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Subclasses of p-valent meromorphic functions involving certain operator

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Abstract. In this paper we investigate some inclusion relationships of two new subclassses of meromorphically p-valent functions, defined by means of a linear operator. We also study some integral preserving properties and convolution properties of these classes.

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1. Introduction

Let $\sum_p$ denote the class of all meromorphic functions f defined by:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in a punctured unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = \mathbb{U} \backslash \{0\}$.

The class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , z \in \mathbb{U},$$

is denoted by $\mathbb{A}$. The functions of this class is called starlike of order $\gamma, 0 \leq \gamma < 1$ if

$$\Re \frac{zf'(z)}{f(z)} > \gamma$$

and called prestarlike of order $\gamma, \gamma < 1$ if

$$\frac{z}{(1-z)^{2(1-\gamma)}} * f(z) \in S^*(\gamma),$$

we denote by $S^*(\gamma)$ and $R(\gamma)$ the classes of stalike and prestarlike of order γ.

If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z)),\ z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [2, 5, 6]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z) \in \sum_p$ given by (1.1) and $g(z) \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p},$$

the Hadamard product of $f(z)$ and $g(z)$ is given by:

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{1.2}$$

Using the operator $Q^{\alpha}_{\beta,p} : \sum_p \to \sum_p$ defined by (see [1]):

$$Q^{\alpha}_{\beta,p} f(z) = \begin{cases} z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum\limits_{k=1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)} a_{k-p} z^{k-p} & (\alpha > 0; \beta > -1) \\ f(z) & (\alpha = 0\ ; \beta > -1)\ . \end{cases}$$

Mostafa [8] defined the operator $H^{\alpha}_{p,\beta,\mu} : \Sigma_p \to \Sigma_p$ as follows:
First put

$$G^{\alpha}_{\beta,p}(z) = z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)} z^{k-p} \quad (p \in \mathbb{N}) \tag{1.3}$$

and let $G^{\alpha *}_{\beta,p,\mu}$ be defined by

$$G^{\alpha}_{\beta,p}(z) * G^{\alpha *}_{\beta,p,\mu}(z) = \frac{1}{z^p(1-z)^{\mu}} \quad (\mu > 0; p \in \mathbb{N})\,. \tag{1.4}$$

Then

$$H^{\alpha}_{p,\beta,\mu} f(z) = G^{\alpha *}_{\beta,p}(z) * f(z) \quad (f \in \Sigma_p)\,. \tag{1.5}$$

Using (1.3)-(1.5), we have

$$H^{\alpha}_{p,\beta,\mu} f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+\alpha)(\mu)_k}{\Gamma(k+\beta)(1)_k} a_{k-p} z^{k-p}\,, \tag{1.6}$$

where $f \in \Sigma_p$ is in the form (1.1) and $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N})\,. \end{cases}$$

It is readily verified from (1.6) that (see [8])

$$z(H^{\alpha}_{p,\beta,\mu} f(z))' = (\alpha+\beta) H^{\alpha+1}_{p,\beta,\mu} f(z) - (\alpha+\beta+p) H^{\alpha}_{p,\beta,\mu} f(z) \tag{1.7}$$

and

$$z(H^{\alpha}_{p,\beta,\mu} f(z))' = \mu H^{\alpha}_{p,\beta,\mu+1} f(z) - (\mu+p) H^{\alpha}_{p,\beta,\mu} f(z)\,. \tag{1.8}$$

It is noticed that, putting $\mu = 1$ in (1.6), we obtain the operator

$$H^{\alpha}_{p,\beta,1}f(z) = H^{\alpha}_{p,\beta}f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)} a_{k-p} z^{k-p}. \tag{1.9}$$

Let $\mathbb{P}$ be the class of functions $h(z)$ with $h(0) = 1$, $\mathrm{Re}h(z) > 0$ which are convex univalent in $\mathbb{U}$.

