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# Existence and stability of Langevin equations with two Hilfer-Katugampola fractional derivatives

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**Abstract.** In this note, we debate the existence, uniqueness and stability results for a general class of Langevin equations. We suggest the generalization via the Hilfer-Katugampola fractional derivative. We introduce some conditions for existence and uniqueness of solutions. We utilize the concept of fixed point theorems (Krasnoselskii fixed point theorem (KFPT), Banach contraction principle (BCP)). Moreover, we illustrate definitions of the Ulam type stability. These definitions generalize the fractional Ulam stability.

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**Keywords:** Fractional calculus, fractional differential operator, fractional differential equation, Ulam stability.

## 1. Introduction

The field of an arbitrary calculus (fractional calculus) is the extension of the ordinary calculus of fractional powers. It plays a significant field in the mathematical analysis. In addition, it is more than three centuries old, yet it only receives much attention and interest in last three decades [7, 17, 20]. The Langevin equation describes the stochastic problem in many fluctuating situations. A modified type of this equation used in various functional approaches of fractal mediums. Another modification requires replacing of ordinary differential equations into fractional differential equations (FDE), which yields the fractional Langevin equation [2, 4, 5, 6, 3, 21].

In recent times, Katugampola [13] introduced a new fractional differential operator which studied extensively by many researchers [14, 15, 25, 26]. Moreover, this operator has been compounded with Hilfer fractional differential operator introduced by Hilfer [7] to develop a new fractional differential operator, so called Hilfer-Katugampola fractional differential operator [19]. For the wide knowledge of fractional differential operators, one can refer to [22, 23, 24]. Rassias imposed the Hyers-Ulam stability (UHS) for both cases linear and nonlinear studies. This outcome of Rassias attracted many investigators worldwide who began to be motivated to study the stability problems of differential equations [1, 12, 18, 29]. The fractional Ulam stability (FUS) introduced by Wang [28], [27] and Ibrahim [8]-[11]. In our investigation, we focus on the following fractional differential equation containing two Hilfer-Katugampola fractional differential operators

$$\begin{cases} {}^{\rho}D^{\alpha_1,\beta}\left({}^{\rho}D^{\alpha_2,\beta}+\lambda\right)x(t)=f(t,x(t)), & t\in J:=(a,b], \\ I^{1-\gamma}x(a)=x_a, \quad \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta. \end{cases} \tag{1.1}$$

where ${}^{\rho}D^{\alpha_1,\beta}$ and ${}^{\rho}D^{\alpha_2,\beta}$ is Hilfer-Katugampola fractional differential operator of orders $\alpha_1$ and $\alpha_2$ and type $\beta$, $\rho>0$ and $\lambda$ is any real number. Let $f: J\times R\to R$ is given continuous function.

The effort is systematic as follows: In Section 2, we submit preliminaries that utilized throughout the paper. In Section 3, we set up the existence and uniqueness for a special formula of multi-power FDE covering the Hilfer-Katugampola fractional differential operator. In Section 4, we discuss some types of fractional Ulam stability.

## 2. Preliminaries

Some basic definitions and results introduced in the recent section. The following observations selected from [17, 14, 19]. Let $C[a,b]$ be a space of all continuous functions subject to the sup. norm$\|\psi\|=\sup\{|\psi(t)|: t\in J\}$. The weighted space $C_{\gamma,\rho}[a,b]$ of functions $f$ on $(a,b]$ is defined by

$$C_{\gamma,\rho}[a,b]=\left\{f:(a,b]\to R:\left(\frac{t^\rho-a^\rho}{\rho}\right)^\gamma f(x)\in C[a,b]\right\}, 0\le\gamma<1,$$

with the norm

$$\|g\|_{C_{\gamma,\rho}}=\left\|\left(\frac{t^\rho-a^\rho}{\rho}\right)^\gamma f(x)\right\|_C=\max_{t\in J}\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^\gamma f(x)\right|, C_{0,\rho}[a,b]=C[a,b].$$

