

Ostrowski type fractional integral operators for generalized $(r; g, s, m, \varphi)$ -preinvex functions

Artion Kashuri and Rozana Liko

Abstract. In the present paper, the notion of generalized $(r; g, s, m, \varphi)$ -preinvex function is applied for establish some new generalizations of Ostrowski type inequalities via fractional integral operators. These results not only extend the results appeared in the literature ([38]) but also provide new estimates on these type. At the end, some applications to special means are given.

Mathematics Subject Classification (2010): 26A33, 26A51, 26D07, 26D10, 26D15.

Keywords: Ostrowski' type inequality, Hölder's inequality, Minkowski's inequality, power mean inequality, Riemann-Liouville fractional integral, fractional integral operator, s -convex function in the second sense, m -invex.

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I . For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K . $\mathbb{R}^n$ is used to denote a n -dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_0 = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following result is known in the literature as the Ostrowski inequality ([23]) and the references cited therein, which gives an upper bound for the approximation of the integral average $\frac{1}{b-a} \int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 1.1. *Let $f : I \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$, be a mapping differentiable in I° and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \quad \forall x \in [a, b]. \quad (1.1)$$

For other recent results concerning Ostrowski type inequalities ([3]-[25],[35]) and the references cited therein. Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and n -times differentiable mappings etc. appeared in a number of papers ([4]-[26],[28]-[39]). In recent years, one more dimension has been added to this studies, by introducing a number of integral inequalities involving various fractional operators like Riemann-Liouville, Erdelyi-Kober, Katugampola, conformable fractional integral operators etc. by many authors ([1]-[34]). Riemann-Liouville fractional integral operators are the most central between these fractional operators.

Fractional calculus ([24]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.2. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Ostrowski type inequalities for functions of different classes ([24]) and the references cited therein.

In ([32]), Raina introduced a class of functions defined formally by

$$\mathcal{F}_{\rho, \lambda}^\sigma(x) = \mathcal{F}_{\rho, \lambda}^{\sigma(0), \sigma(1), \dots}(x) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathbb{R}), \quad (1.2)$$

where the coefficients $(\sigma(k), k \in \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers. With the help of (1.2), Raina ([32]) and Agarwal et al. ([2]) defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$(\mathcal{J}_{\rho, \lambda, a+; w}^\sigma \varphi)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma[w(x-t)^\rho] \varphi(t) dt \quad (x > a > 0), \quad (1.3)$$

$$(\mathcal{J}_{\rho, \lambda, b-; w}^\sigma \varphi)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma[w(t-x)^\rho] \varphi(t) dt \quad (0 < x < b), \quad (1.4)$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exists. It is easy to verify that $\mathcal{J}_{\rho, \lambda, a+; w}^\sigma \varphi(x)$ and $\mathcal{J}_{\rho, \lambda, b-; w}^\sigma \varphi(x)$ are bounded integral operators on $L_1(a, b)$, if

$$\mathfrak{R} := \mathcal{F}_{\rho, \lambda+1}^\sigma[w(b-a)^\rho] < \infty.$$

In fact, for $\varphi \in L_1(a, b)$, we have

$$\|\mathcal{J}_{\rho, \lambda, a+; w}^\sigma \varphi(x)\|_1 \leq \mathfrak{R}(b-a)^\lambda \|\varphi\|_1$$

and

$$\|\mathcal{J}_{\rho, \lambda, b-; w}^\sigma \varphi(x)\|_1 \leq \mathfrak{R}(b-a)^\lambda \|\varphi\|_1$$

where

$$\|\varphi\|_p := \left(\int_a^b |\varphi(t)|^p dt \right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals J_{a+}^α and J_{b-}^α of order α follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (1.3) and (1.4).

Definition 1.3. ([16]) A function $f : \mathbb{R}_o \rightarrow \mathbb{R}$ is said to be s -convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \quad (1.5)$$

for all $x, y \geq 0$, $\lambda \in [0, 1]$ and $s \in (0, 1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_o$ as usual. The s -convex functions in the second sense have been investigated in ([16]).

Definition 1.4. ([5]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \rightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true ([5], [37]).

Definition 1.5. ([29]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η , if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The aim of this paper is to establish some generalizations of Ostrowski type inequalities using new identity given in Section 2 for generalized $(r; g, s, m, \varphi)$ -preinvex functions via generalized fractional integral operators. In Section 3, some applications to special means are given. In Section 4, some conclusions and future research are given. These results not only extend the results appeared in the literature ([38]) but also provide new estimates on these type.

2. Main Results

Definition 2.1. ([15]) A set $K \subseteq \mathbb{R}$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m -invex set degenerates an invex set on K .

We next give new definition, to be referred as generalized $(r; g, s, m, \varphi)$ -preinvex function.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m -invex set with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$, $g : [0, 1] \rightarrow [0, 1]$ be a differentiable function and $\varphi : I \rightarrow K$ is a continuous function. The function $f : K \rightarrow (0, +\infty)$ is said to be generalized $(r; g, s, m, \varphi)$ -preinvex with respect to η , if

$$f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \leq M_r(f(\varphi(x)), f(\varphi(y)), m, s; g(t)) \quad (2.1)$$

holds for some fixed $s, m \in (0, 1]$ and for all $x, y \in I, t \in [0, 1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m, s; g(t)) = \begin{cases} [m(1 - g(t))^s f^r(\varphi(x)) + g^s(t) f^r(\varphi(y))]^{\frac{1}{r}}, & r \neq 0; \\ [f(\varphi(x))]^{m(1 - g(t))^s} [f(\varphi(y))]^{g^s(t)}, & r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2.4. In Definition 2.3, it is worthwhile to note that the class of generalized $(r; g, s, m, \varphi)$ -preinvex function is a generalization of the class of s -convex in the second sense function given in Definition 1.3. For $g(t) = t$, we get the notion of generalized $(r; s, m, \varphi)$ -preinvex function ([18]). Also, for $r = 1, g(t) = t$ and $\varphi(x) = x, \forall x \in I$, we get the notion of generalized (s, m) -preinvex function ([15]).