For $p, n \in \mathbb{N}$, $\in_n = e^{2\pi/n}$, let

$$f^{\mu}_{n}(\alpha)(z) = \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon_n^{jp} H^{\alpha}_{p,\beta,\mu} f(\varepsilon_n^{j} z) = z^{-p} + ..., f \in \sum\nolimits_p . \tag{1.10}$$

By (1.7) and (1.8), $f^{\mu}_{n}(\alpha)(z)$ satisfies:

$$z(f^{\mu}_{n}(\alpha)(z))' = (\alpha+\beta) f^{\mu}_{n}(\alpha+1)(z) - (\alpha+\beta+p) f^{\mu}_{n}(\alpha)(z) \tag{1.11}$$

and

$$z(f^{\mu}_{n}(\alpha)(z))' = \mu f^{\mu+1}_{n}(\alpha)(z) - (\mu+p) f^{\mu}_{n}(\alpha)(z). \tag{1.12}$$

Definition 1.1. For $h \in \mathbb{P}$,$f \in \sum_p$, $f^{\mu}_{n}(\alpha)(z) \neq 0, z \in \mathbb{U}^*$, $S^{\mu}_{n}(\alpha, h)$ is the class of functions f satisfying:

$$-\frac{z(H^{\alpha}_{p,\beta,\mu} f(z))'}{p f^{\mu}_{n}(\alpha)(z)} \prec h(z) \tag{1.13}$$

and $K^{\mu}_{n}(\alpha, h)$ is the class of functions f satisfying:

$$-\frac{z(H^{\alpha}_{p,\beta,\mu} f(z))'}{p g^{\mu}_{n}(\alpha)(z)} \prec h(z), \tag{1.14}$$

where $g^{\mu}_{n}(\alpha)(z) \neq 0$, is defined as in (1.10).

To prove our results, we need the following Lemmas.

Lemma 1.2. [3] *Let $\beta, \gamma \in \mathbb{C}, \beta \neq 0, h$ be convex univalent with $\Re\{\beta h(z) + \gamma\} > 0$ and q be an analytic function such that $q(0) = h(0)$. If*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z),$$

then

$$q(z) \prec h(z).$$

Lemma 1.3. [7] *Let h be convex univalent and w be analytic, $\Re w \geq 0$. If the analytic function q satisfies $q(0) = h(0)$ and*

$$q(z) + w(z)zq'(z) \prec h(z),$$

then $q(z) \prec h(z)$.

Lemma 1.4. [9] *For $\alpha < 1, f \in R(\alpha)$ and $\varphi \in S^*(\alpha)$, we have for any analytic function F in $\mathbb{U}$,*

$$\frac{f * (\varphi F)}{f * \varphi}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U}),$$

where $\overline{co}(F(\mathbb{U})$ is the convex hull of $(F(\mathbb{U})$.

2. Main results

Theorem 2.1. *If $f \in S_n^\mu(\alpha,h)$, then*

$$-\frac{z\left(f_n^\mu(\alpha)(z)\right)'}{pf_n^\mu(\alpha)(z)} \prec h(z), \tag{2.1}$$

where $f_n^\mu(\alpha)(z)$ is defined as in (1.10).

Proof. From (1.10), we have:

$$\begin{aligned} f_n^\mu(\alpha)(\in_n^j z) &= \frac{1}{n}\sum_{t=0}^{n-1} \varepsilon_n^{jt} H_{p,\beta,\mu}^\alpha f(\varepsilon_n^{j+t} z) \\ &= \frac{\varepsilon_n^{-jp}}{n}\sum_{t=0}^{k-1} \varepsilon_n^{(j+t)p} H_{p,\beta,\mu}^\alpha f(\varepsilon_n^{j+t} z) \\ &= \varepsilon_n^{-jp} f_n^\mu(\alpha)(z) \end{aligned} \tag{2.2}$$

and

$$\left(f_n^\mu(\alpha)(z)\right)' = \frac{1}{n}\sum_{j=0}^{n-1} \varepsilon_n^{j(p+1)} \left(H_{p,\beta,\mu}^\alpha f(\varepsilon_n^{j+t} z)\right)'. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned} -\frac{z\left(f_n^\mu(\alpha)(z)\right)'}{pf_n^\mu(\alpha)(z)} &= -\frac{1}{n}\sum_{j=0}^{n-1} \frac{\varepsilon_n^{j(p+1)}\left(H_{p,\beta,\mu}^\alpha f(\varepsilon_k^j z)\right)'}{pf_n^\mu(\alpha)(z)} \\ &= -\frac{1}{n}\sum_{j=0}^{n-1} \frac{\varepsilon_n^{j}\left(H_{p,\beta,\mu}^\alpha f(\varepsilon_n^j z)\right)'}{pf_n^\mu(\alpha)(z)}. \end{aligned} \tag{2.4}$$