Let $\delta_\rho=\left(t^\rho\frac{d}{dt}\right)$ and for $n\in N$, the notion $C^n_{\delta_{\rho,\gamma}}[a,b]$, be the Banach space of all functions $f$ which are continuously differentiable. Suppose that the operator $\delta_\rho$, is on $[a,b]$ of $(n-1)-$order and the derivative $\delta^n_\rho f$ of $n-$order on $(a,b]$ such that $\delta^n_\rho f\in C_{\gamma,\rho}[a,b]$. This leads to

$$C^n_{\delta_{\rho,\gamma}}[a,b]=\left\{\delta^k_\rho f\in C[a,b], \delta^n_\rho f\in C_{\gamma,\rho}[a,b], k=0,1,...,n-1\right\}$$

with the norm

$$\|f\|_{C^n_{\delta_{\rho,\gamma}}}=\sum_{k=0}^{n-1}\left\|\delta^k_\rho f\right\|_C+\left\|\delta^n_\rho f\right\|_{C_{\gamma,\rho}}, \quad \|f\|_{C^n_{\delta_\rho}}=\sum_{k=0}^{n}\max_{x\in R}\left|\delta^k_\rho f(x)\right|.$$

For $n=0$, we have

$$C^0_{\delta_{\rho,\gamma}}[a,b]=C_{\gamma,\rho}[a,b].$$

**Definition 2.1.** Let $\alpha, c \in R$ with $\alpha > 0$ and $f \in X^p_c(a,b)$, where $f \in X^p_c(a,b)$ consists of the Lebesgue measurable functions. The generalized left-sided fractional integral ${}^\rho I^\alpha_{a+} f$ of order $\alpha \in C(\Re(\alpha))$ is defined by

$$\left({}^\rho I^\alpha_{a+} f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s) ds, \;\; t > a. \tag{2.1}$$

The extended fractional derivative analog to the extended fractional integral (2.1), is given by

$$\left({}^\rho D^\alpha_{a+} f\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_a^t (t^\rho - s^\rho)^{n-\alpha+1} s^{\rho-1} f(s) ds. \tag{2.2}$$

**Definition 2.2.** The Hilfer-Katugampola fractional operator with respect to $t$, of order $\rho > 0$, is defined by

$$\begin{aligned} \left({}^\rho D^{\alpha,\beta}_{a\pm} f\right)(t) &= \left(\pm {}^\rho I^\alpha_{a\pm} \left(t^{\rho-1} \frac{d}{dt}\right) {}^\rho I^{(1-\beta)(1-\alpha)}_{a\pm}\right)(t) \\ &= \left(\pm {}^\rho I^\alpha_{a\pm} \delta_\rho {}^\rho I^{(1-\beta)(1-\alpha)}_{a\pm}\right)(t). \end{aligned} \tag{2.3}$$

- The operator ${}^\rho D^{\alpha,\beta}_{a+}$ can be written as

$${}^\rho D^{\alpha,\beta}_{a+} = {}^\rho I^{\beta(1-\alpha)}_{a+} \delta_\rho {}^\rho I^{1-\gamma}_{a+} = {}^\rho I^{\beta(1-\alpha)}_{a+} {}^\rho D^\gamma_{a+}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

- The fractional derivative ${}^\rho D^{\alpha,\beta}_{a+}$ is considered as interpolation, with the convenient parameters, of the following fractional derivatives, Hilfer fractional operator when $\rho \to 1$, Hilfer-Hadamard operator when $\rho \to 0$, generalized fractional operator when $\beta = 0$, Caputo-type fractional derivative when $\beta = 1$, Riemann-Liouville fractional derivative when $\beta = 0, \rho \to 1$, Hadamard operator when $\beta = 0, \rho \to 0$, Caputo operator when $\beta = 1, \rho \to 1$, Caputo-Hadamard operator when $\beta = 1, \rho \to 0$, Liouville fractional derivative when $\beta = 0, \rho \to 1, a = 0$ and Hadamard fractional derivative when $\beta = 0, \rho \to 1, a = -\infty$. We consider the following parameters $\alpha, \beta, \gamma, \mu,$:

$$\gamma = \alpha + \beta - \alpha\beta, \;, \; 0 \le \mu < 1, \; \alpha > 0, \; \beta < 1, 0 \le \gamma < 1.$$