Throughout this paper we denote $\alpha^* = \frac{x - m\varphi(a)}{\eta(\varphi(b), \varphi(a), m)}$ and

$$\begin{aligned} & I_{f, g, \eta, \varphi}(x; \lambda, \rho, w, m, a, b) \\ &= g^\lambda(\alpha^*) \mathcal{F}_{\rho, \lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)g^\rho(\alpha^*)] \\ & \quad \times \frac{f(m\varphi(a) + g(\alpha^*)\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\ & - g^\lambda(0) \mathcal{F}_{\rho, \lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)g^\rho(0)] \frac{f(m\varphi(a) + g(0)\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\ & \quad + (1 - g(1))^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)(1 - g(1))^\rho] \\ & \quad \times \frac{f(m\varphi(a) + g(1)\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\ & - (1 - g(\alpha^*))^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)(1 - g(\alpha^*))^\rho] \end{aligned}$$

$$\begin{aligned}
& \times \frac{f(m\varphi(a) + g(\alpha^*)\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\
& - \frac{1}{\eta^{\lambda+1}(\varphi(b), \varphi(a), m)} \\
& \times \left\{ \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(\alpha^*)\eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^{\lambda-1} \mathcal{F}_{\rho, \lambda}^{\sigma}[w(x - m\varphi(a))^{\rho}] f(x) dx \right. \\
& + \int_{m\varphi(a)+g(\alpha^*)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^{\lambda-1} \\
& \left. \times \mathcal{F}_{\rho, \lambda}^{\sigma}[w(m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^{\rho}] f(x) dx \right\}. \quad (2.2)
\end{aligned}$$

In this section, in order to prove our main results regarding some generalizations of Ostrowski type inequalities for generalized $(r; g, s, m, \varphi)$ -preinvex functions via generalized fractional integral operators, we need the following new lemma:

Lemma 2.5. *Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ be a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ and let $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : K \rightarrow \mathbb{R}$ is a differentiable function on K° . If $f' \in L_1[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$, then we have the following identity involving generalized fractional integral operators:*

$$I_{f, g, \eta, \varphi}(x; \lambda, \rho, w, m, a, b) = \int_0^1 \theta_g(t) f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \quad (2.3)$$

for each $t \in [0, 1]$, where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and

$$\theta_g(t) = \begin{cases} g^\lambda(t) \mathcal{F}_{\rho, \lambda+1}^{\sigma}[w\eta^\rho(\varphi(b), \varphi(a), m)g^\rho(t)], & t \in [0, \alpha^*]; \\ (1 - g(t))^\lambda \mathcal{F}_{\rho, \lambda+1}^{\sigma}[w\eta^\rho(\varphi(b), \varphi(a), m)(1 - g(t))^\rho], & t \in [\alpha^*, 1]. \end{cases}$$

Proof. Integrating by parts, we get

$$\begin{aligned}
& \int_0^1 \theta_g(t) f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
& = \int_0^{\alpha^*} g^\lambda(t) \mathcal{F}_{\rho, \lambda+1}^{\sigma}[w\eta^\rho(\varphi(b), \varphi(a), m)g^\rho(t)] \\
& \quad \times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
& + \int_{\alpha^*}^1 (1 - g(t))^\lambda \mathcal{F}_{\rho, \lambda+1}^{\sigma}[w\eta^\rho(\varphi(b), \varphi(a), m)(1 - g(t))^\rho] \\
& \quad \times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
& = t^\lambda \mathcal{F}_{\rho, \lambda+1}^{\sigma}[w\eta^\rho(\varphi(b), \varphi(a), m)t^\rho] \frac{f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \Bigg|_{g(0)}^{g(\alpha^*)}
\end{aligned}$$

$$\begin{aligned}
& -\lambda \int_{g(0)}^{g(\alpha^*)} t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)t^\rho] \frac{f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} dt \\
& + (1-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)(1-t)^\rho] \frac{f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \Big|_{g(\alpha^*)}^{g(1)} \\
& - \lambda \int_{g(\alpha^*)}^{g(1)} (1-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)(1-t)^\rho] \\
& \quad \times \frac{f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} dt \\
& = \lambda g^\lambda(\alpha^*) \mathcal{F}_{\rho,\lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)g^\rho(\alpha^*)] \frac{f(m\varphi(a) + g(\alpha^*)\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\
& - \lambda g^\lambda(0) \mathcal{F}_{\rho,\lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)g^\rho(0)] \frac{f(m\varphi(a) + g(0)\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\
& \quad + \lambda(1-g(1))^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)(1-g(1))^\rho] \\
& \quad \times \frac{f(m\varphi(a) + g(1)\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \times \\
& - \lambda(1-g(\alpha^*))^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [w\eta^\rho(\varphi(b), \varphi(a), m)(1-g(\alpha^*))^\rho] \\
& \quad \times \frac{f(m\varphi(a) + g(\alpha^*)\eta(\varphi(b), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\
& \quad \times \frac{\lambda}{\eta^{\lambda+1}(\varphi(b), \varphi(a), m)} \\
& \times \left\{ \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(\alpha^*)\eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [w(x - m\varphi(a))^\rho] f(x) dx \right. \\
& \quad + \int_{m\varphi(a)+g(\alpha^*)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^{\lambda-1} \\
& \quad \times \mathcal{F}_{\rho,\lambda}^\sigma [w(m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^\rho] f(x) dx \Big\}. \quad \square
\end{aligned}$$

Remark 2.6. Under the same conditions as in Lemma 2.5 for $g(t) = t$, we get

$$\begin{aligned}
& I_{f,\eta,\varphi}(x; \lambda, \rho, w, m, a, b) \\
& = \left[\frac{(x - m\varphi(a))^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [w(x - m\varphi(a))^\rho]}{\eta^{\lambda+1}(\varphi(b), \varphi(a), m)} \right. \\
& \quad + \frac{(m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [w(m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^\rho]}{\eta^{\lambda+1}(\varphi(b), \varphi(a), m)} \Big] f(x) \\
& \quad \times \frac{\lambda}{\eta^{\lambda+1}(\varphi(b), \varphi(a), m)} \\
& \times \left[(\mathcal{J}_{\rho,\lambda,x-;w}^\sigma f)(m\varphi(a)) + (\mathcal{J}_{\rho,\lambda,x+;w}^\sigma f)(m\varphi(a) + \eta(\varphi(b), \varphi(a), m)) \right]. \quad (2.4)
\end{aligned}$$

By using Lemma 2.5, one can extend to the following results.

