), (3.5) we obtain that $\forall z^0 \in \mathbb{B}^n$

$$N(F, z^0, \mathbf{L}) \leq \frac{-\sum_{j=1}^2 \ln(1 - r'_j)}{\ln \min\{r''_1, r''_2\}} + \frac{\ln p_1(R', R'')}{\ln \min\{r''_1, r''_2\}}.$$

This means that F has bounded **L**-index in joint variables, if $\mathbf{0} < R' < \mathbf{1} < R''$, $|R''| < \beta$.

Now we will prove the theorem for any $\mathbf{0} < R' < R''$, $|R''| < \beta$. From (3.1) with $\mathbf{0} < R_1 < R_2$ it follows that

$$\begin{aligned} &\max\left\{|F(z)| : z \in \mathbb{T}^n \left(z^0, \frac{2R''}{R' + R''} \frac{R' + R''}{2\mathbf{L}(z^0)}\right)\right\} \\ &\leq P_1 \max\left\{|F(z)| : z \in \mathbb{T}^n \left(z^0, \frac{2R'}{R' + R''} \frac{R' + R''}{2\mathbf{L}(z^0)}\right)\right\}. \end{aligned}$$

Denoting $\tilde{\mathbf{L}}(z) = \frac{2\mathbf{L}(z)}{R'+R''}$, we obtain

$$\begin{aligned} & \max \left\{ |F(z)| : z \in \mathbb{T}^n \left(z^0, \frac{2R''}{(R'+R'')\tilde{\mathbf{L}}(z^0)} \right) \right\} \\ & \leq P_1 \max \left\{ |F(z)| : z \in \mathbb{T}^n \left(z^0, \frac{2R''}{(R'+R'')\tilde{\mathbf{L}}(z^0)} \right) \right\}, \end{aligned}$$

where $\mathbf{0} < \frac{2R'}{R'+R''} < \mathbf{1} < \frac{2R''}{R'+R''}$. Taking into account the first part of the proof, we conclude that the function F has bounded $\tilde{\mathbf{L}}$ -index in joint variables. By Theorem 2.2, the function F is of bounded \mathbf{L} -index in joint variables. \square

Also the corresponding necessary conditions are valid.

Theorem 3.2. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. If an analytic in \mathbb{B}^n function F has bounded \mathbf{L} -index in joint variables then for any $R', R'' \in \mathbb{R}_+^n$, $R' < R''$, $|R''| \leq \beta$, there exists a number $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{B}^n$ inequality (3.1) holds.*

Proof. Let $N(F, \mathbf{L}) = N < +\infty$. Suppose that inequality (3.1) does not hold i.e. there exist R', R'' , $0 < |R'| < |R''| < \beta$, such that for each $p_* \geq 1$ and for some $z^0 = z^0(p_*)$

$$M \left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F \right) > p_* M \left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F \right). \tag{3.6}$$

By Theorem 2.1, there exists a number $p_0 = p_0(R'') \geq 1$ such that for every $z^0 \in \mathbb{B}^n$ and some $K^0 \in \mathbb{Z}_+^n$, $\|K^0\| \leq N$, (i.e. $n_0 = N$, see proof of necessity of Theorem 2.1 in [8]) one has

$$M \left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F^{(K^0)} \right) \leq p_0 |F^{(K^0)}(z^0)|. \tag{3.7}$$

We put

$$\begin{aligned} b_1 &= p_0 \left(\prod_{j=2}^n \lambda_{2,j}^N(R'') \right) (N!)^{n-1} \left(\sum_{j=1}^N \frac{(N-j)!}{(r_1'')^j} \right) \left(\frac{r_1'' r_2'' \dots r_n''}{r_1' r_2' \dots r_n'} \right)^N, \\ b_2 &= p_0 \left(\prod_{j=3}^n \lambda_{2,j}^N(R'') \right) (N!)^{n-2} \left(\sum_{j=1}^N \frac{(N-j)!}{(r_2'')^j} \right) \left(\frac{r_2'' \dots r_n''}{r_2' \dots r_n'} \right)^N \left\{ 1, \frac{1}{(r_1')^N} \right\}, \\ & \dots \\ b_{n-1} &= p_0 \lambda_{2,n}^N(R') N! \left(\sum_{j=1}^N \frac{(N-j)!}{(r_{n-1}'')^j} \right) \left(\frac{r_{n-1}'' r_n''}{r_{n-1}' r_n'} \right)^N \max \left\{ 1, \frac{1}{(r_1' \dots r_{n-2}')^N} \right\}, \\ b_n &= p_0 \left(\sum_{j=1}^N \frac{(N-j)!}{(r_n'')^j} \right) \left(\frac{r_n''}{r_n'} \right)^N \max \left\{ 1, \frac{1}{(r_1' \dots r_{n-1}')^N} \right\} \end{aligned}$$

and

$$p_* = (N!)^n p_0 \left(\frac{r_1'' r_2'' \dots r_n''}{r_1' r_2' \dots r_n'} \right)^N + \sum_{k=1}^n b_k + 1.$$

Let $z^0 = z^0(p_*)$ be a point for which inequality (3.6) holds and K^0 be such that (3.7) holds and

$$M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) = |F(z^*)|, \quad M\left(\frac{r''}{\mathbf{L}(z^0)}, z^0, F^{(J)}\right) = |F^{(J)}(z_J^*)|$$

for every $J \in \mathbb{Z}_+^n$, $\|J\| \leq N$. We apply Cauchy's inequality

$$|F^{(J)}(z^0)| \leq J! \left(\frac{\mathbf{L}(z^0)}{R'}\right)^J |F(z^*)| \tag{3.8}$$

for estimate the difference

$$\begin{aligned} & |F^{(J)}(z_{J,1}^*, z_{J,2}^*, \dots, z_{J,n}^*) - F^{(J)}(z_1^0, z_{J,2}^*, \dots, z_{J,n}^*)| \\ &= \left| \int_{z_1^0}^{z_{J,1}^*} \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}}(\xi, z_{J,2}^*, \dots, z_{J,n}^*) d\xi \right| \\ &\leq \left| \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}}(z_{(j_1+1, j_2, \dots, j_n)}^*) \right| \frac{r_1''}{l_1(z^0)}. \end{aligned} \tag{3.9}$$

