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Radii of harmonic mapping with fixed second coefficients in the plane

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Abstract. In this paper we investigate the radii problem for harmonic functions with a fixed coefficient and determine the radii of univalence, stable starlikness, stable convexity, fully starlikness and fully convexity of order α for these type of functions. All results are sharp. Also these results generalize and improve some results in the literature.

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1. Introduction and Preliminaries

Denote by $\mathcal{H}$ the class of all complex-valued harmonic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) - 1$. Each $f \in \mathcal{H}$ can be decomposed as $f = h + \overline{g}$, where g and h are analytic in $\mathbb{D}$ so that

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

Let $\mathcal{S}_H$ denote the class of univalent and orientation- preserving functions $f \in \mathcal{H}$. Then the Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$.
A necessary and sufficient condition (see Lewy [11]) for a harmonic function f to be locally univalent in $\mathbb{D}$ is that $J_f(z) > 0$ in $\mathbb{D}$.
Let $\mathcal{K}_H, \mathcal{S}^*_H$ and $\mathcal{C}_H$ be the subclass of $\mathcal{S}_H$ mapping $\mathbb{D}$ onto convex, starlike and close-to-convex domains, respectively. Also denote by $\mathcal{K}^0_H, {\mathcal{S}^*}^0_H, \mathcal{C}^0_H$ and $\mathcal{S}^0_H$ the class consisting of the functions f in $\mathcal{K}_H, \mathcal{S}^*_H, \mathcal{C}_H$ and $\mathcal{S}_H$ respectively, for which $f_{\overline{z}}(0) = b_1 = 0$.
One of the important questions in the study of class $\mathcal{S}^0_H$ and its subclasses is related to coefficient bounds. In 1984, Clunie and Sheil-Small [5], it was conjectured that

the Taylor coefficients of the the series h and g, namely, a_n and b_n are satisfy the conditions $|a_n| \leq A_n$ and $|b_n| \leq B_n$, where

$$A_n = \frac{1}{6}(2n+1)(n+1) \qquad \text{and} \qquad B_n = \frac{1}{6}(2n-1)(n-1). \tag{1.2}$$

The harmonic function $f(z) = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=2}^{\infty} B_n z^n}$ is known as harmonic Koebe function and maps the unit disk $\mathbb{D}$ onto the slit plane $\mathbb{C} \setminus \{u + iv : u \leq -1/6, v = 0\}$ which is starlike function in $\mathbb{D}$.

Although, the coefficients conjecture remains an open problem for the full class $\mathcal{S}_H^0$, the same has been verified for all functions $f \in \mathcal{S}_H^0$ with real coefficients and all function $f \in \mathcal{S}_H^0$ for which either $f(\mathbb{D})$ is starlike with respect to the origin, close-to-convex function and for convex in one direction (see [5], [19], [20]). The extremal function is the harmonic Koebe function. If $f \in \mathcal{K}_H^0$, Clunie and Sheil-Small [5] proved the Taylor coefficients of h and g satisfy the inequality

$$|a_n| \leq \frac{n+1}{2} \qquad \text{and} \qquad |b_n| \leq \frac{n-1}{2}, \tag{1.3}$$

and equality occurs for the harmonic half-plane mapping

$$\begin{aligned} L(z) &= \frac{2z - z^2}{2(1-z)^2} + \overline{\frac{-z^2}{2(1-z)^2}} \\ &= \sum_{n=1}^{\infty} \frac{n+1}{2} z^n - \overline{\sum_{n=1}^{\infty} \frac{n-1}{2} z^n}. \end{aligned}$$

Chaqui et al. [4] introduced the notion of fully starlike and fully convex harmonic function that do inherit the properties of starlikeness and convexity, respectively. A harmonic mapping f of $\mathbb{D}$ is said to be fully convex of order α, $0 \leq \alpha < 1$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve satisfying

$$\frac{\partial}{\partial \theta}(\arg\left(\frac{\partial}{\partial \theta} f(re^i\theta))\right) > \alpha, \quad 0 \leq \theta < t\pi, \quad 0 < r < 1.$$

Similarly, a harmonic mapping f of $\mathbb{D}$ is said to be fully starlike of order α, $0 \leq \alpha < 1$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bound a domain starlike with respect to the origin satisfying

$$\frac{\partial}{\partial \theta}(\arg(f(re^i\theta))) > \alpha, \quad 0 \leq \theta < t\pi, \quad 0 < r < 1.$$

In [5] Clunie and Sheil-Small proved the following result

Lemma 1.1. *If h, g are analytic in $\mathbb{D}$ with $|h'(0)| > |g'(0)|$ and $h + \epsilon g$ is close-to-convex for each ϵ, $|\epsilon| = 1$, then $f = h + \overline{g}$ is close-to-convex in $\mathbb{D}$.*

This lemma has been used to obtain many important results. Motivated by this result Hernandes et al. [7] introduced the notion of stable univalent stable starlike, stable convex and stable close-to-convex harmonic functions.