Since $f \in S_n^\mu(\alpha,h)$, we have,

$$-\frac{\varepsilon_n^{j}\left(H_{p,\beta,\mu}^\alpha f(\varepsilon_n^j z)\right)'}{pf_n^\mu(\alpha)(z)} \prec h(z),$$

which leads to (2.1). □

Theorem 2.2. *For $\alpha+\beta>0$, $h \in \mathbb{P}$ with $\Re\{\alpha+\beta+p-ph(z)\}>0$ and for $f \in S_n^\mu(\alpha+1,h)$, $g_n^\mu(\alpha) \neq 0$, we have, $f \in S_n^\mu(\alpha,h)$.*

Proof. Since $f \in S_n^\mu(\alpha+1,h)$, then the function

$$q(z) = -\frac{z\left(H_{p,\beta,\mu}^\alpha f(z)\right)'}{pf_n^\mu(\alpha)(z)}, \tag{2.5}$$

is analytic and $q(0)=1$. Applying (1.8) in (2.5), we have

$$q(z)f_n^\mu(\alpha_1)(z) = -\frac{1}{p}[(\alpha+\beta)H_{p,\beta,\mu}^{\alpha+1}f(z) - (\alpha+\beta+p)H_{p,\beta,\mu}^\alpha f(z)]. \tag{2.6}$$

Differentiating (2.6) and using (1.8) again, we have

$$\left(\alpha+\beta+p+\frac{z\left(f_n^{\mu}(\alpha)(z)\right)'}{f_n^{\mu}(\alpha)(z)}\right)q(z)+zq'(z)=-\frac{(\alpha+\beta)z\left(H_{p,\beta,\mu}^{\alpha+1}f(z)\right)'}{pf_n^{\mu}(\alpha)(z)}. \tag{2.7}$$

Taking

$$\phi(z)=-\frac{z\left(f_n^{\mu}(\alpha)(z)\right)'}{pf_n^{\mu}(\alpha)(z)}, \tag{2.8}$$

we see that $\phi(z)$ is analytic, $\phi(0)=1$ and (2.7) can be written as

$$(\alpha+\beta+p-p\phi(z))\,q(z)+zq'(z)=-\frac{(\alpha+\beta)z\left(H_{p,\beta,\mu}^{\alpha+1}f(z)\right)'}{pf_n^{\mu}(\alpha)(z)}, \tag{2.9}$$

that is

$$q(z)+\frac{zq'(z)}{\alpha+\beta+p-p\phi(z)}=-\frac{z\left(H_{p,\beta,\mu}^{\alpha+1}f(z)\right)'}{pf_n^{\mu}(\alpha+1)(z)}. \tag{2.10}$$

Since $f\in S_n^{\mu}(\alpha\ +1,h)$, (2.10) implies

$$q(z)+\frac{zq'(z)}{\alpha+\beta+p-p\phi(z)}\prec h(z). \tag{2.11}$$

Combining (2.11) and (2.8), we have

$$\alpha+\beta+p-p\phi(z)=\frac{(\alpha+\beta)f_n^{\mu}(\alpha+1)(z)}{pf_n^{\mu}(\alpha)(z)}. \tag{2.12}$$

Differentiating (2.12), we get

$$\phi(z)+\frac{z\phi'(z)}{\alpha+\beta+p-p\phi(z)}=-\frac{z\left(f_n^{\mu}(\alpha+1)(z)\right)'}{pf_n^{\mu}(\alpha+1)(z)}. \tag{2.13}$$

By Theorem 2.1, we have

$$-\frac{z\left(f_n^{\mu}(\alpha+1)(z)\right)'}{pf_n^{\mu}(\alpha+1)(z)}\prec h(z),$$

which yields

$$\phi(z)+\frac{z\phi'(z)}{\alpha+\beta+p-p\phi(z)}\prec h(z).$$

Since $\Re\{\alpha+\beta+p-ph(z)\}>0$, by Lemma 1.2, we have $\phi(z)\prec h(z)$, which implies $\Re\{\alpha+\beta+p-p\phi(z)\}>0$. Applying Lemma 1.3 and from (2.10), we have $q(z)\prec h(z)$ that is $f\in S_n^{\mu}(\alpha,h)$. □