Taking into account $(z_1^0, z_{J,2}^*, \dots, z_{J,n}^*) \in \mathbb{D}^n[z^0, \frac{R''}{\mathbf{L}(z^0)}]$, for all $k \in \{1, \dots, n\}$,

$$|z_{J,k}^* - z_k^0| = \frac{r_k''}{l_k(z^0)}, \quad l_k(z_1^0, z_{J,2}^*, \dots, z_{J,n}^*) \leq \lambda_{2,k}(R'') l_k(z^0)$$

and (3.8) with $J = K^0$, by Theorem 2.1 we have

$$\begin{aligned} & |F^{(J)}(z_1^0, z_{J,2}^*, \dots, z_{J,n}^*)| \\ &\leq \frac{J! l_1^{j_1}(z_1^0, z_{J,2}^*, \dots, z_{J,n}^*) \prod_{k=2}^n l_k^{j_k}(z_1^0, z_{J,2}^*, \dots, z_{J,n}^*)}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 |F^{(K^0)}(z^0)| \\ &\leq \frac{J! \mathbf{L}^J(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 K^0! \left(\frac{\mathbf{L}(z^0)}{R'}\right)^{K^0} |F(z^*)| \\ &= \frac{p_0 J! \mathbf{L}^J(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{(R')^{K^0}} |F(z^*)|. \end{aligned} \tag{3.10}$$

From inequalities (3.9) and (3.10) it follows that

$$\begin{aligned} & \left| \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}}(z_{(j_1+1, j_2, \dots, j_n)}^*) \right| \\ &\geq \frac{l_1(z^0)}{r_1''} \left\{ |F^{(J)}(z_j^*)| - |F^{(J)}(z_1^0, z_{J,2}^*, \dots, z_{J,n}^*)| \right\} \\ &\geq \frac{l_1(z_1^0)}{r_1''} |F^{(J)}(z_j^*)| - \frac{p_0 J! \mathbf{L}^{(j_1+1, j_2, \dots, j_n)}(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{r_1'' (R')^{K^0}} |F(z^*)|. \end{aligned}$$

Then

$$|F^{(K^0)}(z_{K^0}^*)| \geq \frac{l_1(z^0)}{r_1''} \left| \frac{\partial^{\|K^0\|-1} f}{\partial z_1^{k_1^0-1} \partial z_2^{k_2^0} \dots \partial z_n^{k_n^0}}(z_{(k_1^0-1, k_2^0, \dots, k_n^0)}^*) \right|$$

$$\begin{aligned}
 & \frac{p_0(k_1^0 - 1)!k_2^0! \dots k_n^0! \mathbf{L}^{K^0}(z^0) \prod_{i=2}^n \lambda_{2,i}^{k_i^0}(R'')}{r_1''(R')^{K^0}} |F(z^*)| \\
 & \geq \frac{l_1^2(z^0)}{(r_1'')^2} \left| \frac{\partial^{\|K^0\|-2} f}{\partial z_1^{k_1^0-2} \partial z_2^{k_2^0} \dots \partial z_n^{k_n^0}}(z_{(k_1^0-2, k_2^0, \dots, k_n^0)}^*) \right| \\
 & \frac{p_0(k_1^0 - 2)!k_2^0! \dots k_n^0! \mathbf{L}^{K^0}(z^0) \prod_{i=2}^n \lambda_{2,i}^{k_i^0}(R'')}{(r_1'')^2(R')^{K^0}} |F(z^*)| \\
 & \frac{p_0(k_1^0 - 1)!k_2^0! \dots k_n^0! \mathbf{L}^{K^0}(z^0) \prod_{i=2}^n \lambda_{2,i}^{k_i^0}(r_i'')}{r_1''(R')^{K^0}} |F(z^*)| \\
 & \dots \\
 & \geq \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}} \left| \frac{\partial^{\|K^0\|-k_1^0} f}{\partial z_2^{k_2^0} \dots \partial z_n^{k_n^0}}(z_{(0, k_2^0, \dots, k_n^0)}^*) \right| \\
 & - \frac{p_0}{(R')^{K^0}} \mathbf{L}^{K^0}(z^0) \left(\prod_{i=2}^n \lambda_{2,i}^{k_i^0}(R'') \right) k_2^0! \dots k_n^0! \sum_{j_1=1}^{k_1^0} \frac{(k_1^0 - j_1)!}{(r_1'')^{j_1}} |F(z^*)| \dots \\
 & \geq \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}} \frac{l_2^{k_2^0}(z^0)}{(r_2'')^{k_2^0}} \left| \frac{\partial^{\|K^0\|-k_1^0-k_2^0} f}{\partial z_3^{k_3^0} \dots \partial z_n^{k_n^0}}(z_{(0,0, k_3^0, \dots, k_n^0)}^*) \right| \\
 & - \frac{l_1^{k_1^0}(z^0) p_0 L^{(0, k_2^0, \dots, k_n^0)}(z^0)}{(r_1'')^{k_1^0} (R')^{K^0}} \left(\prod_{i=3}^n \lambda_{2,i}^{k_i^0}(R'') \right) k_3^0! \dots k_n^0! \sum_{j_2=1}^{k_2^0} \frac{(k_2^0 - j_2)!}{(r_2'')^{j_2}} |F(z^*)| \\
 & - \frac{p_0}{(R')^{K^0}} \mathbf{L}^{K^0}(z^0) \left(\prod_{i=2}^n \lambda_{2,i}^{k_i^0}(R'') \right) k_2^0! \dots k_n^0! \sum_{j_1=1}^{k_1^0} \frac{(k_1^0 - j_1)!}{(r_1'')^{j_1}} |F(z^*)| \\
 & \dots \\
 & \geq \left(\frac{L(z^0)}{R''} \right) |F(z_{\mathbf{0}^*})| - |F(z^*)| \sum_{i=1}^b \tilde{b}_i, \tag{3.11}
 \end{aligned}$$

where in view of the inequalities $\lambda_{2,i}(R'') \geq 1$ and $R'' \geq R'$ we have

$$\begin{aligned}
 \tilde{b}_1 &= \frac{p_0}{(R')^{K^0}} \mathbf{L}^{K^0}(z^0) \left(\prod_{i=2}^n \lambda_{2,i}^{k_i^0}(R'') \right) k_2^0! \dots k_n^0! \sum_{j_1=1}^{k_1^0} \frac{(k_1^0 - j_1)!}{(r_1'')^{j_1}} \\
 &= \left(\frac{\mathbf{L}(z^0)}{R''} \right)^{K^0} \left(\frac{R''}{R'} \right)^{K^0} p_0 \left(\prod_{i=2}^n \lambda_{2,i}^{k_i^0}(R'') \right) k_2^0! \dots k_n^0! \sum_{j_1=1}^{k_1^0} \frac{(k_1^0 - j_1)!}{(r_1'')^{j_1}} \leq \left(\frac{\mathbf{L}(z^0)}{R''} \right)^{K^0} b_1,
 \end{aligned}$$

$$\tilde{b}_2 = \frac{p_0}{(R')^{K^0}} \mathbf{L}^{K^0}(z^0) \left(\prod_{i=3}^n \lambda_{2,i}^{k_i^0}(R'') \right) \frac{k_3^0! \dots k_n^0!}{(r_1'')^{k_1^0}} \sum_{j_2=1}^{k_2^0} \frac{(k_2^0 - j_2)!}{(r_2'')^{j_2}} \leq \left(\frac{\mathbf{L}(z^0)}{R''} \right)^{K^0} b_2,$$