We say that the (sense-preserving) harmonic mapping $f = h + \overline{g}$ is stable univalent (resp. stable convex, stable starlike with respect to origin, or stable close-to-convex)

if all the mapping $f_\lambda = h + \lambda\overline{g}$ with $|\lambda| = 1$ are univalent (resp. convex, starlike with respect to origin, or close-to-convex) in $\mathbb{D}$.
Analogs to Lemma 1.1 it is proved [7] that the harmonic mapping $f = h + \overline{g}$ is stable univalent (resp. stable convex, stable starlike with respect to origin, or stable close-to-convex) if and only if all the mapping $f_\lambda = h + \lambda g$ with $|\lambda| = 1$ are univalent (resp. convex, starlike with respect to origin, or close-to-convex) in $\mathbb{D}$.
We note that fully starlike function need not be univalent, but stable starlike function is univalent. Also it is easy to see that the stable starlike and stable convex functions are fully starlike and fully convex functions, respectively. It is easy to see that a function $f = h + \overline{g}$ whose coefficients satisfy the conditions $|a_n| \leq A_n$ and $|b_n| \leq B_n$ is harmonic in $\mathbb{D}$, however it need not be univalent. For example the function $f(z) = z + 2\overline{z}^4$ satisfy the above mentioned conditions but it is not sense-preserving in $\mathbb{D}$. It is therefore of interest to determine the largest subdisk $|z| < \rho < 1$ in which the function f satisfying the condition (1.2) and (1.3) (or others) influence the univalency of f. Recall that given two subsets $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ the $\mathcal{N}$ radius in $\mathcal{M}$ is the largest R such that, for every $f \in \mathcal{M}$, $r^{-1}f(rz) \in \mathcal{N}$ for every $r \leq R$.
The radius of fully convexity of the class $\mathcal{K}_H^0$ is $\sqrt{2} - 1$, while the radius of fully convexity of the class ${\mathcal{S}^*}_H^0$ is $3 - \sqrt{8}$ (see [18], [19]). The corresponding problem for the radius of fully starlikeness are still unsolved. In [10], the radius of close-to-convexity of harmonic mapping was determined. These results are generalized in context of fully starlike and fully convex functions of order α $(0 \leq \alpha < 1)$ in [14]. Also, we remark that many authors studied the radii problem in the analytic functions (see [1], [2], [11], [13], [15], [17], [16], [21]) There is a challenge in fixing the second coefficient which is due to that removal of natural extremal function from the class. In this paper such as [12] we investigate the radii of univalence, stable starlikeness, stable convexity, fully starlikeness and fully convexity for these types of functions. Also we determine the Bloch constant for harmonic functions with fixed second coefficient.
For proving our result we shall need the following result due to Jahangiri [8], [9].

Theorem 1.1. *Let $f = h + \overline{g}$, where h and g are given by (1.1) and let $0 \leq \alpha < 1$. Then we have the following*
(i) If

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha}|b_n| \leq 1,$$

then f is harmonic univalent and f is fully starlike of order α.
(ii) If

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 1,$$

then f is harmonic univalent and f is fully convex of order α.

By making use of Theorem 1.1 we conclude the following result.

Corollary 1.2. *Let $f = h + \overline{g}$, where h and g are given by (1.1), then we have the following* *(i) If*

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 2 \qquad with \quad a_1 = 1,$$

then f is harmonic univalent and f is stable starlike.
(ii) If

$$\sum_{n=1}^{\infty} n^2(|a_n| + |b_n|) \leq 2 \qquad with \quad a_1 = 1,$$

then f is harmonic univalent and f is stable convex.

2. Radii of Univalence

In this section, the sharp stable starlike (convex) radius and the sharp fully starlike (convex) of order $\alpha, (0 \leq \alpha < 1)$ are obtained for harmonic functions.