The following results can be found in [19]:

**Lemma 2.3.** *Let $\alpha, \beta > 0$, $0 < a < b < \infty$, $\rho, c \in R$, $1 \le p \le \infty$ and $\rho \ge c$. Then, for $f \in X^p_c(a,b)$ the semi group property is valid. This is,*

$$({}^\rho I^\alpha_{a+} {}^\rho I^\beta_{a+} f)(x) = ({}^\rho I^{\alpha+\beta}_{a+})(x),$$

*and*

$$({}^\rho D^\alpha_{a+} {}^\rho I^\alpha_{a+} f)(x) = f(x).$$

**Lemma 2.4.** *Assume that $x > a$, ${}^{\rho}I^{\alpha}_{a+}$ and ${}^{\rho}D^{\alpha}_{a+}$ are according on Eq.* (2.1) *and* (2.2), *respectively. Then*

$$
\begin{aligned}
{}^{\rho}I^{\alpha}_{a+}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}(x) &= \left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta-1}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \alpha \geq 0,\\
{}^{\rho}D^{\alpha}_{a+}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}(x) &= 0, \quad \alpha \in (0,1),\ \beta \in (0,\infty).
\end{aligned}
$$

**Lemma 2.5.** *Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $f \in C_{\gamma}[a,b]$ and ${}^{\rho}I^{1-\alpha}_{a+}f \in C^{1}_{\gamma}[a,b]$, then*

$$
\left({}^{\rho}I^{\alpha}_{a+}{}^{\rho}D^{\alpha}_{a+}f\right)(x) = f(x) - \frac{\left({}^{\rho}I^{1-\alpha}_{a+}f\right)(a)}{\Gamma(\alpha)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1},
$$

*for all $x \in (a,b]$.*

**Lemma 2.6.** *Let $0 < a < b < \infty$, $\alpha > 0$, $0 \leq \gamma < 1$ and $f \in C_{\gamma,\rho}[a,b]$. If $\alpha > \gamma$, then ${}^{\rho}I^{\alpha}_{a+}f$ is continuous on $[a,b]$ and*

$$
\left({}^{\rho}I^{\alpha}_{a+}f\right)(a) = \lim_{t\to a^{+}}\left({}^{\rho}I^{\alpha}_{a+}f\right)(t) = 0.
$$

We present some spaces as follows:

$$
C^{\alpha,\beta}_{1-\gamma,\rho}[a,b] = \left\{f \in C_{1-\gamma,\rho}[a,b], {}^{\rho}D^{\alpha,\beta}_{a+}f \in C_{\mu,\rho}[a,b]\right\}
$$

and

$$
C^{\gamma}_{1-\gamma,\rho}[a,b] = \left\{f \in C_{1-\gamma,\rho}[a,b], {}^{\rho}D^{\gamma}_{a+}f \in C_{1-\gamma,\rho}[a,b]\right\}.
$$

Clearly, we have

$$
C^{\gamma}_{1-\gamma,\rho}[a,b] \subset C^{\alpha,\beta}_{1-\gamma,\rho}[a,b].
$$

**Lemma 2.7.** *If $C^{\gamma}_{1-\gamma}[a,b]$, then*

$$
{}^{\rho}I^{\gamma}_{a+}{}^{\rho}D^{\gamma}_{a+}f = {}^{\rho}I^{\alpha}_{a+}{}^{\rho}D^{\alpha,\beta}_{a+}f \tag{2.4}
$$