...

$$\begin{aligned} \tilde{b}_{n-1} &= \frac{p_0}{(R')^{K^0}} \mathbf{L}^{K^0}(z^0) \lambda_{2,n}^{k_n^0}(R'') \frac{k_n^0!}{(r_1'')^{k_1^0} \dots (r_{n-2}'')^{k_{n-2}^0}} \times \\ &\quad \times \sum_{j_{n-1}=1}^{k_{n-1}^0} \frac{(k_{n-1}^0 - j_{n-1})!}{(r_{n-1}'')^{j_{n-1}}} \leq \left(\frac{\mathbf{L}(z^0)}{R''} \right)^{K^0} b_{n-1}, \\ \tilde{b}_n &= \frac{p_0}{(R')^{K^0}} \mathbf{L}^{K^0}(z^0) \frac{1}{(r_1'')^{k_1^0} \dots (r_{n-1}'')^{k_{n-1}^0}} \sum_{j_n=1}^{k_n^0} \frac{(k_n^0 - j_n)!}{(r_n'')^{j_n}} \leq \left(\frac{\mathbf{L}(z^0)}{R''} \right)^{K^0} b_n. \end{aligned}$$

Thus, (3.11) implies that

$$|F^{(K^0)}(z_{K^0}^*)| \geq \left(\frac{\mathbf{L}(z^0)}{R''} \right)^{K^0} |F(z^*)| \left\{ \frac{|F(z_0^*)|}{|F(z^*)|} - \sum_{j=1}^n b_j \right\}.$$

But in view of (3.6) and a choice of p_* we have

$$\frac{|F(z_0^*)|}{|F(z^*)|} \geq p_* > \sum_{j=1}^n b_j.$$

Thus, (3.7) and (3.8) imply

$$\begin{aligned} |F^{(K^0)}(z_{K^0}^*)| &\geq \left(\frac{\mathbf{L}(z^0)}{R''} \right)^{K^0} |F(z^*)| \left\{ p_* - \sum_{j=1}^n b_j \right\} \\ &\geq \left(\frac{\mathbf{L}(z^0)}{R''} \right)^{K^0} \left\{ p_* - \sum_{j=1}^n b_j \right\} \frac{|F^{(K^0)}(z^0)|(R')^{K^0}}{K^0! \mathbf{L}^{K^0}(z^0)} \\ &\geq \left(\frac{r_1' \dots r_n'}{r_1'' \dots r_n''} \right)^N \left\{ p_* - \sum_{j=1}^n b_j \right\} \frac{|F^{(K^0)}(z_{K^0}^*)|}{p_0(n!)^n}. \end{aligned}$$

Hence, we have $p_* \leq p_0 \left(\frac{r_1' \dots r_n'}{r_1'' \dots r_n''} \right)^N (N!)^n + \sum_{j=1}^n b_j$, but this contradicts the choice of p_* . □

4. Analogue of Theorem of Hayman for analytic in a ball function of bounded \mathbf{L} -index in joint variables

Theorem 4.1. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. An analytic function F in \mathbb{B}^n has bounded \mathbf{L} -index in joint variables if and only if there exist $p \in \mathbb{Z}_+$ and $c \in \mathbb{R}_+$ such that for each $z \in \mathbb{B}^n$*

$$\max \left\{ \frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} : \|J\| = p + 1 \right\} \leq c \cdot \max \left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \|K\| \leq p \right\}. \tag{4.1}$$

Proof. Let $N = N(F, \mathbf{L}, \mathbb{B}^n) < +\infty$. The definition of the boundedness of \mathbf{L} -index in joint variables yields the necessity with $p = N$ and $c = ((N + 1)!)^n$.

We prove the sufficiency. For $F \equiv 0$ theorem is obvious. Thus, we suppose that $F \not\equiv 0$. Denote $\beta = (\frac{\beta}{\sqrt{n}}, \dots, \frac{\beta}{\sqrt{n}})$.

Assume that (4.1) holds, $z^0 \in \mathbb{B}^n$, $z \in \mathbb{D}^n[z^0, \frac{\beta}{\mathbf{L}(z^0)}]$. For all $J \in \mathbb{Z}_+^n$, $\|J\| \leq p+1$, one has

$$\begin{aligned} \frac{|F^{(J)}(z)|}{\mathbf{L}^J(z^0)} &\leq \Lambda_2^J(\beta) \frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} \leq c \cdot \Lambda_2^J(\beta) \max \left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \|K\| \leq p \right\} \\ &\leq c \cdot \Lambda_2^J(\beta) \max \left\{ \Lambda_1^{-K}(2) \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z^0)} : \|K\| \leq p \right\} \leq BG(z), \end{aligned} \tag{4.2}$$

where $B = c \cdot \max\{\Lambda_2^K(\beta) : \|K\| = p+1\} \max\{\Lambda_1^{-K}(\beta) : \|K\| \leq p\}$, and

$$G(z) = \max \left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z^0)} : \|K\| \leq p \right\}.$$

We choose

$$z^{(1)} = (z_1^{(1)}, \dots, z_n^{(1)}) \in \mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)})$$

and

$$z^{(2)} = (z_1^{(2)}, \dots, z_n^{(2)}) \in \mathbb{T}^n(z^0, \frac{\beta}{\mathbf{L}(z^0)})$$

such that $F(z^{(1)}) \neq 0$ and

$$|F(z^{(2)})| = M \left(\frac{\beta}{\mathbf{L}(z^0)}, z^0, F \right) \neq 0. \tag{4.3}$$

These points exist, otherwise if $F(z) \equiv 0$ on skeleton

$$\mathbb{T}^n \left(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)} \right) \quad \text{or} \quad \mathbb{T}^n \left(z^0, \frac{\beta}{\mathbf{L}(z^0)} \right)$$

then by the uniqueness theorem $F \equiv 0$ in \mathbb{B}^n . We connect the points $z^{(1)}$ and $z^{(2)}$ with plane

$$\alpha = \begin{cases} z_2 = k_2 z_1 + c_2, \\ z_3 = k_3 z_1 + c_3, \\ \dots \\ z_n = k_n z_1 + c_n, \end{cases}$$

where

$$k_i = \frac{z_i^{(2)} - z_i^{(1)}}{z_1^{(2)} - z_1^{(1)}}, \quad c_i = \frac{z_i^{(1)} z_1^{(2)} - z_i^{(2)} z_1^{(1)}}{z_1^{(2)} - z_1^{(1)}}, \quad i = 2, \dots, n.$$

It is easy to check that $z^{(1)} \in \alpha$ and $z^{(2)} \in \alpha$. Let $\tilde{G}(z_1) = G(z)|_\alpha$ be a restriction of the function G onto α .