Theorem 2.1. *Let $f = h + \overline{g} \in \mathcal{H}$, where h, g is given by (1.1). Let*

$$b_1 = 0, \ |a_2| + |b_2| = 2b \quad (0 \leq b \leq 1) \quad and \quad |a_n| + |b_n| \leq n \ (n \geq 3). \tag{2.1}$$

Then for f,
(i) the radius of stable starlikeness is r_s, where $r_s = r_s(b)$ is the smallest root in $(0, 1)$ of the equation

$$1 + r = 2[1 + 2(1 - b)r](1 - r)^3. \tag{2.2}$$

(ii) the radius of stable convexity is r_c, where $r_c = r_c(b)$ is the smallest root in $(0, 1)$ of the equation

$$2[1 + 4(1 - b)r](1 - r)^4 = 1 + 4r + r^2. \tag{2.3}$$

Furthermore, all results are sharp.

Proof. First we prove the case (i). For $r \in (0, 1)$ with $r \leq r_s$, it is sufficient to show that F_r is stable starlike, where

$$F_r(z) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}. \tag{2.4}$$

According to Corollary 1.1, it is sufficient to show that

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) r^{n-1} \leq 1. \tag{2.5}$$

Considering condition (2.1), we have

$$\begin{aligned} \sum_{n=2}^{\infty} n(|a_n| + |b_n|) r^{n-1} &\leq 4br_s + \sum_{n=3}^{\infty} n^2 r_s^{n-1} \\ &= 4br_s + \frac{1 + r_s}{(1 - r_s)^3} - 1 - 4r_s = 1, \end{aligned}$$

provided r_s is the root of the equation (2.2) in the hypothesis of the theorem. Set

$$m(r) = 1 + r - 2(1 + 2(1 - b)r)(1 - r)^3.$$

Then $m(0) = -1$ and $m(1) = 2$ and so intermediate value theorem shows that the equation (2.2) has a root in the interval $(0, 1)$. Next for proving the sharpness part we consider the function $f_0 = h_0 + \overline{g_0}$, where

$$h_0(z) = 2z + \frac{3}{2}z^2 - \frac{2z - z^2}{2(1-z)^2} \qquad \text{and} \qquad g_0(z) = 2bz^2 - \frac{1}{2}z^2 + \frac{z^2}{2(1-z)^2}.$$

Direct calculation leads to

$$h_0'(z) = 2 + 3z - \frac{1}{(1-z)^3} \qquad \text{and} \qquad g_0'(z) = 4bz - z + \frac{z}{(1-z)^3}.$$

Now from (2.2), we have

$$[h_0'(r) - g_0'(r)]_{r=r_s} = \frac{1}{(1-r)^3}[2(1+2r(1-b))(1-r)^3 - (1+r)]|_{r=r_s} = 0.$$

Hence,

$$J_{f_0}(r)|_{r=r_s} = [(h_0'(r) + g_0'(r))(h_0'(r) - g_0'(r))]_{r=r_s} = 0.$$

Therefore, in view of Lewy's theorem, the function f_0 is not univalent in $|z| < r$ if $r > r_s$. This shows the radius r_s is sharp.
Case (ii). The proof of (2.3) is similar to the case (i) and we omit the details. To proof sharpness, we take the function $f_1 = h_1 + \overline{g_1}$, where

$$h_1(z) = 2z + 2z^2 - \frac{z}{(1-z)^2} \quad \text{and} \quad g_1(z) = -2bz^2$$

Then if we define $F_1(z) = h_1(z) + g_1(z)$, it yields

$$\operatorname{Re}\left(1 + \frac{zF_1''(z)}{F_1'(z)}\right)|_{r=r_c} = \left\{\frac{2(1-r^4)[1+4(1-b)r] - (1+4r+r^2)}{(1-r)[(2+4(1-b)r)(1-r^3) - (1+r)]}\right\}|_{r=r_c} = 0.$$

The denominator of the rational function in the middle of the equation above is greater than the numerator for all $0 \le r < 1$ and $0 \le b \le 1$. Therefore, if we take the smallest root of the denominator with r_p in $(0, 1)$ then, we have $r_c < r_p$. Hence for $r_c < r < r_p$ the denominator of the rational function in the middle of the equation above is the positive while the numerator is negative, and this means the expression in the middle is negative for $r_c < r < r_p$. So we observe that the function F_1 is not convex for $|z| < r$, where $r > r_c$. Now the functions $F_\lambda = h_1(z) + \lambda g_1(z)$ for all λ with $|\lambda| = 1$ are not convex for $|z| < r$, where $r > r_c$, or the functions $F_\lambda = h_1(z) + \lambda\overline{g_1(z)}$ for all λ with $|\lambda| = 1$ are not convex for $|z| < r$, where $r > r_c$. This shows that the radius r_c is sharp. □

