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A note on the Wang-Zhang and Schwarz inequalities

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Abstract. In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

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1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex inner product space and $x, y \in H$ two nonzero vectors. One can define the *angle* between the vectors x, y either by

$$\Phi_{x,y} = \arccos \left(\frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \, \|y\|} \right) \text{ or by } \Psi_{x,y} = \arccos \left(\frac{|\langle x, y \rangle|}{\|x\| \, \|y\|} \right).$$

The function $\Psi_{x,y}$ is a natural metric on complex projective space [6].

In 1969 M. K. Kreĭn [5] obtained the following inequality for angles between two vectors

$$\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{z,y} \tag{1.1}$$

for any $x, y, z \in H \setminus \{0\}$.

By using the representation

$$\Psi_{x,y} = \inf_{\alpha,\beta \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, \beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, y} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \Phi_{x, \beta y} \tag{1.2}$$

and Kreĭn's inequality (1.1), M. Lin [6] has shown recently that the following triangle inequality is also valid

$$\Psi_{x,y} \leq \Psi_{x,z} + \Psi_{z,y} \tag{1.3}$$

for any $x, y, z \in H \setminus \{0\}$.

The following inequality has been obtained by Wang and Zhang in [9] (see also [11, p. 195])

$$\sqrt{1-\frac{|\langle x,y\rangle|^2}{\|x\|^2\|y\|^2}} \le \sqrt{1-\frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}} + \sqrt{1-\frac{|\langle y,z\rangle|^2}{\|y\|^2\|z\|^2}} \tag{1.4}$$

for any $x,y,z \in H\setminus\{0\}$. Using the above notations it can be written as [6]

$$\sin\Psi_{x,y} \le \sin\Psi_{x,z} + \sin\Psi_{z,y} \tag{1.5}$$

for any $x,y,z \in H\setminus\{0\}$. It also provides another triangle type inequality complementing the Kreĭn and Lin inequalities above.

In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

2. Reverse of Schwarz inequality

In the sequel we assume that $(H,\langle\cdot,\cdot\rangle)$ is a complex inner product space. The inequality

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2 \text{ for } x,y\in H \tag{2.1}$$

is well know in the literature as the *Schwarz inequality*. The equality holds in (2.1) iff x and y are linearly dependent.

Theorem 2.1. *Let $x,y,z\in H$ with $\|z\|=1$ and $\alpha,\beta\in\mathbb{C}$, $r,s>0$ such that*

$$\|x-\alpha z\|\le r \text{ and } \|y-\beta z\|\le s. \tag{2.2}$$

Then

$$(0\le)\, \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \le (r\|y\|+s\|x\|)^2. \tag{2.3}$$

Proof. If we multiply (1.4) by $\|x\|\|y\|\|z\|>0$, then we get

$$\begin{aligned} &\|z\|\sqrt{\|x\|^2\|y\|^2-|\langle x,y\rangle|^2} \\ &\le \|y\|\sqrt{\|x\|^2\|z\|^2-|\langle x,z\rangle|^2} + \|x\|\sqrt{\|y\|^2\|z\|^2-|\langle y,z\rangle|^2} \end{aligned} \tag{2.4}$$

for any $x,y,z\in H\setminus\{0\}$.

We observe that, if either $x=0$ or $y=0$, then the inequality (2.4) reduces to an equality.

Let $z\in H$ with $\|z\|=1$, and since (see for instance [2, Lemma 2.4])

$$\|x\|^2-|\langle x,z\rangle|^2 = \inf_{\lambda\in\mathbb{C}}\|x-\lambda z\|^2 \text{ and } \|y\|^2-|\langle y,z\rangle|^2 = \inf_{\mu\in\mathbb{C}}\|y-\mu z\|^2$$

then by (2.4) we have

$$\sqrt{\|x\|^2\|y\|^2-|\langle x,y\rangle|^2} \le \|y\|\inf_{\lambda\in\mathbb{C}}\|x-\lambda z\| + \|x\|\inf_{\mu\in\mathbb{C}}\|y-\mu z\|, \tag{2.5}$$

for any $x,y,z\in H$ with $\|z\|=1$.