Theorem 2.3. *Let $\alpha+\beta>0$, $h\in\mathbb{P}$ with $\Re\{\alpha+\beta+p-ph(z)\}>0$ and $f\in K_n^{\mu}(\alpha+1,h)$ with $g\in S_n^{\mu}(\alpha+1,h)$. Then, $f\in K_n^{\mu}(\alpha,h)$ provided $g_n^{\mu}(\alpha)(z)\neq 0$.*

Proof. By Theorem 2.2, $g\in S_n^{\mu}(\alpha+1,h)\Rightarrow g\in S_n^{\mu}(\alpha,h)$ and by Theorem 2.1, we have

$$\psi(z)=-\frac{z\left(g_n^{\mu}(\alpha)(z)\right)'}{pg_n^{\mu}(\alpha)(z)}\prec h(z). \tag{2.14}$$

Let

$$q(z) = -\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)'}{pg^{\mu}_{n}(\alpha)(z)}. \tag{2.15}$$

Then, from (1.8), we have

$$q(z)g^{\mu}_{n}(\alpha)(z) = -\frac{1}{p}[(\alpha+\beta)H^{\alpha+1}_{p,\beta,\mu}f(z) - (\alpha+\beta+p)H^{\alpha}_{p,\beta,\mu}f(z)]. \tag{2.16}$$

Differentiating (2.16), we have

$$(\alpha+\beta+p-p\psi(z))\,q(z) + zq'(z) = -\frac{(\alpha+\beta)z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{pg^{\mu}_{n}(\alpha)(z)}. \tag{2.17}$$

Applying (1.11) for g, (2.17) is equivalent to

$$q(z) + \frac{zq'(z)}{\alpha_1+p-p\psi(z)} = -\frac{z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{pg^{\mu}_{n}(\alpha+1)(z)}. \tag{2.18}$$

Since $f \in K^{\mu}_{n}(\alpha+1,h)$, the above equation leads to

$$q(z) + \frac{zq'(z)}{\alpha_1+p-p\psi(z)} \prec h(z). \tag{2.19}$$

We have $\Re\{\alpha+\beta+p-p\psi(z)\} > 0$ because $\Re\{\alpha+\beta+p-ph(z)\} > 0$. Applying Lemma 1.3, for (2.19), we have $q(z) \prec h(z)$. That is $f \in K^{\mu}_{n}(\alpha,h)$. □

Theorem 2.4. *Let $h \in \mathbb{P}, \Re\{\mu+p-ph(z)\} > 0$ and $f \in S^{\mu+1}_{n}(\alpha,h)$ such that $f^{\mu+1}_{n}(\alpha)(z) \neq 0$. Then $f \in S^{\mu}_{n}(\alpha,h)$.*

Proof. Let $f \in S^{\mu+1}_{n}(\alpha,h)$,

$$q(z) = -\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)'}{pf^{\mu}_{n}(\alpha)(z)}. \tag{2.20}$$

Applying (1.9) in (2.20), we have

$$q(z)f^{\mu}_{n}(\alpha)(z) = -\frac{\mu}{p}[H^{\alpha}_{p,\beta,\mu+1}f(z) + (\frac{\mu+p}{p})H^{\alpha}_{p,\beta,\mu}f(z)]. \tag{2.21}$$

Differentiating (2.21) and putting

$$\Phi(z) = -\frac{z\left(f^{\mu}_{n}(\alpha)(z)\right)'}{pf^{\mu}_{n}(\alpha)(z)}, \tag{2.22}$$

simple computations leads to

$$[\mu+p-p\Phi(z)]\,q(z) + zq'(z) = -\left(\frac{\mu}{p}\right)\frac{z\left(H^{\alpha}_{p,\beta,\mu+1}f(z)\right)'}{pf^{\mu}_{n}(\alpha)(z)}. \tag{2.23}$$

Using (1.12), we have

$$\mu+p-p\Phi(z) = \frac{\mu f^{\mu+1}_{n}(\alpha)(z)}{f^{\mu}_{n}(\alpha)(z)}. \tag{2.24}$$