For example let us consider the half-plane harmonic function $L = h + \overline{g}$, where

$$h(z) = \frac{2z - z^2}{2(1-z)^2}, \qquad g(z) = \frac{-z^2}{2(1-z)^2}.$$

This function maps $\mathbb{D}$ harmonically onto domain $\Omega = \{z \in \mathbb{C} : \operatorname{Re}\{z\} > -1/2\}$ and so it is convex function. We remark this function is not stable starlike function. Since if we consider the function

$$h(z) = \frac{2z - z^2}{2(1-z)^2}$$

we obtain

$$\mathrm{Re}\frac{zh'(z)}{h(z)} = 2\mathrm{Re}\left(\frac{1}{1-z} - \frac{1}{2-z}\right),$$

and with calculation one can see that which is zero in the point $z_0 = \frac{\sqrt{14}}{4}e^{icos^{-1}\frac{3}{\sqrt{14}}}$. Hence h is not starlike and by result in [7] the half plane map L is not stable starlike. On the other hand it is easy to see that

$$L(z) = z + \sum_{n=2}^{\infty}\frac{n+1}{2}z^n - \overline{\sum_{n=2}^{\infty}\frac{n-1}{2}z^n},$$

so by taking $a_2 = 3/2, b_2 = 1/2$ we observe that L satisfies the condition of Theorem 2.1. Therefore L is stable starlike function in the disk $|z| \leq r_0$ where $r_0 \simeq 0.164878$ is the real root of the equation $2r^3 - 6r^2 + 7r - 1 = 0$, in the interval $(0, 1)$.

Theorem 2.2. *Let* $f = h + \overline{g} \in \mathcal{H}$, *where* h, g *is given by* (1.1). *Let*

$$|b_1| < 1,\ |a_2|+|b_2| = 2b\ (0 \leq b \leq \frac{M}{2}) \quad \text{and} \quad |a_n|+|b_n| \leq M\ (M > 0)\ (n \geq 3). \tag{2.6}$$

Then for f,
(i) the radius of stable starlikeness is r_s, *where* $r_s = r_s(b)$ *is the smallest root in* $(0, 1)$ *of the equation*

$$M - [1 + M - |b_1| + 2(M - 2b)r](1 - r)^2 = 0. \tag{2.7}$$

(ii) the radius of stable convexity is r_c, *where* $r_c = r_c(b)$ *is the smallest root in* $(0, 1)$ *of the equation*

$$(1 - r)^3[1 + M - |b_1| + 4r(M - 2b)] = M(1 + r). \tag{2.8}$$

Furthermore, all results are sharp.

Proof. The proof of case (i) is similar to the proof of case (i) in the Theorem 2.1 and so is omitted. The function $f_0 = h_0 + \overline{g_0}$, where

$$h_0(z) = z - \frac{Mz^3}{2(1-z)} \qquad \text{and} \qquad g_0(z) = -|b_1|z - 2bz^2 - \frac{Mz^3}{2(1-z)} \tag{2.9}$$

shows that the result is sharp. Indeed, in view of (2.9) and direct computation imply that

$$h_0'(z) = 1 - M\frac{3z^2 - 2z^3}{2(1-z)^2} \qquad \text{and} \qquad g_0'(z) = -|b_1| - 4bz - M\frac{3z^2 - 2z^3}{2(1-z)^2},$$

and so

$$J_{f_0}(r) = |h_0'(r)|^2 - |g_0'(r)|^2 = (1 + 4br)\left[1 - 4br - |b_1| - Mr^2\left(\frac{3-2r}{(1-r)^2}\right)\right],$$

which shows that $J_{f_0}(r_s) = 0$ and $J_{f_0}(r) < 0$ for $r > r_s$. Thus the proof of case (i) is complete.
Case (ii). For $r \in (0, 1)$ with $r \leq r_c$, it is sufficient to show that F_r is stable convex, where

$$F_r(z) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=1}^{\infty} b_n r^{n-1} z^n}. \tag{2.10}$$

In view of corollary 1.1, it is sufficient to show that

$$|b_1| + \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|)r^{n-1} \leq 1. \tag{2.11}$$

Considering condition (2.6), we have

$$\begin{aligned} |b_1| + \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|)r^{n-1} &\leq |b_1| + 8br_c + M\sum_{n=3}^{\infty} n^2 r_c^{n-1} \\ &= |b_1| + 8br_c + M\left[\frac{1+r_c}{(1-r_c)^3} - 1 - 4r_c\right] = 1, \end{aligned}$$

provided r_c is the root of the Equation (2.8) in the hypothesis of the theorem. Set

$$m(r) = M[1+r] - [1 + M - |b_1| + 4(M-2b)r](1-r)^3.$$

Then $m(0) = -(1-|b_1|)$ and $m(1) = 2M$ and so intermediate value theorem shows that the Equation (2.8) has a root in the interval $(0,1)$. To proof of sharpness, we define the function $f_1 = h_1 + \overline{g_1}$, where

$$h_1(z) = z - M\frac{z^3}{2(1-z)} \quad \text{and} \quad g_1(z) = -|b_1|z - 2bz^2 - M\frac{z^3}{2(1-z)}.$$