Theorem 2.7. *Let $\varphi : I \longrightarrow A$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ a differentiable function. Suppose $A \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in (0, 1]$ and let $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : A \longrightarrow (0, +\infty)$ is a differentiable function on A° . If $0 < r \leq 1$ and f' is a generalized $(r; g, s, m, \varphi)$ -preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}
 |I_{f,g,\eta,\varphi}(x; \lambda, \rho, w, m, a, b)| &\leq \mathcal{F}_{\rho,\lambda+1}^\sigma [|w|\eta^\rho(\varphi(b), \varphi(a), m)] \\
 &\times \left\{ m f'(\varphi(a))^r B_{g(\alpha^*)}^r \left(\lambda + \rho k + 1, \frac{s}{r} + 1 \right) \right. \\
 &\left. + f'(\varphi(b))^r \left(\frac{g^{\lambda+\rho k+\frac{s}{r}+1}(\alpha^*) - g^{\lambda+\rho k+\frac{s}{r}+1}(0)}{\lambda + \rho k + \frac{s}{r} + 1} \right)^r \right\}^{\frac{1}{r}} \\
 &+ \left\{ m f'(\varphi(a))^r \left(\frac{(1 - g(\alpha^*))^{\lambda+\rho k+\frac{s}{r}+1} - (1 - g(1))^{\lambda+\rho k+\frac{s}{r}+1}}{\lambda + \rho k + \frac{s}{r} + 1} \right)^r \right. \\
 &\left. + f'(\varphi(b))^r \left[B_{g(1)} \left(\frac{s}{r} + 1, \lambda + \rho k + 1 \right) - B_{g(\alpha^*)} \left(\frac{s}{r} + 1, \lambda + \rho k + 1 \right) \right]^r \right\}^{\frac{1}{r}}, \quad (2.5)
 \end{aligned}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$, $k = 0, 1, 2, \dots$, and

$$B_{g(x)}(a, b) = \int_{g(0)}^{g(x)} t^{a-1} (1-t)^{b-1} dt.$$

Proof. From Lemma 2.5, generalized $(r; g, s, m, \varphi)$ -preinvexity of f' , Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
 &|I_{f,g,\eta,\varphi}(x; \lambda, \rho, w, m, a, b)| \\
 &\leq \int_0^{\alpha^*} g^\lambda(t) \mathcal{F}_{\rho,\lambda+1}^\sigma [|w|\eta^\rho(\varphi(b), \varphi(a), m) g^\rho(t)] \\
 &\quad \times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
 &+ \int_{\alpha^*}^1 |1 - g(t)|^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [|w|\eta^\rho(\varphi(b), \varphi(a), m) |1 - g(t)|^\rho] \\
 &\quad \times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
 &\leq \int_0^{\alpha^*} g^\lambda(t) \mathcal{F}_{\rho,\lambda+1}^\sigma [|w|\eta^\rho(\varphi(b), \varphi(a), m) g^\rho(t)] \\
 &\quad \times [m(1 - g(t))^s f'(\varphi(a))^r + g^s(t) f'(\varphi(b))^r]^{\frac{1}{r}} d[g(t)] \\
 &+ \int_{\alpha^*}^1 (1 - g(t))^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [|w|\eta^\rho(\varphi(b), \varphi(a), m) (1 - g(t))^\rho] \\
 &\quad \times [m(1 - g(t))^s f'(\varphi(a))^r + g^s(t) f'(\varphi(b))^r]^{\frac{1}{r}} d[g(t)]
 \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ m f'(\varphi(a))^r \left(\int_0^{\alpha^*} g^\lambda(t) (1-g(t))^{\frac{s}{r}} \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) g^\rho(t)] d[g(t)] \right)^r \right. \\
&\quad \left. + f'(\varphi(b))^r \left(\int_0^{\alpha^*} g^{\lambda+\frac{s}{r}}(t) \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) g^\rho(t)] d[g(t)] \right)^r \right\}^{\frac{1}{r}} \\
&\quad + \left\{ m f'(\varphi(a))^r \left(\int_{\alpha^*}^1 (1-g(t))^{\lambda+\frac{s}{r}} \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) (1-g(t))^\rho] d[g(t)] \right)^r \right. \\
&\quad \left. + f'(\varphi(b))^r \left(\int_{\alpha^*}^1 g^{\frac{s}{r}}(t) (1-g(t))^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) (1-g(t))^\rho] d[g(t)] \right)^r \right\}^{\frac{1}{r}} \\
&\quad = \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m)] \\
&\quad \times \left[\left\{ m f'(\varphi(a))^r B_{g(\alpha^*)}^r \left(\lambda + \rho k + 1, \frac{s}{r} + 1 \right) \right. \right. \\
&\quad \left. \left. + f'(\varphi(b))^r \left(\frac{g^{\lambda+\rho k+\frac{s}{r}+1}(\alpha^*) - g^{\lambda+\rho k+\frac{s}{r}+1}(0)}{\lambda + \rho k + \frac{s}{r} + 1} \right)^r \right\}^{\frac{1}{r}} \right. \\
&\quad \left. + \left\{ m f'(\varphi(a))^r \left(\frac{(1-g(\alpha^*))^{\lambda+\rho k+\frac{s}{r}+1} - (1-g(1))^{\lambda+\rho k+\frac{s}{r}+1}}{\lambda + \rho k + \frac{s}{r} + 1} \right)^r \right. \right. \\
&\quad \left. \left. + f'(\varphi(b))^r \left[B_{g(1)} \left(\frac{s}{r} + 1, \lambda + \rho k + 1 \right) - B_{g(\alpha^*)} \left(\frac{s}{r} + 1, \lambda + \rho k + 1 \right) \right]^r \right\}^{\frac{1}{r}} \right]. \quad \square
\end{aligned}$$