*and*

$$
{}^{\rho}D^{\gamma}_{a+}{}^{\rho}I^{\alpha}_{a+}f = {}^{\rho}D^{\beta(1-\alpha)}_{a+}f. \tag{2.5}
$$

**Lemma 2.8.** *If ${}^{\rho}D^{\beta(1-\alpha)}_{a+}f$ exists on $L^{1}(a,b)$, then*

$$
{}^{\rho}D^{\alpha,\beta}_{a+}{}^{\rho}I^{\alpha}_{a+}f = {}^{\rho}I^{\beta(1-\alpha)}_{a+}{}^{\rho}D^{\beta(1-\alpha)}_{a+}f.
$$

**Lemma 2.9.** *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}[a,b]$ and ${}^{\rho}I^{1-\beta(1-\alpha)}_{a+} \in C^{1}_{1-\gamma}[a,b]$, then ${}^{\rho}D^{\alpha,\beta}_{a+}I^{\alpha}_{a+}$ exists on $(a,b]$ and*

$$
{}^{\rho}D^{\alpha,\beta}_{a+}I^{\alpha}_{a+}f = f.
$$

**Lemma 2.10.** [19] *Let $\gamma = \alpha + \beta - \alpha\beta$. If $f : (a,b] \times R \to R$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\rho}[a,b]$ for all $x \in C_{1-\gamma,\rho}[a,b]$ then a function $x \in C^{\gamma}_{1-\gamma,\rho}[a,b]$ is the outcome of the problem*

$$
\begin{cases}
{}^{\rho}D^{\alpha_1,\beta}_{a+}\left({}^{\rho}D^{\alpha_2,\beta}_{a+}+\lambda\right)x(t) = f(t,x(t)),\ t \in (a,b],\\
{}^{\rho}I^{1-\gamma}_{a_+}x(a) = x_a,
\end{cases}
$$

*if and only if $x$ achieves the following formula:*

$$x(t) = \frac{x_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1}x(s)ds$$
$$+ \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1}f(s,x(s))ds.$$

The proof of the lemma is similar ([30], Lemma 3.1).

**Theorem 2.11.** *(KFPT) Suppose that $\Sigma$ is a Banach space, $\Theta$ is a closed, bounded and convex subset of $\Sigma$ and two functions $\Gamma_1, \Gamma_2 : \Theta \to \Sigma$ such that $\Gamma_1\chi + \Gamma_2\eta \in \Theta$ for all $\chi, \eta \in \Theta$. If $\Gamma_1$ is a contraction function and $\Gamma_2$ is completely continuous, then $\Gamma_1\chi + \Gamma_2\chi = \chi$ admits a solution in $\Theta$.*

**Theorem 2.12.** *(Arzela-Ascoli theorem)* [16] *A subset $F$ of C(X) is relatively compact if and only if it is closed, bounded and equicontinuous.*

## 3. Existence and uniqueness results

For our setting, we deliver the following assumptions:

(H1) Let $f(\cdot, x(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\rho}[a,b]$ for any $x \in C_{1-\gamma,\rho}[a,b]$. There exists a positive constant $\ell$ such that
$$|f(t,\chi) - f(t,\eta)| \le \ell\,|\chi - \eta|, \quad \text{for all } \chi, \eta \in R.$$

(H2) The constant
$$\Omega = \left(\frac{\ell B(\gamma, \alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2} + \frac{\lambda B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_2}\right) < 1.$$

(H3) There exist a nondecreasing function $\varphi : J \to R^+$ and $\lambda_\varphi > 0$ such that for $t \in J$,
$${}^\rho I^{\alpha_1+\alpha_2}_{a+}\varphi(t) \le \lambda_\varphi \varphi(t).$$

By applying Theorem 2.11, we have the following result:

**Theorem 3.1.** *(Existence) Suppose that [H1] and [H2] are achieved. Then, Eq.* (1.1) *admits at least one outcome in $C^\gamma_{1-\gamma,\rho}[a,b] \subset C^{\alpha,\beta}_{1-\gamma,\rho}[a,b]$.*