For every $K \in \mathbb{Z}_+^n$ the function $F^{(K)}(z)|_\alpha$ is analytic function of variable z_1 and $\tilde{G}(z_1^{(1)}) = G(z^{(1)})|_\alpha \neq 0$ because $F(z^{(1)}) \neq 0$. Hence, all zeros of the function $F^{(K)}(z)|_\alpha$ are isolated as zeros of a function of one variable. Thus, zeros of the function $\tilde{G}(z_1)$ are isolated too. Therefore, we can choose piecewise analytic curve γ onto α as following

$$z = z(t) = (z_1(t), k_2 z_1(t) + c_2, \dots, k_n z_1(t) + c_n), \quad t \in [0, 1],$$

which connect the points $z^{(1)}, z^{(2)}$ and such that $G(z(t)) \neq 0$ and

$$\int_0^1 |z'_1(t)| dt \leq \frac{2\beta}{\sqrt{n}l_1(z_1^0)}.$$

For a construction of the curve we connect $z_1^{(1)}$ and $z_1^{(2)}$ by a line

$$z_1^*(t) = (z_1^{(2)} - z_1^{(1)})t + z_1^{(1)}, \quad t \in [0, 1].$$

The curve γ can cross points z_1 at which the function $\tilde{G}(z_1) = 0$. The number of such points $m = m(z^{(1)}, z^{(2)})$ is finite. Let $(z_{1,k}^*)$ be a sequence of these points in ascending order of the value $|z_1^{(1)} - z_{1,k}^*|$, $k \in \{1, 2, \dots, m\}$. We choose

$$r < \min_{1 \leq k \leq m-1} \left\{ |z_{1,k}^* - z_{1,k+1}^*|, |z_{1,1}^* - z_1^{(1)}|, |z_{1,m}^* - z_1^{(2)}|, \frac{2\beta^2 - 1}{2\pi\sqrt{n}\beta l_1(z^0)} \right\}.$$

Now we construct circles with centers at the points $z_{1,k}^*$ and corresponding radii $r'_k < \frac{r}{2^k}$ such that $\tilde{G}(z_1) \neq 0$ for all z_1 on the circles. It is possible, because $F \neq 0$.

Every such circle is divided onto two semicircles by the line $z_1^*(t)$. The required piecewise-analytic curve consists with arcs of the constructed semicircles and segments of line $z_1^*(t)$, which connect the arcs in series between themselves or with the points $z_1^{(1)}, z_1^{(2)}$. The length of $z_1(t)$ in \mathbb{C} (but not $z(t)$ in $\mathbb{C}^n!$) is lesser than

$$\frac{\beta/\sqrt{n}}{l_1(z^0)} + \frac{1}{2\sqrt{n}\beta l_1(z^0)} + \pi r \leq \frac{2\beta}{\sqrt{n}l_1(z^0)}.$$

Then

$$\begin{aligned} \int_0^1 |z'_s(t)| dt &= |k_s| \int_0^1 |z'_1(t)| dt \leq \frac{|z_s^{(2)} - z_s^{(1)}|}{|z_1^{(2)} - z_1^{(1)}|} \frac{2\beta}{\sqrt{n}l_1(z^0)} \\ &\leq \frac{2\beta^2 + 1}{2\sqrt{n}\beta l_s(z^0)} \frac{2\sqrt{n}\beta l_1(z^0)}{2\beta^2 - 1} \frac{2\beta}{\sqrt{n}l_1(z^0)} \leq \frac{2\beta(2\beta^2 + 1)}{(2\beta^2 - 1)\sqrt{n}l_s(z^0)}, \quad s \in \{2, \dots, n\}. \end{aligned}$$

Hence,

$$\int_0^1 \sum_{s=1}^n l_s(z^0) |z'_s(t)| dt \leq \frac{2\beta(2\beta^2 + 1)\sqrt{n}}{2\beta^2 - 1} = S. \tag{4.4}$$

Since the function $z = z(t)$ is piece-wise analytic on $[0, 1]$, then for arbitrary $K \in \mathbb{Z}_+^n$, $J \in \mathbb{Z}_+^n$, $\|K\| \leq p$, either

$$\frac{|F^{(K)}(z(t))|}{\mathbf{L}^K(z^0)} \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}, \tag{4.5}$$

or the equality

$$\frac{|F^{(K)}(z(t))|}{\mathbf{L}^K(z^0)} = \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)} \tag{4.6}$$

holds only for a finite set of points $t_k \in [0; 1]$.

Then for function $G(z(t))$ as maximum of such expressions $\frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ by all $\|J\| \leq p$ two cases are possible:

1. In some interval of analyticity of the curve γ the function $G(z(t))$ identically equals simultaneously to some derivatives, that is (4.5) holds. It means that $G(z(t)) \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ for some $J, \|J\| \leq p$. Clearly, the function $F^{(J)}(z(t))$ is analytic. Then $|F^{(J)}(z(t))|$ is continuously differentiable function on the interval of analyticity except points where this partial derivative equals zero $|F^{(j_1, j_2)}(z_1(t), z_2(t))| = 0$. However, there are not the points, because in the opposite case $G(z(t)) = 0$. But it contradicts the construction of the curve γ .
2. In some interval of analyticity of the curve γ the function $G(z(t))$ equals simultaneously to some derivatives at a finite number of points t_k , that is (4.6) holds. Then the points t_k divide interval of analyticity onto a finite number of segments, in which of them $G(z(t))$ equals to one from the partial derivatives, i. e. $G(z(t)) \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ for some $J, \|J\| \leq p$. As above, in each from these segments the functions $|F^{(J)}(z(t))|$, and $G(z(t))$ are continuously differentiable except the points t_k .