Corollary 2.8. *Under the same conditions as in Theorem 2.7, if we choose $m = s = 1$, $\eta(\varphi(b), \varphi(a), m) = \varphi(b) - m\varphi(a)$, $g(t) = t$ and $\varphi(x) = x$, we get*

$$\begin{aligned}
&\left| \left[\frac{(x-a)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [w(x-a)^\rho] + (b-x)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [w(b-x)^\rho]}{(b-a)^{\lambda+1}} \right] f(x) \right. \\
&\quad \left. - \frac{\lambda}{(b-a)^{\lambda+1}} \left[(\mathcal{J}_{\rho, \lambda, x-; w}^\sigma f)(a) + (\mathcal{J}_{\rho, \lambda, x+; w}^\sigma f)(b) \right] \right| \\
&\leq \left\{ f'(a)^r \left(\mathcal{F}_{\rho, \lambda+1}^{\sigma_1^*} [|w|(b-a)^\rho] \right)^r + f'(b)^r \left(\mathcal{F}_{\rho, \lambda+1}^{\sigma_2^*} [|w|(x-a)^\rho] \right)^r \right\}^{\frac{1}{r}} \\
&\quad + \left\{ f'(a)^r \left(\mathcal{F}_{\rho, \lambda+1}^{\sigma_3^*} [|w|(b-x)^\rho] \right)^r + f'(b)^r \left(\mathcal{F}_{\rho, \lambda+1}^{\sigma_4^*} [|w|(b-a)^\rho] \right)^r \right\}^{\frac{1}{r}}, \quad (2.6)
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1^*(k) &= \sigma(k) \beta \left(\frac{x-a}{b-a}; \lambda + \rho k + 1, \frac{1}{r} + 1 \right); \\
\sigma_2^*(k) &= \sigma(k) \left(\frac{x-a}{b-a} \right)^{\lambda+\frac{1}{r}+1} \frac{1}{\lambda + \rho k + \frac{1}{r} + 1};
\end{aligned}$$

$$\sigma_3^*(k) = \sigma(k) \left(\frac{b-x}{b-a} \right)^{\lambda + \frac{1}{r} + 1} \frac{1}{\lambda + \rho k + \frac{1}{r} + 1};$$

$$\sigma_4^*(k) = \sigma(k) \beta \left(\frac{b-x}{b-a}; \lambda + \rho k + 1, \frac{1}{r} + 1 \right).$$

Corollary 2.9. *If we choose $r = \sigma(0) = 1, w = 0$ in Corollary 2.8, the inequality (2.6) reduces to inequality (2.1) ([38] Theorem 2.1).*

Theorem 2.10. *Let $\varphi : I \longrightarrow A$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ a differentiable function. Suppose $A \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in (0, 1]$ and let $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : A \longrightarrow (0, +\infty)$ is a differentiable function on A° . If $0 < r \leq 1$ and f'^q is a generalized $(r; g, s, m, \varphi)$ -preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then the following inequality for generalized fractional integral operators holds:*

$$|I_{f,g,\eta,\varphi}(x; \lambda, \rho, w, m, a, b)| \leq \left(\frac{r}{s+r} \right)^{\frac{1}{q}} \mathcal{F}_{\rho, \lambda+1}^{\sigma^*} [|w| \eta^\rho(\varphi(b), \varphi(a), m)]$$

$$\times \left\{ \left(g^{(\lambda+\rho k)p+1}(\alpha^*) - g^{(\lambda+\rho k)p+1}(0) \right)^{\frac{1}{p}} \right.$$

$$\times \left[m \left((1 - g(0))^{\frac{s}{r}+1} - (1 - g(\alpha^*))^{\frac{s}{r}+1} \right)^r f'(\varphi(a))^{rq} \right.$$

$$\left. + \left(g^{\frac{s}{r}+1}(\alpha^*) - g^{\frac{s}{r}+1}(0) \right)^r f'(\varphi(b))^{rq} \right]^{\frac{1}{rq}}$$

$$+ \left((1 - g(\alpha^*))^{(\lambda+\rho k)p+1} - (1 - g(1))^{(\lambda+\rho k)p+1} \right)^{\frac{1}{p}}$$

$$\times \left[m \left((1 - g(\alpha^*))^{\frac{s}{r}+1} - (1 - g(1))^{\frac{s}{r}+1} \right)^r f'(\varphi(a))^{rq} \right.$$

$$\left. + \left(g^{\frac{s}{r}+1}(1) - g^{\frac{s}{r}+1}(\alpha^*) \right)^r f'(\varphi(b))^{rq} \right]^{\frac{1}{rq}} \Bigg\}, \quad (2.7)$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$, $k = 0, 1, 2, \dots$, and

$$\sigma^*(k) = \sigma(k) \left(\frac{1}{(\lambda + \rho k)p + 1} \right)^{\frac{1}{p}}.$$