*Proof.* Define the operator $N : C_{1-\gamma,\rho}[a,b] \to C_{1-\gamma,\rho}[a,b]$, it is well defined and given by

$$(Nx)(t) = \begin{cases} \frac{x_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1}x(s)ds \\ +\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1}\, f(s,x(s))ds \end{cases} \tag{3.1}$$

Set $\widetilde{f}(s) = f(s, 0)$ and

$$\omega = \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2}\frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\|f\|_{C_{1-\gamma,\rho}} + \frac{x_a}{\Gamma(\gamma)}$$

Consider the ball $B_r = \Big\{\chi \in C_{1-\gamma,\rho}[a,b] : \|\chi\|_{C_{1-\gamma,\rho}} \leq r\Big\}$.
Now we subdivide the operator $N$ into two operator $A$ and $B$ on $B_r$ as follows:

$$(Ax)(t) = \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s,x(s))ds$$

and

$$(Bx)(t) = \frac{x_a}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1} x(s)ds.$$

The proof is as follows:
Step 1. $Ax + By \in B_r$ for every $x, y \in B_r$.

$$\begin{aligned}
&\left|(Ax)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \times f(s,x(s))ds \\
&\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right) ds \\
&\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \left(\ell\, |x(s)| + \left|\widetilde{f}(s)\right|\right) ds \\
&\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{B(\gamma, \alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2+\gamma-1} \left(\ell\, \|x\|_{C_{1-\gamma}} + \left\|\widetilde{f}\right\|_{C_{1-\gamma,\rho}}\right).
\end{aligned}$$

This gives

$$\|Ax\|_{C_{1-\gamma,\rho}} \leq \frac{B(\gamma, \alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2} \left(\ell\, \|x\|_{C_{1-\gamma,\rho}} + \left\|\widetilde{f}\right\|_{C_{1-\gamma,\rho}}\right). \tag{3.2}$$

For operator $B$

$$\begin{aligned}
&\left|(Bx)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\leq \frac{x_a}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1} x(s)ds \\
&\leq \frac{x_a}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{\lambda B(\gamma, \alpha_2)}{\Gamma(\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha_2+\gamma-1} \|x\|_{C_{1-\gamma,\rho}}.
\end{aligned}$$

Thus, we obtain

$$\|(Bx)\|_{C_{1-\gamma}} \leq \frac{x_a}{\Gamma(\gamma)} + \frac{\lambda B(\gamma, \alpha_2)}{\Gamma(\alpha_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_2} \|x\|_{C_{1-\gamma,\rho}}. \tag{3.3}$$

Linking (3.2) and (3.3), for every $x, y \in B_r$, we get

$$\|Ax + By\|_{C_{1-\gamma,\rho}} \leq \|Ax\|_{C_{1-\gamma,\rho}} + \|By\|_{C_{1-\gamma,\rho}} \leq \Omega r + \omega.$$

Step 2. $A$ is a contraction mapping.
For any $x, y \in B_r$, we observe the conclusion

$$\left|((Ax)(t)-(Ay)(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\right|$$
$$\leq \left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1}\left|f(s,x(s))-f(s,y(s))\right|ds$$
$$\leq \left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\frac{\ell}{\Gamma(\alpha_1+\alpha_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha_1+\alpha_2+\gamma-1}\|x-y\|_{C_{1-\gamma,\rho}}.$$

This gives

$$\|(Ax)-(Ay)\| \leq \frac{\ell B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_1+\alpha_2}\|x-y\|_{C_{1-\gamma,\rho}}.$$

In view of [H2], the operator $A$ is a contraction mapping.