The inequality

$$\frac{d}{dt}|f(t)| \leq \left| \frac{df(t)}{dt} \right|$$

holds for complex-valued functions of real argument outside a countable set of points. In view of this fact and (4.2) we have

$$\begin{aligned} \frac{d}{dt}G(z(t)) &\leq \max \left\{ \frac{1}{\mathbf{L}^J(z^0)} \left| \frac{d}{dt}F^{(J)}(z(t)) \right| : \|J\| \leq p \right\} \\ &\leq \max \left\{ \sum_{s=1}^n \left| \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1} \dots \partial z_s^{j_s+1} \dots \partial z_n^{j_n}}(z(t)) \right| \frac{|z'_s(t)|}{\mathbf{L}^J(z^0)} : \|J\| \leq p \right\} \\ &\leq \max \left\{ \sum_{s=1}^n \left| \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1} \dots \partial z_s^{j_s+1} \dots \partial z_n^{j_n}}(z(t)) \right| \frac{l_s(z^0)|z'_s(t)|}{l_1^{j_1}(z^0) \dots l_s^{j_s+1}(z^0) \dots l_n^{j_n}(z^0)} : \right. \\ &\quad \left. \|J\| \leq p \right\} \leq \left(\sum_{s=1}^n l_s(z^0)|z'_s(t)| \right) \max \left\{ \frac{|F^{(j)}(z(t))|}{\mathbf{L}^J(z^0)} : \|J\| \leq p+1 \right\} \\ &\leq \left(\sum_{s=1}^n l_s(z^0)|z'_s(t)| \right) BG(z(t)). \end{aligned}$$

Therefore, (4.4) yields

$$\left| \ln \frac{G(z^{(2)})}{G(z^{(1)})} \right| = \left| \int_0^1 \frac{1}{G(z(t))} \frac{d}{dt}G(z(t)) dt \right| \leq B \int_0^1 \sum_{s=1}^n l_s(z^0)|z'_s(t)| dt \leq S \cdot B.$$

Using (4.3), we deduce

$$M \left(\frac{\beta}{\mathbf{L}(z^0)}, z^0, F \right) \leq G(z^{(2)}) \leq G(z^{(1)})e^{SB}.$$

Since $z^{(1)} \in \mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)})$, the Cauchy inequality holds

$$\frac{|F^{(J)}(z^{(1)})|}{\mathbf{L}^J(z^0)} \leq J!(2\beta\sqrt{n})^{\|J\|} M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right).$$

for all $J \in \mathbb{Z}_+^n$. Therefore, for $\|J\| \leq p$ we obtain

$$G(z^{(1)}) \leq (p!)^n (2\beta\sqrt{n})^p M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right),$$

$$M\left(\frac{\beta}{\mathbf{L}(z^0)}, z^0, F\right) \leq e^{SB} (p!)^n (2\beta\sqrt{n})^p M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right).$$

Hence, by Theorem 3.1 the function F has bounded \mathbf{L} -index in joint variables. \square

The following result was also obtained for other classes of holomorphic functions in [21, 11, 7].

Theorem 4.2. *Let $\mathbf{L} \in Q(\mathbb{B}^n)$. An analytic function F in \mathbb{B}^n has bounded \mathbf{L} -index in joint variables if and only if there exist $c \in (0; +\infty)$ and $N \in \mathbb{N}$ such that for each $z \in \mathbb{B}^n$ the inequality*

$$\sum_{\|K\|=0}^N \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \geq c \sum_{\|K\|=N+1}^{\infty} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}. \tag{4.7}$$

Proof. Let $\frac{1}{\beta} < \theta_j < 1$, $j \in \{1, \dots, n\}$, $\Theta = (\theta_1, \dots, \theta_n)$. If the function F has bounded \mathbf{L} -index in joint variables then by Theorem 2.2 F has bounded $\tilde{\mathbf{L}}$ -index in joint variables, where $\tilde{\mathbf{L}} = (\tilde{l}_1(z), \dots, \tilde{l}_n(z))$, $\tilde{l}_j(z) = \theta_j l_j(z)$, $j \in \{1, \dots, n\}$. Let $\tilde{N} = N(F, \tilde{\mathbf{L}}, \mathbb{B}^n)$. Therefore,

$$\begin{aligned} & \max \left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq \tilde{N} \right\} = \max \left\{ \frac{\Theta^K |F^{(K)}(z)|}{K!\tilde{\mathbf{L}}^K(z)} : \|K\| \leq \tilde{N} \right\} \\ & \geq \prod_{s=1}^n \theta_s^{\tilde{N}} \max \left\{ \frac{|F^{(K)}(z)|}{K!\tilde{\mathbf{L}}^K(z)} : \|K\| \leq \tilde{N} \right\} \geq \prod_{s=1}^n \theta_s^{\tilde{N}} \frac{|F^{(J)}(z)|}{J!\tilde{\mathbf{L}}^J(z)} = \prod_{s=1}^n \theta_s^{\tilde{N}-j_s} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \end{aligned}$$

for all $J \geq \mathbf{0}$ and

$$\begin{aligned} & \sum_{\|J\|=\tilde{N}+1}^{\infty} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \leq \max \left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq \tilde{N} \right\} \sum_{\|J\|=\tilde{N}+1}^{\infty} \theta_s^{j_s-\tilde{N}} \\ & = \prod_{i=1}^n \frac{\theta_s}{1-\theta_s} \max \left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq \tilde{N} \right\} \leq \prod_{i=1}^n \frac{\theta_s}{1-\theta_s} \sum_{\|K\|=0}^{\tilde{N}} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}. \end{aligned}$$

Hence, we obtain (4.7) with $N = \tilde{N}$ and

$$c = \prod_{i=1}^n \frac{\theta_s}{1-\theta_s}.$$

On the contrary, inequality (4.7) implies

$$\begin{aligned} \max \left\{ \frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} : \|J\| = N + 1 \right\} &\leq \sum_{\|K\|=N+1}^{\infty} \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} \leq \frac{1}{c} \sum_{\|K\|=0}^N \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} \\ &\leq \frac{1}{c} \sum_{i=0}^N C_{n+i-1}^i \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : \|K\| \leq N \right\} \end{aligned}$$

and by Theorem 4.1 F is of bounded \mathbf{L} -index in joint variables. □

5. Some application for PDE: a scheme

Here we present a scheme of application of Hayman’s Theorem to PDE. This is also applicable in a more general situation.