Then if we consider the function $F_1(z) = h_1(z) + g_1(z)$, direct calculation gives that $F_1'(r_c) + r_cF_1''(r_c) = 0$. In the other words we conclude that

$$\left[\mathrm{Re}\left(1 + \frac{zF_1''(z)}{F_1'(z)}\right)\right]_{|z=r_c} = 0.$$

The rest of proof is exactly the same as proof of sharpness part of case (ii) of Theorem 2.1 and we omit the details. ☐

There are two important constants, one is the radius of univalencs, while the other is the Bloch constant. Many authors have been studied these problems. (see [3]). By making use of Theorem 2.2 we conclude the following result.

Corollary 2.3. *Let $f = h + \overline{g} \in \mathcal{H}$, where h, g is given by* (1.1). *Let*

$$b_1 = 0,\ |a_2| + |b_2| = 2b, \quad 0 \leq b \leq \frac{2c}{\pi} \quad \text{and} \quad |f(z)| < c. \tag{2.12}$$

Then for f, the radius of univalence is r_0, where r_0 is the smallest root in $(0,1)$ of the equation

$$\frac{4c}{\pi} - \left[1 + \frac{4c}{\pi} + 4\left(\frac{2c}{\pi} - b\right)r\right](1-r)^2 = 0. \tag{2.13}$$

Furthermore, $f(\mathbb{D}_{r_0})$ contains a univalent disk of radius at least

$$R_0 = r_0 - 2b{r_0}^2 - \frac{4cr_0^3}{\pi(1-r_0)}, \tag{2.14}$$

where $\mathbb{D}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

Proof. According to [3] we can obtain the sharp estimates

$$|a_n| + |b_n| \le \frac{4c}{\pi}$$

for any $n \ge 3$. By Theorem 2.2 with $M = \frac{4c}{\pi}$, we conclude that the radius of univalence is r_0. Furthermore, for $|z| = r_0$, we have

$$\begin{aligned}
|f(z)| &= |z + \sum_{n=2}^{\infty}(a_n z^n + \overline{b_n z^n})| \\
&\ge |z| - |\sum_{n=2}^{\infty}(a_n z^n + \overline{b_n z^n})| \\
&\ge r_0 - \sum_{n=2}^{\infty}(|a_n| + |b_n|)r_0^n \\
&\ge r_0 - 2br_0^2 - \frac{4c}{\pi}\sum_{n=3}^{\infty} r_0^n \\
&= r_0 - 2br_0^2 - \frac{4cr_0^3}{\pi(1-r_0)} = R_0
\end{aligned}$$

and the proof is complete. □

Theorem 2.4. *Let $f = h + \overline{g} \in \mathcal{H}$, where h, g is given by* (1.1)*. Let*

$$b_1 = 0,\ |a_2| = a, |b_2| = b,\ |a_n| \le \frac{n+1}{2} \quad \text{and} \quad |b_n| \le \frac{n-1}{2} \quad \text{for} \quad n \ge 3, \tag{2.15}$$

where $0 \le a \le \frac{3}{2}$ and $0 \le b \le \frac{1}{2}$. Then for f,
(i) the radius of fully starlikeness is r_s, where $r_s = r_s(\alpha, b)$ is the smallest root in $(0,1)$ *of the equation*

$$(1+r) - \alpha(1-r)^2 = (1-r)^3\{r[-2(a+b) + \alpha(a-b) + 4 - \alpha] + 2(1-\alpha)\}. \tag{2.16}$$

(ii) the radius of fully convexity is r_c, where $r_c = r_c(\alpha, b)$ is the smallest root in $(0,1)$ *of the equation*

$$1+4r+r^2-\alpha(1-r)^2 = (1-r)^4\{r[(-4(a+b)+2\alpha(a-b)+8-2\alpha)]+2(1-\alpha)\}. \tag{2.17}$$

Furthermore, all results are sharp.