So, (2.23), reduces to

$$q(z)+\frac{zq'(z)}{\mu+p-p\Phi(z)}=-\frac{z\left(H^{\alpha}_{p,\beta,\mu+1}f(z)\right)'}{pf^{\mu+1}_n(\alpha)(z)}\prec h(z), \tag{2.25}$$

where $f\in S^{\mu+1}_n(\alpha,h)$. Also differentating (2.24), we have

$$\Phi(z)+\frac{z\Phi'(z)}{\mu+p-p\Phi(z)}=-\frac{z\left(f^{\mu+1}_n(\alpha)f(z)\right)'}{pf^{\mu+1}_n(\alpha)(z)}. \tag{2.26}$$

By Theorem 2.1, we have

$$-\frac{z\left(f^{\mu+1}_n(\alpha)f(z)\right)'}{pf^{\mu+1}_n(\alpha)(z)}\prec h(z). \tag{2.27}$$

Combining (2.26), (2.27) and the condition $\Re\{\mu+p-ph(z)\}>0$, we have $\Phi(z)\prec h(z)$, which leads to $\Re\{\mu+p-p\Phi(z)\}>0$ and so applying Lemma 1.3 to (2.25). we have $q(z)\prec h(z)$ which completes the proof of Theorem 2.4. □

Theorem 2.5. *Let $h\in\mathbb{P}$ with $\Re\{\mu+p-ph(z)\}>0$ and $f\in K^{\mu+1}_n(\alpha,h)$ with $g\in S^{\mu+1}_n(\alpha,h)$. Then, $f\in K^{\mu}_n(\alpha,h)$ provided $g^{\mu}_n(\alpha)(z)\neq 0$.*

Proof. By Theorem 2.4, $g\in S^{\mu+1}_n(\alpha,h)\Rightarrow g\in S^{\mu}_n(\alpha,h)$ and by Theorem 2.1, we have

$$\Psi(z)=-\frac{z\left(g^{\mu}_n(\alpha)(z)\right)'}{pg^{\mu}_n(\alpha)(z)}\prec h(z),$$

and letting

$$q(z)=-\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)'}{pg^{\mu}_n(\alpha)(z)},$$

we can complete the proof as in Theorem 2.4. Next, let

$$F_{p,\delta}(f(z))=\frac{\delta-p}{z^{\delta}}\int_0^z t^{\delta-1}f(t)dt\quad(\delta>0), \tag{2.28}$$

which by using (1.6) gives

$$\delta H^{\alpha}_{p,\beta,\mu}F_{p,\delta}f(z)+z\left(H^{\alpha}_{p,\beta,\mu+1})F_{p,\delta}f(z)\right)'=(\delta-p)H^{\alpha}_{p,\beta,\mu}f(z). \tag{2.29}$$

The operator $F_{p,\delta}$ was investigated by many authors (see [10, 11]). □

Theorem 2.6. *Let $h\in\mathbb{P}$ with $\Re\{\delta-ph(z)\}>0$ and $f\in S^{\mu}_n(\alpha,h)$, then $F_{p,\delta}(f)\in S^{\mu}_n(\alpha,h)$ provided $F^{\mu}_n(\alpha)\neq 0$, where $F^{\mu}_n(\alpha)$ is defined as in (1.10) .*

Proof. From (2.29), we have

$$\delta F^{\mu}_n(\alpha)(z)+z\left(F^{\mu}_n(\alpha)(z)\right)'=(\delta-p)f^{\mu}_n(\alpha)(z). \tag{2.30}$$

Let

$$q(z)=-\frac{z\left(H^{\alpha}_{p,\beta,\mu}F_{p,\delta}(f(z))\right)'}{pF^{\mu}_n(\alpha)(z)}$$

and

$$w(z) = -\frac{z\left(F_n^\mu(\alpha)(z)\right)'}{pF_n^\mu(\alpha)(z)}. \tag{2.31}$$

Using (2.30) in (2.31), we have

$$\delta - pw(z) = (\delta - p)\frac{f_n^\mu(\alpha)(z)}{F_n^\mu(\alpha)(z)}.$$

Differentiating and using Theorem 2.1, we obtain

$$w(z) + \frac{zw'(z)}{\delta - pw(z)} = -\frac{z\left(f_n^\mu(\alpha)(z)\right)'}{pf_n^\mu(\alpha)(z)} \prec h(z). \tag{2.32}$$

By Lemma 1.2, (2.32) implies $w(z) \prec h(z)$. The remaining part of the proof is similar to that of Theorem 2.2, so we omit it. □

The proof of the following theorem is similar to that of Theorems 2.3 and 2.5, so we omit it.