Let us consider the following system of partial differential equations:

$$\begin{cases} F^{(2,0)}(z_1, z_2) = 2\pi z_2 \tan(\pi z_1 z_2) F^{(1,0)}(z_1, z_2), \\ F^{(0,2)}(z_1, z_2) = 2\pi z_1 \tan(\pi z_1 z_2) F^{(0,1)}(z_1, z_2). \end{cases}$$

Differentiate in variables z_1 and z_2 we deduce

$$\left\{ \begin{aligned} F^{(3,0)}(z_1, z_2) &= \frac{2\pi^2 z_2^2}{\cos^2(\pi z_1 z_2)} F^{(1,0)}(z_1, z_2) + 2\pi z_2 \tan(\pi z_1 z_2) F^{(2,0)}(z_1, z_2), \\ F^{(2,1)}(z_1, z_2) &= 2\pi \tan(\pi z_1 z_2) F^{(1,0)}(z_1, z_2) + \frac{2\pi^2 z_1 z_2}{\cos^2(\pi z_1 z_2)} F^{(1,0)}(z_1, z_2) + \\ &\quad + 2\pi z_2 \tan(\pi z_1 z_2) F^{(1,1)}(z_1, z_2), \\ F^{(1,2)}(z_1, z_2) &= 2\pi \tan(\pi z_1 z_2) F^{(0,1)}(z_1, z_2) + \frac{2\pi^2 z_1 z_2}{\cos^2(\pi z_1 z_2)} F^{(1,0)}(z_1, z_2) + \\ &\quad + 2\pi z_1 \tan(\pi z_1 z_2) F^{(1,1)}(z_1, z_2), \\ F^{(0,3)}(z_1, z_2) &= \frac{2\pi^2 z_1^2}{\cos^2(\pi z_1 z_2)} F^{(1,0)}(z_1, z_2) + 2\pi z_1 \tan(\pi z_1 z_2) F^{(2,0)}(z_1, z_2), \end{aligned} \right. \tag{5.1}$$

Let

$$\mathbf{L}(z_1, z_2) = (l_1(z_1, z_2), l_2(z_1, z_2)) = \left(\frac{|z_2| + 1}{(1 - |z|)|\frac{1}{2} - z_1 z_2|}, \frac{|z_1| + 1}{(1 - |z|)|\frac{1}{2} - z_1 z_2|} \right),$$

where $z = (z_1, z_2)$, $|z| = \sqrt{|z_1|^2 + |z_2|^2}$. Now we will estimate all third order partial derivatives of the function $F(z_1, z_2)$ by its first and second order partial derivatives. From the first equation of system (5.1) we have for all $z \in \mathbb{B}^2$:

$$\begin{aligned} \frac{|F^{(3,0)}(z_1, z_2)|}{l_1^3(z_1, z_2)} &\leq \frac{2\pi^2 |z_2|^2 |F^{(1,0)}(z_1, z_2)|}{|\cos^2(\pi z_1 z_2)| l_1^3(z_1, z_2)} + 2\pi |z_2 \tan(\pi z_1 z_2)| \frac{|F^{(2,0)}(z_1, z_2)|}{l_1^3(z_1, z_2)} \\ &\leq \left(\frac{2\pi^2 |z_2|^2}{|\cos^2(\pi z_1 z_2)| l_1^2(z_1, z_2)} + \frac{2\pi |z_2 \tan(\pi z_1 z_2)|}{l_1(z_1, z_2)} \right) \max_{j \in \{1,2\}} \left\{ \frac{|F^{(j,0)}(z_1, z_2)|}{l_1^j(z_1, z_2)} \right\} \\ &\leq \left(2\pi^2 \frac{(1 - |z|)^2 |\frac{1}{2} - z_1 z_2|^2}{|\cos^2(\pi z_1 z_2)|} + 2\pi |\tan(\pi z_1 z_2)| (1 - |z|) \left| \frac{1}{2} - z_1 z_2 \right| \right) \end{aligned}$$

$$\begin{aligned} & \times \max \left\{ \frac{|F^{(j,0)}(z_1, z_2)|}{l_1^j(z_1, z_2)} : j \in \{1, 2\} \right\} \\ & = \left(\frac{(1 - |z|)^2 \left| \frac{\pi}{2} - \pi z_1 z_2 \right|^2}{\left| \sin^2 \left(\frac{\pi}{2} - \pi z_1 z_2 \right) \right|} + 2 \left| \sin(\pi z_1 z_2) \right| (1 - |z|) \frac{\left| \frac{\pi}{2} - \pi z_1 z_2 \right|}{\left| \sin \left(\frac{\pi}{2} - \pi z_1 z_2 \right) \right|} \right) \\ & \times \max \left\{ \frac{|F^{(j,0)}(z_1, z_2)|}{l_1^j(z_1, z_2)} : j \in \{1, 2\} \right\} \leq C \max \left\{ \frac{|F^{(j,0)}(z_1, z_2)|}{l_1^j(z_1, z_2)} : j \in \{1, 2\} \right\}. \end{aligned}$$

Similarly, the second equation of system (5.1) yields

$$\begin{aligned} \frac{|F^{(2,1)}(z_1, z_2)|}{l_1^2(z_1, z_2) l_2(z_1, z_2)} & \leq \left(\frac{2\pi \left| \tan(\pi z_1 z_2) \right|}{l_1(z_1, z_2) l_2(z_1, z_2)} + \frac{2\pi^2 |z_1 z_2|}{\left| \cos^2(\pi z_1 z_2) \right| l_1(z_1, z_2) l_2(z_1, z_2)} \right) \\ & \times \frac{|F^{(1,0)}(z_1, z_2)|}{l_1(z_1, z_2)} + \frac{2\pi |z_2 \tan(\pi z_1 z_2)|}{l_1(z_1, z_2)} \frac{|F^{(1,1)}(z_1, z_2)|}{l_1(z_1, z_2) l_2(z_1, z_2)} \\ & \leq \left(\frac{2\pi \left| \sin(\pi z_1 z_2) \right| (1 - |z|)^2 \left| \frac{1}{2} - z_1 z_2 \right|^2}{\left| \cos(\pi z_1 z_2) \right|} \right. \\ & \left. + \frac{2\pi^2 (1 - |z|)^2 \left| \frac{1}{2} - z_1 z_2 \right|^2}{\left| \cos^2(\pi z_1 z_2) \right|} + \frac{2\pi \left| \sin(\pi z_1 z_2) \right| (1 - |z|) \left| \frac{1}{2} - z_1 z_2 \right|}{\left| \cos(\pi z_1 z_2) \right|} \right) \\ & \times \max \left\{ \frac{|F^{(1,j)}(z_1, z_2)|}{l_1(z_1, z_2) l_2^j(z_1, z_2)} : j \in \{0, 1\} \right\} \\ & \leq \left(\frac{2\pi \left| \sin(\pi z_1 z_2) \right| (1 - |z|)^2 \left| \frac{1}{2} - z_1 z_2 \right|^2}{\left| \sin \left(\frac{\pi}{2} - \pi z_1 z_2 \right) \right|} + \frac{2(1 - |z|)^2 \left| \frac{\pi}{2} - \pi z_1 z_2 \right|^2}{\left| \sin^2 \left(\frac{\pi}{2} - \pi z_1 z_2 \right) \right|} \right. \\ & \left. + \frac{2 \left| \sin(\pi z_1 z_2) \right| (1 - |z|) \left| \frac{\pi}{2} - \pi z_1 z_2 \right|}{\left| \sin \left(\frac{\pi}{2} - \pi z_1 z_2 \right) \right|} \right) \max \left\{ \frac{|F^{(1,j)}(z_1, z_2)|}{l_1(z_1, z_2) l_2^j(z_1, z_2)} : j \in \{0, 1\} \right\} \\ & \leq C \max \left\{ \frac{|F^{(1,j)}(z_1, z_2)|}{l_1(z_1, z_2) l_2^j(z_1, z_2)} : j \in \{0, 1\} \right\}. \end{aligned}$$