Proof. The proof is similar to the proof of Theorem 2.1. Let $r \in (0,1)$, it is sufficient to prove that $F_r(z)$ is fully starlike of order α, where $F_r(z)$ is given by (2.10). According to Theorem 1.1 and assumption (2.15), it is enough to show that

$$(2-\alpha)ar + (2+\alpha)br + \sum_{n=3}^{\infty}[n^2 - \alpha]r^{n-1} \le 1 - \alpha.$$

By making use of identities,

$$\sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{1+r}{(1-r)^3} \quad \text{and} \quad \sum_{n=1}^{\infty} r^{n-1} = \frac{1}{(1-r)},$$

the last inequality reduces to

$$(1+r)-\alpha(1-r)^2-(1-r)^3[r(-2(a+b)+\alpha(a-b)+4-\alpha)+2(1-\alpha)]\geq 0.$$

Set

$$m(r)=(1+r)-\alpha(1-r)^2-(1-r)^3[r(-2(a+b)+\alpha(a-b)+4-\alpha)+2(1-\alpha)].$$

Then $m(0)=-(1-\alpha)$ and $m(1)=2$ and so intermediate value theorem shows that the Equation (2.16) has a root in the interval $(0,1)$. Therefore, $F_r(z)$ is fully starlike of order α for $r\leq r_s$, where r_s is smallest root of equation (2.16) in $(0,1)$. To prove sharpness, we take $f_0=h_0+\overline{g_0}$, where

$$h_0(z)=2z-az^2+\frac{3}{2}z^2-\frac{2z-z^2}{2(1-z)^2}\quad\text{and}\quad g_0(z)=bz^2-\frac{1}{2}z^2+\frac{z^2}{2(1-z)^2}.$$

Direct computation leads to

$$h_0'(z)=2-2az+3z-\frac{1}{(1-z)^3}\quad,\qquad g_0'(z)=2bz-z+\frac{z}{(1-z)^3},$$

and

$$\frac{\partial}{\partial\theta}\left(\arg\left(f_0(re^{i\theta})\right)\right)|_{\theta=0}=\frac{rh_0'(r)-rg_0'(r)}{h_0(r)+g_0(r)}=\frac{2-2ar+4r-2br-\frac{1+r}{(1-r)^3}}{2-ar+r+br-\frac{1}{1-r}}. \tag{2.18}$$

Also, from equation (2.16) we have

$$\frac{[2-2ar+4r-2br](1-r)^3-(1+r)}{[2-ar+r+br](1-r)^3-(1-r)^2}=\alpha. \tag{2.19}$$

Thus in view of (2.18) and (2.19) we conclude that

$$\frac{\partial}{\partial\theta}\left(\arg\left(f_0(re^{i\theta})\right)\right)|_{\theta=0,r=r_s}=\alpha,$$

and this shows that the bound r_s is best possible. The proof of first part of case (ii) is the same as the case (i) and we omit the details. For the sharpness, we take $f_1=h_1+\overline{g_1}$, where

$$h_1(z)=2z-az^2+\frac{3}{2}z^2-\frac{2z-z^2}{2(1-z)^2}\quad\text{and}\quad g_1(z)=-bz^2+\frac{1}{2}z^2-\frac{z^2}{2(1-z)^2}.$$

Direct computation leads to

$$h_1'(z)=2-2az+3z-\frac{1}{(1-z)^3}\quad,\qquad g_1'(z)=-2bz+z-\frac{z}{(1-z)^3},$$

$$h_1''(z)=-2a+3-\frac{3}{(1-z)^4}\quad,\qquad g_1''(z)=-2b+1-\frac{1+2z}{(1-z)^4},$$

and

$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}f_1(re^{i\theta})\right)\right)|_{\theta=0} \tag{2.20}$$
$$= \frac{h_1'(r)+g_1'(r)+r(h_1''(r)+g_1''(r))}{h_1'(r)-g_1'(r)}$$
$$= \frac{2-4ar+8r-4br-\frac{1+4r+r^2}{(1-r)^4}}{2-2ar+2r+2br-\frac{1}{(1-r)^2}}.$$

Also, from equation (2.18) we have

$$\frac{[2-4ar+8r-4br](1-r)^4-(1+4r+r^2)}{[2-2ar+2r+2br](1-r)^4-(1-r)^2}=\alpha. \tag{2.21}$$

Thus in view of (2.20) and (2.21) we conclude that

$$\frac{\partial}{\partial\theta}(\arg(\frac{\partial}{\partial\theta}f_1(re^{i\theta})))|_{\theta=0,r=r_c}=\alpha,$$

and this shows that the bound r_c is best possible. Hence the proof is complete. □

Remark 2.5. By compering Theorems 2.1 and 2.3 one can observe that by putting $\alpha=0$ and $a+b=2c$ on the Theorem 2.3 we obtain the assumption of Theorem 2.1. Also, by putting $\alpha=0$ on the equations (2.16) and (2.17) we obtain the equations (2.2) and (2.3), respectively. So in this case the radius of stable starlikeness and stable convexity is the the same as radius of fully starlikeness and fully convexity of order zero, respectively.