Proof. From Lemma 2.5, generalized $(r; g, s, m, \varphi)$ -preinvexity of f'^q , Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$|I_{f,g,\eta,\varphi}(x; \lambda, \rho, w, m, a, b)|$$

$$\leq \int_0^{\alpha^*} g^\lambda(t) \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) g^\rho(t)]$$

$$\times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)]$$

$$+ \int_{\alpha^*}^1 |1 - g(t)|^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m)] |1 - g(t)|^\rho$$

$$\times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)]$$

$$\begin{aligned}
& \leq \sum_{k=0}^{+\infty} \frac{\sigma(k)|w|^k \eta^{\rho k}(\varphi(b), \varphi(a), m)}{\Gamma(\lambda + \rho k + 1)} \times \left\{ \left(\int_0^{\alpha^*} g^{(\lambda + \rho k)p}(t) d[g(t)] \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\int_0^{\alpha^*} f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))^q d[g(t)] \right)^{\frac{1}{q}} \\
& \quad \left. + \left(\int_{\alpha^*}^1 (1 - g(t))^{(\lambda + \rho k)p} dt \right)^{\frac{1}{p}} \right. \\
& \quad \left. \times \left(\int_{\alpha^*}^1 f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))^q d[g(t)] \right)^{\frac{1}{q}} \right\} \\
& \leq \sum_{k=0}^{+\infty} \frac{\sigma(k)|w|^k \eta^{\rho k}(\varphi(b), \varphi(a), m)}{\Gamma(\lambda + \rho k + 1)} \times \left\{ \left(\int_0^{\alpha^*} g^{(\lambda + \rho k)p}(t) d[g(t)] \right)^{\frac{1}{p}} \right. \\
& \quad \times \left(\int_0^{\alpha^*} [m(1 - g(t))^s f'(\varphi(a))^{rq} + g^s(t) f'(\varphi(b))^{rq}]^{\frac{1}{r}} d[g(t)] \right)^{\frac{1}{q}} \\
& \quad \left. + \left(\int_{\alpha^*}^1 (1 - g(t))^{(\lambda + \rho k)p} d[g(t)] \right)^{\frac{1}{p}} \right. \\
& \quad \left. \times \left(\int_{\alpha^*}^1 [m(1 - g(t))^s f'(\varphi(a))^{rq} + g^s(t) f'(\varphi(b))^{rq}]^{\frac{1}{r}} d[g(t)] \right)^{\frac{1}{q}} \right\} \\
& \leq \sum_{k=0}^{+\infty} \frac{\sigma(k)|w|^k \eta^{\rho k}(\varphi(b), \varphi(a), m)}{\Gamma(\lambda + \rho k + 1)} \times \left\{ \left(\int_0^{\alpha^*} g^{(\lambda + \rho k)p}(t) d[g(t)] \right)^{\frac{1}{p}} \right. \\
& \quad \times \left[m f'(\varphi(a))^{rq} \left(\int_0^{\alpha^*} (1 - g(t))^{\frac{s}{r}} d[g(t)] \right)^r + f'(\varphi(b))^{rq} \left(\int_0^{\alpha^*} g^{\frac{s}{r}}(t) d[g(t)] \right)^r \right]^{\frac{1}{rq}} \\
& \quad \left. + \left(\int_{\alpha^*}^1 (1 - g(t))^{(\lambda + \rho k)p} d[g(t)] \right)^{\frac{1}{p}} \right. \\
& \quad \left. \times \left[m f'(\varphi(a))^{rq} \left(\int_{\alpha^*}^1 (1 - g(t))^{\frac{s}{r}} d[g(t)] \right)^r + f'(\varphi(b))^{rq} \left(\int_{\alpha^*}^1 g^{\frac{s}{r}}(t) d[g(t)] \right)^r \right]^{\frac{1}{rq}} \right\} \\
& = \left(\frac{r}{s + r} \right)^{\frac{1}{q}} \mathcal{F}_{\rho, \lambda + 1}^{\sigma^*} [|w| \eta^{\rho}(\varphi(b), \varphi(a), m)] \\
& \quad \times \left\{ \left(g^{(\lambda + \rho k)p + 1}(\alpha^*) - g^{(\lambda + \rho k)p + 1}(0) \right)^{\frac{1}{p}} \right. \\
& \quad \times \left[m \left((1 - g(0))^{\frac{s}{r} + 1} - (1 - g(\alpha^*))^{\frac{s}{r} + 1} \right)^r f'(\varphi(a))^{rq} \right. \\
& \quad \left. \left. + \left(g^{\frac{s}{r} + 1}(\alpha^*) - g^{\frac{s}{r} + 1}(0) \right)^r f'(\varphi(b))^{rq} \right]^{\frac{1}{rq}} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left((1 - g(\alpha^*))^{(\lambda + \rho k)p + 1} - (1 - g(1))^{(\lambda + \rho k)p + 1} \right)^{\frac{1}{p}} \\
& \times \left[m \left((1 - g(\alpha^*))^{\frac{s}{r} + 1} - (1 - g(1))^{\frac{s}{r} + 1} \right)^r f'(\varphi(a))^{rq} \right. \\
& \quad \left. + \left(g^{\frac{s}{r} + 1}(1) - g^{\frac{s}{r} + 1}(\alpha^*) \right)^r f'(\varphi(b))^{rq} \right]^{\frac{1}{rq}} \Bigg\}. \quad \square
\end{aligned}$$

Corollary 2.11. *Under the same conditions as in Theorem 2.10, if we choose $m = s = 1$, $\eta(\varphi(b), \varphi(a), m) = \varphi(b) - m\varphi(a)$, $g(t) = t$ and $\varphi(x) = x$, we get*