By analogy, we can prove similar estimates for the third and the fourth equation of system (5.1). Combining all estimates, one has

$$\begin{aligned} & \max \left\{ \frac{|F^{(k,3-k)}(z_1, z_2)|}{l_1^k(z_1, z_2) l_2^{3-k}(z_1, z_2)} : k \in \{0, 1, 2, 3\} \right\} \\ & \leq C \max \left\{ \frac{|F^{(k,j)}(z_1, z_2)|}{l_1^k(z_1, z_2) l_2^j(z_1, z_2)} : 0 \leq k + j \leq 2 \right\}. \end{aligned}$$

Hence, by Theorem 4.1 every analytic solution in \mathbb{B}^2 of system (5.1) has bounded **L**-index in joint variables with

$$\mathbf{L}(z_1, z_2) = \left(\frac{|z_2| + 1}{(1 - |z|) \left| \frac{1}{2} - z_1 z_2 \right|}, \frac{|z_1| + 1}{(1 - |z|) \left| \frac{1}{2} - z_1 z_2 \right|} \right).$$

Particularly, the function $F(z_1, z_2) = \tan(\pi z_1 z_2)$ has the bounded \mathbf{L} -index in joint variables. Indeed, it is easy to see that the function F is analytic solution in \mathbb{B}^2 of system (5.1).

6. Boundedness of l_j -index in every direction $\mathbf{1}_j$

This section shows another application of Theorem 3.1. The boundedness of l_j -index of a function F in every variable z_j , generally speaking, does not imply the boundedness of \mathbf{L} -index in joint variables (see example in [4]). But, if F has bounded l_j -index in every direction $\mathbf{1}_j, j \in \{1, \dots, n\}$, then F is a function of bounded \mathbf{L} -index in joint variables.

Let $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $L : \mathbb{B}^n \rightarrow \mathbb{R}_+$ be a continuous function such that for all $z \in \mathbb{B}^n$ $L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}, \beta > 1$.

For $\eta \in [0, \beta], z \in \mathbb{B}^n$, we define

$$\lambda_1^{\mathbf{b}}(z, \eta, L) = \inf\{L(z + t\mathbf{b})/L(z) : |t| \leq \frac{\eta}{L(z)}\},$$

$$\lambda_1^{\mathbf{b}}(\eta, L) = \inf\{\lambda_1^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}^n\},$$

$$\lambda_2^{\mathbf{b}}(z, \eta, L) = \sup\{L(z + t\mathbf{b})/L(z) : |t| \leq \frac{\eta}{L(z)}\},$$

$$\lambda_2^{\mathbf{b}}(\eta, L) = \sup\{\lambda_2^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}^n\}.$$

By $Q_{\mathbf{b},\beta}(\mathbb{B}^n)$ we denote the class of all functions L satisfying $\forall \eta \in [0, \beta]$,

$$0 < \lambda_1^{\mathbf{b}}(\eta, L) \leq \lambda_2^{\mathbf{b}}(\eta, L) < +\infty.$$

Analytic in \mathbb{B}^n function $F(z)$ is called a function of *bounded L -index in the direction \mathbf{b}* , if there exists $m_0 \in \mathbb{Z}_+$ that for every $m \in \mathbb{Z}_+$ and for every $z \in \mathbb{B}^n$ the following inequality is valid

$$\frac{1}{m!L^m(z)} \left| \frac{\partial^m F(z)}{\partial \mathbf{b}^m} \right| \leq \max \left\{ \frac{1}{k!L^k(z)} \left| \frac{\partial^k F(z)}{\partial \mathbf{b}^k} \right| : 0 \leq k \leq m_0 \right\}, \tag{6.1}$$

where

$$\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z), \frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j, \bar{\mathbf{b}}, \frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}} \left(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}} \right), k \geq 2.$$

The least such integer m_0 is called the *L -index in the direction \mathbf{b} of the analytic function F* and is denoted by $N_{\mathbf{b}}(F, L) = m_0$. In the case $n = 1, \mathbf{b} = 1$ and $L = l$ we obtain a definition of analytic in an unit disc function of bounded l -index [22, 21].

We need the following theorem.

Theorem 6.1 ([3]). *Let $\beta > 1, L \in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$. Analytic in \mathbb{B}^n function $F(z)$ is of bounded L -index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if for any r_1 and any r_2 with $0 < r_1 < r_2 \leq \beta$, there exists number $P_1 = P_1(r_1, r_2) \geq 1$ such that for each $z^0 \in \mathbb{B}^n$*

$$\max \left\{ |F(z^0 + t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)} \right\} \leq P_1 \max \left\{ |F(z^0 + t\mathbf{b})| : |t| = \frac{r_1}{L(z^0)} \right\}. \tag{6.2}$$

It is easy to see that if $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ and $\mathbf{L} \in Q(\mathbb{B}^n)$, then

$$l_j \in Q_{\mathbf{1}_j, \beta/\sqrt{n}}(\mathbb{B}^n), j \in \{1, \dots, n\}.$$

Theorem 6.2. Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $\mathbf{L} \in Q(\mathbb{B}^n)$. If an analytic in \mathbb{B}^n function F has bounded l_j -index in the direction $\mathbf{1}_j$ for every $j \in \{1, \dots, n\}$, then F is of bounded \mathbf{L} -index in joint variables.