Corollary 2.6. *Under the hypothesis of Theorem* 2.3 *for f, we have*
(i) the radius of stable starlikeness is r_s, where r_s is the smallest root in $(0,1)$ *of the equation*

$$(1+r)=(1-r)^3[r(-2(a+b)+4)+2]. \tag{2.22}$$

(ii) the radius of fully convexity is r_c, where r_c is the smallest root in $(0,1)$ *of the equation*

$$1+4r+r^2=(1-r)^4\{r[(-4(a+b)+8]+2\}. \tag{2.23}$$

Furthermore, all results are sharp.

Proof. The proof is similar to the proof of Theorem 2.3 and we omit the details. □

Theorem 2.7. *Let $f=h+\overline{g}\in\mathcal{H}$ is given by* (1.1) *and $|a_n|\le A_n$ and $|b_n|\le B_n$ for $n\ge 3$, where A_n and B_n is given by* (1.2). *Also, let $b_1=0, |a_2|=a$ and $|b_2|=b$, where $0\le a\le\frac{5}{2}$ and $0\le b\le\frac{1}{2}$. Then for f,*
(i) the radius of fully starlikeness is r_s, where $r_s=r_s(b)$ is the smallest root in $(0,1)$ *of the equation*

$$(1+r)^2-\alpha(1-r)^2=(1-r)^4\{r[-2(a+b)+\alpha(a-b)+2(3-\alpha)]+2(1-\alpha)\}, \tag{2.24}$$

(ii) the radius of fully convexity is r_c, where $r_c=r_c(b)$ is the smallest root in $(0,1)$ *of the equation*

$$(1+r)[1+6r+r^2-\alpha(1-r)^2]=(1-r)^5\{r[(-4(a+b)+2\alpha(a-b)+12-4\alpha)]+2(1-\alpha)\}, \tag{2.25}$$

Furthermore, all results are sharp.

Proof. Using the same argument of the proof of Theorem 2.3 we can obtain the equations (2.24) and (2.25). Only we prove the sharpness part of theorem. In the first case the results is sharp for the function $f_0 = h_0 + \overline{g_0}$ given by

$$h_0(z) = 2z - az^2 + \frac{5}{2}z^2 - \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \quad \text{and} \quad g_0(z) = bz^2 - \frac{1}{2}z^2 + \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}. \tag{2.26}$$

In view of (2.26), and computation shows that

$$\frac{\partial}{\partial\theta}(\arg(f_0(re^{i\theta})))|_{\theta=0} = \frac{rh_0'(r) - rg_0'(r)}{h_0(r) + g_0(r)} = \frac{2 - 2ar + 6r - 2br - \frac{(1+r)^2}{(1-r)^4}}{2 - ar + 2r + br - \frac{1}{(1-r)^2}}. \tag{2.27}$$

Also, from equation (2.24) we have

$$\frac{[2 - 2(a+b)r + 6r](1-r)^4 - (1+r)^2}{[2 - (a-b)r + 2r](1-r)^4 - (1-r)^2} = \alpha. \tag{2.28}$$

Thus in view of (2.27) and (2.28) we conclude that

$$\frac{\partial}{\partial\theta}(\arg(f_0(re^{i\theta})))|_{\theta=0, r=r_s} = \alpha,$$

and this shows that the bound r_s is best possible. Furthermore, in the second part the result is sharp for the function $f_1 = h_1 + \overline{g_1}$ given by

$$h_1(z) = 2z - az^2 + \frac{5}{2}z^2 - \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \quad \text{and} \quad g_1(z) = -bz^2 + \frac{1}{2}z^2 - \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}. \tag{2.29}$$

According to (2.29), direct calculation yields

$$\begin{aligned} &\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}f_1(re^{i\theta})\right)\right)|_{\theta=0} \qquad (2.30)\\ &= \frac{h_1'(r) + g_1'(r) + r(h_1''(r) + g_1''(r))}{h_1'(r) - g_1'(r)} \\ &= \frac{[2 - 4ar + 12r - 4br](1-r)^5 - (1+r)^2(1-r) - 6r - 8r^2 - 2r^3}{[2 - 2ar + 4r + 2br](1-r)^5 - (1+r)(1-r)^2}. \end{aligned}$$