$$\begin{aligned}
& \left| \left[\frac{(x - a)^\lambda \mathcal{F}_{\rho, \lambda + 1}^\sigma [w(x - a)^\rho] + (b - x)^\lambda \mathcal{F}_{\rho, \lambda + 1}^\sigma [w(b - x)^\rho]}{(b - a)^{\lambda + 1}} \right] f(x) \right. \\
& \quad \left. - \frac{\lambda}{(b - a)^{\lambda + 1}} \left[(\mathcal{J}_{\rho, \lambda, x-; w}^\sigma f)(a) + (\mathcal{J}_{\rho, \lambda, x+; w}^\sigma f)(b) \right] \right| \\
& \leq \left(\frac{r}{r + 1} \right)^{\frac{1}{q}} \frac{1}{(b - a)^{\lambda + \frac{2}{rq} + \frac{1}{p}}} \\
& \times \left\{ \left[[(b - a)^{\frac{1}{r} + 1} - (b - x)^{\frac{1}{r} + 1}]^r f'(a)^{rq} + (x - a)^2 f'(b)^{rq} \right]^{\frac{1}{rq}} \right. \\
& \quad \times (x - a)^{\lambda + \frac{1}{p}} \mathcal{F}_{\rho, \lambda + 1}^{\sigma*} [|w|(x - a)^\rho] \\
& \quad + [(b - x)^2 f'(a)^{rq} + [(b - a)^{\frac{1}{r} + 1} - (x - a)^{\frac{1}{r} + 1}]^r f'(b)^{rq}]^{\frac{1}{rq}} \\
& \quad \left. \times (b - x)^{\lambda + \frac{1}{p}} \mathcal{F}_{\rho, \lambda + 1}^{\sigma*} [|w|(b - x)^\rho] \right\}. \quad (2.8)
\end{aligned}$$

Theorem 2.12. *Let $\varphi : I \longrightarrow A$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ a differentiable function. Suppose $A \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in (0, 1]$ and let $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : A \longrightarrow (0, +\infty)$ is a differentiable function on A° . If $0 < r \leq 1$ and f'^q is a generalized $(r; g, s, m, \varphi)$ -preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$, $q \geq 1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}
& |I_{f, g, \eta, \varphi}(x; \lambda, \rho, w, m, a, b)| \leq \mathcal{F}_{\rho, \lambda + 1}^{\sigma**} [|w|\eta^\rho(\varphi(b), \varphi(a), m)] \\
& \times \left\{ [g^{\rho k + 1}(\alpha^*) - g^{\rho k + 1}(0)]^{1 - \frac{1}{q}} \right. \\
& \times \left[m f'(\varphi(a))^{rq} B_{g(\alpha^*)}^r \left(\lambda q + \rho k + 1, \frac{s}{r} + 1 \right) \right. \\
& \quad \left. + f'(\varphi(b))^{rq} \left(\frac{g^{\lambda q + \rho k + \frac{s}{r} + 1}(\alpha^*) - g^{\lambda q + \rho k + \frac{s}{r} + 1}(0)}{\lambda q + \rho k + \frac{s}{r} + 1} \right)^r \right]^{\frac{1}{rq}} \\
& \quad \left. + [(1 - g(\alpha^*))^{\rho k + 1} - (1 - g(1))^{\rho k + 1}]^{1 - \frac{1}{q}} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left[m f'(\varphi(a))^{rq} \left(\frac{(1 - g(\alpha^*))^{\lambda q + \rho k + \frac{s}{r} + 1} - (1 - g(1))^{\lambda q + \rho k + \frac{s}{r} + 1}}{\lambda q + \rho k + \frac{s}{r} + 1} \right)^r \right. \\
& \quad + f'(\varphi(b))^{rq} \left[B_{g(1)} \left(\frac{s}{r} + 1, \lambda q + \rho k + 1 \right) \right. \\
& \quad \quad \left. \left. - B_{g(\alpha^*)} \left(\frac{s}{r} + 1, \lambda q + \rho k + 1 \right) \right]^r \right]^{\frac{1}{rq}}, \tag{2.9}
\end{aligned}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$, $k = 0, 1, 2, \dots$, and

$$\sigma^{**}(k) = \sigma(k) \left(\frac{1}{\rho k + 1} \right)^{1 - \frac{1}{q}}.$$

Proof. From Lemma 2.5, generalized $(r; g, s, m, \varphi)$ -preinvexity of f'^q , the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
& |I_{f, g, \eta, \varphi}(x; \lambda, \rho, w, m, a, b)| \\
& \leq \int_0^{\alpha^*} g^\lambda(t) \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) g^\rho(t)] \\
& \quad \times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
& + \int_{\alpha^*}^1 |1 - g(t)|^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) |1 - g(t)|^\rho] \\
& \quad \times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
& \leq \left(\int_0^{\alpha^*} \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) g^\rho(t)] d[g(t)] \right)^{1 - \frac{1}{q}} \\
& \quad \times \left(\int_0^{\alpha^*} g^{\lambda q}(t) \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) g^\rho(t)] \right. \\
& \quad \quad \left. \times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))^q d[g(t)] \right)^{\frac{1}{q}} \\
& + \left(\int_{\alpha^*}^1 \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) (1 - g(t))^\rho] d[g(t)] \right)^{1 - \frac{1}{q}} \\
& \quad \times \left(\int_{\alpha^*}^1 (1 - g(t))^{\lambda q} \mathcal{F}_{\rho, \lambda+1}^\sigma [|w| \eta^\rho(\varphi(b), \varphi(a), m) (1 - g(t))^\rho] \right. \\
& \quad \quad \left. \times f'(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))^q d[g(t)] \right)^{\frac{1}{q}} \\
& \leq \left(\sum_{k=0}^{+\infty} \frac{\sigma(k) |w|^k \eta^{\rho k}(\varphi(b), \varphi(a), m)}{\Gamma(\lambda + \rho k + 1)} \int_0^{\alpha^*} g^{\rho k}(t) d[g(t)] \right)^{1 - \frac{1}{q}} \\
& \quad \times \left[\sum_{k=0}^{+\infty} \frac{\sigma(k) |w|^k \eta^{\rho k}(\varphi(b), \varphi(a), m)}{\Gamma(\lambda + \rho k + 1)} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^{\alpha^*} g^{\lambda q + \rho k}(t) [m(1 - g(t))^s f'(\varphi(a))^{rq} + g^s(t) f'(\varphi(b))^{rq}]^{\frac{1}{r}} d[g(t)] \Bigg]^{\frac{1}{q}} \\
& + \left(\sum_{k=0}^{+\infty} \frac{\sigma(k) |w|^k \eta^{\rho k}(\varphi(b), \varphi(a), m)}{\Gamma(\lambda + \rho k + 1)} \int_{\alpha^*}^1 (1 - g(t))^{\rho k} d[g(t)] \right)^{1 - \frac{1}{q}} \\
& \quad \times \left[\sum_{k=0}^{+\infty} \frac{\sigma(k) |w|^k \eta^{\rho k}(\varphi(b), \varphi(a), m)}{\Gamma(\lambda + \rho k + 1)} \right. \\
& \times \int_{\alpha^*}^1 (1 - g(t))^{\lambda q + \rho k} [m(1 - g(t))^s f'(\varphi(a))^{rq} + g^s(t) f'(\varphi(b))^{rq}]^{\frac{1}{r}} d[g(t)] \Bigg]^{\frac{1}{q}} \\
& \leq \mathcal{F}_{\rho, \lambda+1}^{\sigma^{**}} [|w| \eta^{\rho}(\varphi(b), \varphi(a), m)] \times \left\{ [g^{\rho k+1}(\alpha^*) - g^{\rho k+1}(0)]^{1 - \frac{1}{q}} \right. \\
& \quad \times \left[m f'(\varphi(a))^{rq} B_{g(\alpha^*)}^r \left(\lambda q + \rho k + 1, \frac{s}{r} + 1 \right) \right. \\
& \quad \left. + f'(\varphi(b))^{rq} \left(\frac{g^{\lambda q + \rho k + \frac{s}{r} + 1}(\alpha^*) - g^{\lambda q + \rho k + \frac{s}{r} + 1}(0)}{\lambda q + \rho k + \frac{s}{r} + 1} \right)^r \right]^{\frac{1}{rq}} \\
& \quad + [(1 - g(\alpha^*))^{\rho k+1} - (1 - g(1))^{\rho k+1}]^{1 - \frac{1}{q}} \\
& \quad + \left[m f'(\varphi(a))^{rq} \left(\frac{(1 - g(\alpha^*))^{\lambda q + \rho k + \frac{s}{r} + 1} - (1 - g(1))^{\lambda q + \rho k + \frac{s}{r} + 1}}{\lambda q + \rho k + \frac{s}{r} + 1} \right)^r \right. \\
& \quad + f'(\varphi(b))^{rq} \left[B_{g(1)} \left(\frac{s}{r} + 1, \lambda q + \rho k + 1 \right) \right. \\
& \quad \left. \left. - B_{g(\alpha^*)} \left(\frac{s}{r} + 1, \lambda q + \rho k + 1 \right) \right] \right]^{\frac{1}{rq}} \Bigg\}. \quad \square
\end{aligned}$$