Theorem 2.7. *Let $h \in \mathbb{P}$ with $\Re\{\delta - ph(z)\} > 0$ and $f \in K_n^\mu(\alpha, h)$, with respect to $g_n^\mu \in S_n^\mu(\alpha, h)$, then, $F_{p,\delta}(f) \in K_n^\mu(\alpha, h)$ with respect to $G = F_{p,\delta}(g)$ provided $G_n^\mu(\alpha)(z) \neq 0$.*

Note that for $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, we have $\Re h(z) = \frac{1+A}{1+B}$.

Remark 2.8. Taking $h(z) = \frac{1+Az}{1+Bz}$, in Theorems 2.2-2.7 we get corresponding results for the classes $S_n^\mu(\alpha, A, B)$ and $K_n^\mu(\alpha, A, B)$.

Theorem 2.9. *If $h \in \mathbb{P}$, with $\Re\{p + 1 - \gamma - ph(z)\} > 0$, $f \in S_n^\mu(\alpha, h), \varphi \in \sum_p$ and $z^{p+1}\varphi(z) \in R(\gamma), \gamma < 1$, then $f * \varphi \in S_n^\mu(\alpha, h)$.*

Proof. For $f \in S_n^\mu(\alpha, h)$, we have

$$F(z) = -\frac{z\left(H_{p,\beta,\mu}^\alpha f(z)\right)'}{pf_n^\mu(\alpha)(z)} \prec h(z). \tag{2.33}$$

Let

$$\psi(z) = z^{p+1} f_n^\mu(\alpha)(z),$$

then $\psi \in \mathbb{A}$ and

$$\frac{z\psi'(z)}{\psi(z)} = p + 1 + \frac{z\left(f_n^\mu(\alpha)(z)\right)'}{f_n^\mu(\alpha)(z)} \prec p + 1 - ph(z). \tag{2.34}$$

From the hypotheses of the theorem, we see that

$$\Re\frac{z\psi'(z)}{\psi(z)} > \gamma, \tag{2.35}$$

that is $\psi \in S^*\ (\gamma), \gamma < 1$.For $\varphi \in \sum_p$ it is easy to get

$$z^{p+1}H_{p,\beta,\mu}^\alpha(f * \varphi)(\varepsilon_k^j z) = (z^{p+1}\varphi(z)) * H_{p,\beta,\mu}^\alpha f(\varepsilon_k^j z)$$

and

$$z^{p+2}(H_{p,\beta,\mu}^\alpha(f * \varphi)(z))' = (z^{p+1}\varphi(z)) * (z^{p+2}H_{p,\beta,\mu}^\alpha f(z))'.$$

So, we have

$$\begin{aligned}\Psi(z) &= -\frac{(H^{\alpha}_{p,\beta,\mu}(f*\varphi)(z))'}{\frac{p}{k}\sum\limits_{j=0}^{k-1}\varepsilon_k^{jp}H^{\alpha}_{p,\beta,\mu}(f*\varphi)(\varepsilon_k^{j}z)} \\ &= -\frac{(z^{p+1}\varphi(z))*z^{p+2}(H^{\alpha}_{p,\beta,\mu}f(z))'}{pz^{p+1}\varphi(z)*(z^{p+1}f_n^{\mu}(\alpha)(z))} \\ &= \frac{z^{p+1}\varphi(z)*(\psi(z)F(z))}{z^{p+1}\varphi(z)*\psi(z)}. \qquad (2.36)\end{aligned}$$

Since h is convex, univalent, applying Lemma 1.4, it follows $\Psi(z) \prec h(z)$, that is $f*\varphi \in S_n^{\mu}(\alpha, h)$. □

Remark 2.10. Taking $\mu = 1$, in the above results we obtain results concerning the operator $H^{\alpha}_{p,\beta}f(z)$ defined by (1.9).

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