Since, by (2.2)

$$\inf_{\lambda\in\mathbb{C}}\|x-\lambda z\|\le\|x-\alpha z\|\le r \text{ and } \inf_{\mu\in\mathbb{C}}\|y-\mu z\|\le\|y-\beta z\|\le s,$$

then by (2.5) we obtain the desired result (2.3). □

Corollary 2.2. *Let $x,y,z\in H$ with $\|z\|=1$ and $\lambda,\Lambda,\gamma,\Gamma\in\mathbb{C}$ with $\lambda\ne\Lambda$, $\gamma\ne\Gamma$ and such that either*

$$\operatorname{Re}\langle\Lambda z-x,x-\lambda z\rangle\ge 0 \text{ and } \operatorname{Re}\langle\Gamma z-y,y-\gamma z\rangle\ge 0 \tag{2.6}$$

or, equivalently

$$\left\|x-\frac{\lambda+\Lambda}{2}z\right\|\le\frac{1}{2}|\Lambda-\lambda| \text{ and } \left\|y-\frac{\gamma+\Gamma}{2}z\right\|\le\frac{1}{2}|\Gamma-\gamma|$$

are valid. Then

$$(0\le)\,\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\le\frac{1}{4}\left(|\Lambda-\lambda|\,\|y\|+|\Gamma-\gamma|\,\|x\|\right)^2. \tag{2.7}$$

Proof. Follows by Theorem 2.1 on observing that

$$\operatorname{Re}\langle\Delta e-u,u-\delta e\rangle=\frac{1}{4}|\Delta-\delta|^2-\left\|u-\frac{\delta+\Delta}{2}e\right\|^2$$

for any $\delta,\Delta\in\mathbb{C}$ with $\delta\ne\Delta$ and $u,e\in H$ with $\|e\|=1$. □

We give an example for n-tuples of complex numbers.

Let $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$ and $z=(z_1,...,z_n)$ be n-tuples of complex numbers, $p=(p_1,...,p_n)$ a probability distribution, i.e. $p_i>0$ $i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$, with $\sum_{i=1}^n p_i|z_i|^2=1$ and $\lambda,\Lambda,\gamma,\Gamma\in\mathbb{C}$ with $\lambda\ne\Lambda$, $\gamma\ne\Gamma$ and such that

$$\operatorname{Re}\left[(\Lambda z_i-x_i)\left(\overline{x}_i-\overline{\lambda}\overline{z}_i\right)\right]\ge 0 \text{ and } \operatorname{Re}\left[(\Gamma z_i-\overline{y}_i)\left(\overline{y}_i-\overline{\gamma z_i}\right)\right]\ge 0$$

or, equivalently

$$\left|x_i-\frac{\lambda+\Lambda}{2}z_i\right|\le\frac{1}{2}|\Lambda-\lambda| \text{ and } \left|y_i-\frac{\gamma+\Gamma}{2}z_i\right|\le\frac{1}{2}|\Gamma-\gamma|$$

for any $i\in\{1,...,n\}$. Then

$$\sum_{i=1}^n p_i\operatorname{Re}\left[(\Lambda z_i-x_i)\left(\overline{x}_i-\overline{\lambda}\overline{z}_i\right)\right]\ge 0 \text{ and } \sum_{i=1}^n p_i\operatorname{Re}\left[(\Gamma z_i-\overline{y}_i)\left(\overline{y}_i-\overline{\gamma z_i}\right)\right]\ge 0$$

and by applying Corollary 2.2 for the inner product $\langle\cdot,\cdot\rangle_p:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ with

$$\langle x,y\rangle_p=\sum_{i=1}^n p_i x_i\overline{y}_i,$$

we have

$$0 \leq \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 - \left| \sum_{i=1}^{n} p_i x_i \overline{y}_i \right|^2 \tag{2.8}$$
$$\leq \frac{1}{4} \left[|\Lambda - \lambda| \left(\sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} + |\Gamma - \gamma| \left(\sum_{i=1}^{n} p_i |x_i|^2 \right)^{1/2} \right]^2 .$$

If $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$ for any $i \in \{1, ..., n\}$ then by (2.8) we have for any $p = (p_1, ..., p_n)$ a probability distribution that

$$0 \leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \left(\sum_{i=1}^{n} p_i a_i b_i \right)^2 \tag{2.9}$$
$$\leq \frac{1}{4} \left[(A - a) \left(\sum_{i=1}^{n} p_i b_i^2 \right)^{1/2} + (B - b) \left(\sum_{i=1}^{n} p_i a_i^2 \right)^{1/2} \right]^2 .$$

The interested reader may compare this new result with the classical reverses of Schwarz inequality obtained by Diaz and Metcalf [1], Ozeki [4], G. Pólya and G. Szegö [7], Shisha and Mond [8] and Cassels [10].

For other reverses of Schwarz inequality in complex inner product spaces see the monograph [3] and the references therein.

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