Corollary 2.13. *Under the same conditions as in Theorem 2.12, if we choose $r = m = s = 1$, $\eta(\varphi(b), \varphi(a), m) = \varphi(b) - m\varphi(a)$, $g(t) = t$ and $\varphi(x) = x$, we get*

$$\begin{aligned}
& \left| \left[\frac{(x - a)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [w(x - a)^\rho] + (b - x)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [w(b - x)^\rho]}{(b - a)^{\lambda+1}} \right] f(x) \right. \\
& \quad \left. - \frac{\lambda}{(b - a)^{\lambda+1}} \left[(\mathcal{J}_{\rho, \lambda, x-; w}^\sigma f)(a) + (\mathcal{J}_{\rho, \lambda, x+; w}^\sigma f)(b) \right] \right| \\
& \leq \left(\mathcal{F}_{\rho, \lambda+1}^{\sigma_1^*} [|w|(x - a)^\rho] \right)^{1 - \frac{1}{q}} \\
& \quad \times \left[f'(a)^q \mathcal{F}_{\rho, \lambda+1}^{\sigma_2^*} [|w|(b - a)^\rho] + f'(b)^q \mathcal{F}_{\rho, \lambda+1}^{\sigma_3^*} [|w|(x - a)^\rho] \right]^{\frac{1}{q}} \\
& \quad + \left(\mathcal{F}_{\rho, \lambda+1}^{\sigma_4^*} [|w|(b - x)^\rho] \right)^{1 - \frac{1}{q}}
\end{aligned}$$

$$\times \left[f'(a)^q \mathcal{F}_{\rho, \lambda+1}^{\sigma_5^*} [|w|(b-x)^\rho] + f'(b)^q \mathcal{F}_{\rho, \lambda+1}^{\sigma_6^*} [|w|(b-a)^\rho] \right]^{\frac{1}{q}}, \quad (2.10)$$

where

$$\begin{aligned} \sigma_1^*(k) &= \sigma(k) \left(\frac{x-a}{b-a} \right) \frac{1}{\rho k + 1}; \\ \sigma_2^*(k) &= \sigma(k) \beta \left(\frac{x-a}{b-a}; \lambda q + \rho k + 1, 2 \right); \\ \sigma_3^*(k) &= \sigma(k) \left(\frac{x-a}{b-a} \right)^{\lambda q + 2} \frac{1}{\lambda q + \rho k + 2}; \\ \sigma_4^*(k) &= \sigma(k) \left(\frac{b-x}{b-a} \right) \frac{1}{\rho k + 1}; \\ \sigma_5^*(k) &= \sigma(k) \left(\frac{b-x}{b-a} \right)^{\lambda q + 2} \frac{1}{\lambda q + \rho k + 2}; \\ \sigma_6^*(k) &= \sigma(k) \beta \left(\frac{b-x}{b-a}; \lambda q + \rho k + 1, 2 \right). \end{aligned}$$

Corollary 2.14. *If we choose $\sigma(0) = 1, w = 0$ in Corollary 2.13, the inequality (2.10) reduces to inequality (2.4) ([38] Theorem 2.3).*

3. Applications to Special Means

Definition 3.1. ([6]) A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers $\alpha \neq \beta$.