Proof. Let F be an analytic in \mathbb{B}^n function of bounded l_j -index in every direction $\mathbf{1}_j$. Then by Theorem 6.1 for every $j \in \{1, \dots, n\}$ and arbitrary $0 < r'_j < 1 < r''_j \leq \frac{\beta}{\sqrt{n}}$ there exists a number $p_j = p_j(r', r'')$ such that for every $(z_1, \dots, z_{j-1}, z_j^0, z_{j+1}, \dots, z_n) \in \mathbb{B}^n$,

$$\begin{aligned} & \max \left\{ |F(z)| : |z_j - z_j^0| = \frac{r''_j}{l_j(z_1, \dots, z_{j-1}, z_j^0, z_{j+1}, \dots, z_n)} \right\} \leq p_j(r'_j, r''_j) \\ & \times \max \left\{ |F(z)| : |z_j - z_j^0| = \frac{r'_j}{l_j(z_1, \dots, z_{j-1}, z_j^0, z_{j+1}, \dots, z_n)} \right\}. \end{aligned} \tag{6.3}$$

Obviously, if for every $j \in \{1, \dots, n\}$ $l_j \in Q_{\mathbf{1}_j, \beta/\sqrt{n}}(\mathbb{B}^n)$ then $\mathbf{L} \in Q(\mathbb{B}^n)$. Let z^0 be an arbitrary point in \mathbb{B}^n , and a point $z^* \in \mathbb{T}^n(z^0, \frac{R''}{\mathbf{L}(z^0)})$ is such that

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) = |F(z^*)|.$$

We choose R'' and R' such that $\mathbf{1} < R'' \leq (\frac{\beta}{\sqrt{n}}, \dots, \frac{\beta}{\sqrt{n}})$ and $R' < \Lambda_1(R'')$. Then inequality (6.3) implies that

$$\begin{aligned} & M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \leq \max \left\{ |F(z_1, z_2^*, z_3^*, \dots, z_n^*)| : |z_1 - z_1^0| = \frac{r''_1}{l_1(z^0)} \right\} \\ & = \max \left\{ |F(z_1, z_2^*, \dots, z_n^*)| : |z_1 - z_1^0| = \frac{r''_1}{l_1(z_1^0, z_2^*, \dots, z_n^*)} \frac{l_1(z_1^0, z_2^*, \dots, z_n^*)}{l_1(z^0)} \right\} \\ & \leq \max \left\{ |F(z_1, z_2^*, \dots, z_n^*)| : |z_1 - z_1^0| = \frac{r''_1 \lambda_{2,1}(R'')}{l_1(z_1^0, z_2^*, \dots, z_n^*)} \right\} \\ & \leq p_1(r'_1, r''_1 \lambda_{2,1}(R'')) \max \left\{ |F(z_1, z_2^*, \dots, z_n^*)| : |z_1 - z_1^0| = \frac{r'_1}{l_1(z_1^0, z_2^*, \dots, z_n^*)} \right\} \\ & \quad = p_1(r'_1, r''_1 \lambda_{2,1}(R'')) \\ & \quad \times \max \left\{ |F(z_1, z_2^*, \dots, z_n^*)| : |z_1 - z_1^0| = \frac{r'_1}{l_1(z^0)} \frac{l_1(z^0)}{l_1(z_1^0, z_2^*, \dots, z_n^*)} \right\} \\ & \leq p_1(r'_1, r''_1 \lambda_{2,1}(R'')) \max \left\{ |F(z_1, z_2^*, \dots, z_n^*)| : |z_1 - z_1^0| = \frac{r'_1}{\lambda_{1,1}(R'') l_1(z^0)} \right\} \\ & \quad = p_1(r'_1, r''_1 \lambda_{2,1}(R'')) |F(z_1^*, z_2^*, \dots, z_n^*)| \leq p_1(r'_1, r''_1 \lambda_{2,1}(R'')) \\ & \quad \times \max \left\{ |F(z_1^*, z_2, z_3^*, \dots, z_n^*)| : |z_2 - z_2^0| = \frac{r''_2}{l_2(z^0)} \right\} = p_1(r'_1, r''_1 \lambda_{2,1}(R'')) \\ & \quad \times \max \left\{ |F(z_1^*, z_2, \dots, z_n^*)| : |z_2 - z_2^0| = \frac{r''_2}{l_2(z_1^*, z_2^0, \dots, z_n^*)} \frac{l_2(z_1^*, z_2^0, \dots, z_n^*)}{l_2(z^0)} \right\} \\ & \leq p_1(r'_1, r''_1 \lambda_{2,1}(R'')) \max \left\{ |F(z_1^*, z_2, \dots, z_n^*)| : |z_2 - z_2^0| = \frac{r''_2 \lambda_{2,2}(R'')}{l_2(z_1^*, z_2^0, \dots, z_n^*)} \right\} \end{aligned}$$

$$\begin{aligned}
& \leq \prod_{j=1}^2 p_j(r'_j, r''_j \lambda_{2,j}(R'')) \\
& \quad \times \max \left\{ |F(z_1^{**}, z_2, \dots, z_n^*)| : |z_2 - z_2^0| = \frac{r'_2}{l_2(z_1^{**}, z_2^0, \dots, z_n^*)} \right\} \\
& \leq \prod_{j=1}^2 p_j(r'_j, r''_j \lambda_{2,j}(R'')) \max \left\{ |F(z_1^{**}, z_2, \dots, z_n^*)| : |z_2 - z_2^0| = \frac{r'_2}{\lambda_{1,2}(R'') l_2(z^0)} \right\} \\
& = \prod_{j=1}^2 p_j(r'_j, r''_j \lambda_{2,j}(R'')) |F(z_1^{**}, z_2^{**}, z_3^*, \dots, z_n^*)| \leq \dots \leq \prod_{j=1}^n p_j(r'_j, r''_j \lambda_{2,j}(R'')) \\
& \quad \times \max \left\{ |F(z_1, z_2, \dots, z_n)| : |z_j - z_j^0| = \frac{r'_j}{\lambda_{1,j}(R'') l_j(z^0)}, j \in \{1, \dots, n\} \right\} \\
& = \prod_{j=1}^n p_j(r'_j, r''_j \lambda_{2,j}(R'')) M \left(\frac{R'}{\Lambda_1(R'') \mathbf{L}(z^0)}, z^0, F \right).
\end{aligned}$$

Hence, by Theorem 3.1 the function F is of bounded \mathbf{L} -index in joint variables. \square

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