Step 3. The operator $B$ is completely compact.
According to Step 1, we know that

$$\|(Bx)\|_{C_{1-\gamma,\rho}} \leq \frac{x_a}{\Gamma(\gamma)}+\frac{\lambda B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_2}\|x\|_{C_{1-\gamma,\rho}}.$$

Thus, the operator $B$ is uniformly bounded. Next, we show that the operator $B$ is compact. A calculation implies

$$|(Bx)(t_1)-(Bx)(t_2)| \leq \frac{x_a}{\Gamma(\gamma)}\left|\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\gamma-1}-\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right|$$
$$+\frac{\ell B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\|x\|_{C_{1-\gamma,\rho}}\left|\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha_2+\gamma-1}-\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha_2+\gamma-1}\right|,$$

which is tending to zero as $t_1 \to t_2$. Thus $B$ is equicontinuous. Hence, in virtue of the Theorem 2.12, the operator $B$ is compact on $B_r$. It leads by Krasnoselskii fixed point theorem, that the problem (1.1) admits a solution. □

**Theorem 3.2.** *If hypothesis (H1) and (H2) are fulfilled. Then, Eq.* (1.1) *admits a unique solution.*

## 4. Stability outcomes

In the recent section, we shall give the definitions and the criteria of (UHS) and (UHRS) for the generalized Langevin Eq. (1.1). Now for $\epsilon > 0$ and a continuous function $\varphi : J \to R^+$ , we theorize the next inequalities:

$$\left|{}^\rho D_{a^+}^{\alpha_1,\beta}\left({}^\rho D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)-f(t,z(t))\right| \leq \epsilon, \quad t \in J, \tag{4.1}$$

$$\left|{}^\rho D_{a^+}^{\alpha_1,\beta}\left({}^\rho D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)-f(t,z(t))\right| \leq \epsilon\varphi(t), \quad t \in J, \tag{4.2}$$

$$\left|{}^{\rho}D_{a^+}^{\alpha_1,\beta}\left({}^{\rho}D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)-f(t,z(t))\right|\leq\varphi(t),\quad t\in J. \tag{4.3}$$

**Qualifier**

The Eq. (1.1) is UHS if there occurs a number $C_f > 0$ and $\epsilon > 0$ such that for all outcome $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.1) there occurs an outcome $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) satisfying

$$|z(t)-x(t)|\leq C_f\ \epsilon,\quad t\in J.$$

The Eq. (1.1) is generalized UHS if there occurs a function $\varphi \in C_{1-\gamma,\rho}[a,b]$, $\varphi_f(0) = 0$ such that for all outcome $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.1) there occurs an outcome $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) achieving

$$|z(t)-x(t)|\leq \varphi_f\ \epsilon,\quad t\in J.$$

The Eq. (1.1) is UHRS esteeming by $\varphi \in C_{1-\gamma,\rho}[a,b]$ if there occurs a number $C_{f,\varphi} > 0$ for all $\epsilon > 0$ and for every outcome $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.2) there occurs an outcome $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) filing

$$|z(t)-x(t)|\leq C_{f,\varphi}\ \epsilon\varphi(t),\quad t\in J.$$

The Eq. (1.1) is generalized UHRS corresponding to $\varphi \in C_{1-\gamma,\rho}[a,b]$ if there occurs a real number $C_{f,\varphi} > 0$ whenever for every outcome $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.3) there occurs an outcome $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) satisfying

$$|z(t)-x(t)|\leq C_{f,\varphi}\varphi(t),\quad t\in J.$$

**Remark 4.1.** A function $z \in C_{1-\gamma,\rho}[a,b]$ is an outcome of the inequality (4.1) if and only if there exists a function $g \in C_{1-\gamma,\rho}[a,b]$ such that

$$\left|{}^{\rho}D_{a^+}^{\alpha_1,\beta}\left({}^{\rho}D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)-f(t,z(t))\right|\leq\epsilon,\quad t\in J,$$

if and only if there occurs a function $g \in C_{1-\gamma,\rho}[a,b]$ such that

(i) $|g(t)| \leq \epsilon, t \in J$.
(ii) ${}^{\rho}D_{a^+}^{\alpha_1,\beta}\left({}^{\rho}D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)=f(t,z(t))+g(t),\ t\in J.$

Similarly, for the inequalities (4.2) and (4.3).