Meanwhile, from equation (2.25) we have

$$\frac{[2 - 4ar + 12r - 4br](1-r)^5 - (1+r)^2(1-r) - 6r - 8r^2 - 2r^3}{[2 - 2ar + 4r + 2br](1-r)^5 - (1+r)(1-r)^2} = \alpha \tag{2.31}$$

Thus in view of (2.27) and (2.28) we conclude that

$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}f_1(re^{i\theta})\right)\right)|_{\theta=0, r=r_c} = \alpha,$$

and this shows that the bound r_c is best possible. Hence the proof is complete. □

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References

[1] Ali, R.M., Cho, N.E., Jain, N.K., Ravichandran, V., *Radii of starlikeness and convexity of functions defined by subordination with fixed second coefficients*, Filomat, **26**(2012), 553-561.

[2] Ali, R.M., Nagpal, S., Ravichandran, V., *Second-order differential subordinations for analytic functions with fixed initial coefficient*, Bull. Malays. Math. Sci. Soc., **34**(2011), no. 2, 611-629.

[3] Chen, Sh., Ponnusamy, S., Wang, X., *Bloch constants and Landau's theorems for planner p-harmonic mapping*, J. Math. Anal. Appl., **373**(2011), 102-110.

[4] Chuaqui, M., Duren, P.L., Osgood, B., *Curvature properties of planar harmonic mappings*, Comput. Methods Funct. Theory, **4**(2004), no. 1, 127-142.

[5] Clunie, J.G., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A., **9**(1984), 3-25.

[6] Gangadharan, A., Ravichandran, V., Shanmugam, T.N., *Radii of convexity and strong starlikeness for some classes of analytic functions*, Journal of Mathematical Analysis and Applications, **211**(1997), no. 1, 301-313.

[7] Hernandez, R., Martin, M.J., *Stable geometric properties of analytic and harmonic functions*, Math. Proc. of the Cambridge Philosophical Soc., **155**(2013), no. 2, 343-359.

[8] Jahangiri, J.M., *Coefficient bounds and univalence criteria for harmonic functions with negative coefficients*, Ann. Univ. Mariae Curie-Shlodowska, sect. A, **59**(1998), no. 2, 57-66.

[9] Jahangiri, J.M., *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl., **235**(1999), no. 2, 470-477.

[10] Kalaj, D., Ponnusamy, S., Vuorinen, M., *Radius of close-to-convexity of harmonic functions, Complex Var. Elliptic Equ*, **59**(2014), no. 4, 539-552.

[11] Lewy, H., *On the non-vanishing of the Jacobian in certain one-to-one mapping*, Bull. Amer. Math. Soc., **24**(10)(1936), 689-692.

[12] Long, B.Y., Hang, H., *Radii of harmonic mapping in the plane*, J. Aust. Math. Soc., (2016), 1-7.

[13] Ma, W., Minda, D., *Uniformly convex functions*, Ann. Polon. Math., **57**(1992), 165-175.

[14] Nagpal, S., Ravichandran, V., *Fully starlike and convex harmonic mappings of order α*, Ann. Polon. Math., **108**(2012), 1-7.

[15] Rajni, M., Nagpal, S., Ravichandran, V., *Radii of starlikeness and convexity for analytic functions with fixed second coefficient satisfying certain coefficient inequalities*, Kyungpook Mathematical Journal, **55**(2015), no. 2, 395-410.

[16] Ravichandran, V., *Radii of starlikeness and convexity of analytic functions satisfying certain coefficient inequalities*, Mathematica Slovaca, **64**(2014), no. 1, 27-38.

[17] Ravichandran, V., Khan, M.H., Silverman, H., Subramanian, K.G., *Radius problems for a class of analytic functions*, Demonstratio Math., **39**(2006), no. 1, 67-74.

[18] Ruscheweyh, St., Salinas, L., *On the preservation of direction-convexity and Goodman-Saff Conjecture*, Ann. Acad. Sci. Fenn. Ser. A I Math., **14**(1989), 63-73.

[19] Sheil-Small, T., *Constants for plannar harmonic mapping*, J. Lond. Math. Soc., **42**(1990), no. 2, 237-248.

[20] Wang, X.-T., Liang, X.-Q., Zhang, Y.-L., *Precise coefficient for close-to-convex harmonic univalent mapping*, J. Math. Anal. Appl., **263**(2001), no. 2, 501-509.

[21] Yamashita, S., *Radii of univalence, starlikeness and convexity*, Bull. Austral. Math. Soc., **25**(1982), 453-457.

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