**Remark 4.2.** If $z$ is an outcome of (4.1), then $z$ is an outcome of the following formula:

$$\begin{aligned}&\left|z(t)-\frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{\lambda}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}z(s)ds\right.\\&\left.-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1}s^{\rho-1}f(s,z(s))ds\right|\\&\leq\frac{1}{\Gamma(\alpha_1+\alpha_2+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_1+\alpha_2}\epsilon.\end{aligned}$$

It is clear that

$${}^{\rho}D_{a^+}^{\alpha_1,\beta}\left({}^{\rho}D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)=\ f(t,z(t))+g(t),\ t\in J.$$

Then

$$z(t)=\frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}-\frac{\lambda}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}z(s)ds$$
$$+\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1}s^{\rho-1}\left(f(s,z(s))+g(s)\right)ds.$$

Consequently, we obtain

$$\left|z(t)-\frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{\lambda}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}z(s)ds\right.$$
$$\left.-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1}s^{\rho-1}f(s,z(s))ds\right|$$
$$\leq\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1}s^{\rho-1}\left|g(s)\right|ds$$
$$\leq\frac{1}{\Gamma(\alpha_1+\alpha_2+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_1+\alpha_2}\epsilon.$$

We have similar remarks for the inequality (4.2) and (4.3).

Our main result is as follows:

**Theorem 4.3.** *Suppose that the hypotheses [H1] and [H3] achieved. Then Eq.* (1.1) *is a generalized UHRS.*

*Proof.* Let $z$ be a solution of 4.3. In view of Theorem 3.2, there $x$ is a unique outcome of the problem satisfying

$$\begin{aligned}{}^\rho D_{a^+}^{\alpha_1,\beta}\left({}^\rho D_{a^+}^{\alpha_2,\beta}+\lambda\right)x(t)&=f(t,x(t)),\\ I_{a^+}^{1-\gamma}x(a)&=I_{a^+}^{1-\gamma}z(a).\end{aligned}$$

Then we have

$$x(t)=\frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}-\frac{\lambda}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}x(s)ds$$
$$+\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1}s^{\rho-1}f(s,x(s))ds.$$

By differentiating inequality (4.3), we have

$$\left|z(t)-\frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{\lambda}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}z(s)ds\right.$$
$$\left.-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1}s^{\rho-1}f(s,z(s))ds\right|$$

$$\leq \left| \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \varphi(s) ds \right| \leq \lambda_\varphi \varphi(t).$$

Hence it follows that,

$$\begin{aligned}
&|z(t) - x(t)| \\
&\leq \left| z(t) - \frac{z_a}{\Gamma(\gamma)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2-1} s^{\rho-1} x(s) ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s, x(s)) ds \right| \\
&\leq \left| z(t) - \frac{z_a}{\Gamma(\gamma)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2-1} s^{\rho-1} z(s) ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s, z(s)) ds \right| \\
&+ \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2-1} s^{\rho-1} \left| x(s) - z(s) \right| ds \\
&+ \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \left| f(s, z(s)) - f(s, x(s)) \right\| ds \\
&\leq \lambda_\varphi \varphi(t) + \left( \frac{\lambda}{\Gamma(\alpha_2+1)} \left( \frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_2} + \frac{\ell}{\Gamma(\alpha_1+\alpha_2+1)} \left( \frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_1+\alpha_2} \right) |z - x| .
\end{aligned}$$

By Lemma 2.5, there occurs a constant $M^* > 0$ independent of $\lambda_\varphi \varphi(t)$, achieving

$$|z(t) - x(t)| \leq M^* \varphi(t).$$

Thus, Eq. (1.1) is generalized UHRS. □

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