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2} \right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e} \left(\frac{\beta^\beta}{\alpha^\alpha} \right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)}.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)} \right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

8. The weighted p -power mean:

$$M_p \left(\begin{matrix} \alpha_1, & \alpha_2, & \cdots & \alpha_n \\ u_1, & u_2, & \cdots & u_n \end{matrix} \right) = \left(\sum_{i=1}^n \alpha_i u_i^p \right)^{\frac{1}{p}}$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function

$$M := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+,$$

which is one of the above mentioned means and $\varphi : I \longrightarrow A$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ a differentiable function. Therefore one can obtain various inequalities using the results of Section 2 for these means as follows:

Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(y), \varphi(x))$ and setting

$$\eta(\varphi(a), \varphi(b)) = M(\varphi(a), \varphi(b))$$

for value $m = 1$ in (2.5), (2.7) and (2.9), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} |I_{f,g,M(\cdot,\cdot),\varphi}(x; \lambda, \rho, w, 1, a, b)| &\leq \mathcal{F}_{\rho,\lambda+1}^\sigma[|w| M^p(\varphi(a), \varphi(b))] \\ &\times \left[\left\{ f'(\varphi(a))^r B_{g(\alpha_1^*)}^r \left(\lambda + \rho k + 1, \frac{s}{r} + 1 \right) \right. \right. \\ &\quad \left. \left. + f'(\varphi(b))^r \left(\frac{g^{\lambda+\rho k+\frac{s}{r}+1}(\alpha_1^*) - g^{\lambda+\rho k+\frac{s}{r}+1}(0)}{\lambda + \rho k + \frac{s}{r} + 1} \right)^r \right\}^{\frac{1}{r}} \right. \\ &\quad \left. + \left\{ f'(\varphi(a))^r \left(\frac{(1 - g(\alpha_1^*))^{\lambda+\rho k+\frac{s}{r}+1} - (1 - g(1))^{\lambda+\rho k+\frac{s}{r}+1}}{\lambda + \rho k + \frac{s}{r} + 1} \right)^r \right. \right. \\ &\quad \left. \left. + f'(\varphi(b))^r \left[B_{g(1)} \left(\frac{s}{r} + 1, \lambda + \rho k + 1 \right) \right. \right. \right. \\ &\quad \left. \left. \left. - B_{g(\alpha_1^*)} \left(\frac{s}{r} + 1, \lambda + \rho k + 1 \right) \right] \right]^r \right\}^{\frac{1}{r}} \right], \end{aligned} \quad (3.1)$$

where

$$\begin{aligned}
 \alpha_1^* &= \frac{x - \varphi(a)}{M(\varphi(a), \varphi(b))}, \\
 |I_{f,g,M(\cdot,\cdot),\varphi}(x; \lambda, \rho, w, 1, a, b)| &\leq \left(\frac{r}{s+r} \right)^{\frac{1}{q}} \mathcal{F}_{\rho,\lambda+1}^{\sigma_1^*} [|w| M^\rho(\varphi(a), \varphi(b))] \\
 &\quad \times \left\{ \left(g^{(\lambda+\rho k)p+1}(\alpha_1^*) - g^{(\lambda+\rho k)p+1}(0) \right)^{\frac{1}{p}} \right. \\
 &\quad \times \left[\left((1-g(0))^{\frac{s}{r}+1} - (1-g(\alpha_1^*))^{\frac{s}{r}+1} \right)^r f'(\varphi(a))^{rq} \right. \\
 &\quad \left. + \left(g^{\frac{s}{r}+1}(\alpha_1^*) - g^{\frac{s}{r}+1}(0) \right)^r f'(\varphi(b))^{rq} \right]^{\frac{1}{rq}} \\
 &\quad + \left((1-g(\alpha_1^*))^{(\lambda+\rho k)p+1} - (1-g(1))^{(\lambda+\rho k)p+1} \right)^{\frac{1}{p}} \\
 &\quad \times \left[\left((1-g(\alpha_1^*))^{\frac{s}{r}+1} - (1-g(1))^{\frac{s}{r}+1} \right)^r f'(\varphi(a))^{rq} \right. \\
 &\quad \left. + \left(g^{\frac{s}{r}+1}(1) - g^{\frac{s}{r}+1}(\alpha_1^*) \right)^r f'(\varphi(b))^{rq} \right]^{\frac{1}{rq}} \Bigg\}, \tag{3.2}
 \end{aligned}$$

$$\begin{aligned}
 |I_{f,g,M(\cdot,\cdot),\varphi}(x; \lambda, \rho, w, 1, a, b)| &\leq \mathcal{F}_{\rho,\lambda+1}^{\sigma^{**}} [|w| M^\rho(\varphi(a), \varphi(b))] \\
 &\quad \times \left\{ \left[g^{\rho k+1}(\alpha_1^*) - g^{\rho k+1}(0) \right]^{1-\frac{1}{q}} \right. \\
 &\quad \times \left[f'(\varphi(a))^{rq} B_{g(\alpha_1^*)}^r \left(\lambda q + \rho k + 1, \frac{s}{r} + 1 \right) \right. \\
 &\quad \left. + f'(\varphi(b))^{rq} \left(\frac{g^{\lambda q + \rho k + \frac{s}{r} + 1}(\alpha_1^*) - g^{\lambda q + \rho k + \frac{s}{r} + 1}(0)}{\lambda q + \rho k + \frac{s}{r} + 1} \right)^r \right]^{\frac{1}{rq}} \\
 &\quad + \left[(1-g(\alpha_1^*))^{\rho k+1} - (1-g(1))^{\rho k+1} \right]^{1-\frac{1}{q}} \\
 &\quad + \left[f'(\varphi(a))^{rq} \left(\frac{(1-g(\alpha_1^*))^{\lambda q + \rho k + \frac{s}{r} + 1} - (1-g(1))^{\lambda q + \rho k + \frac{s}{r} + 1}}{\lambda q + \rho k + \frac{s}{r} + 1} \right)^r \right. \\
 &\quad + f'(\varphi(b))^{rq} \left[B_{g(1)} \left(\frac{s}{r} + 1, \lambda q + \rho k + 1 \right) \right. \\
 &\quad \left. \left. - B_{g(\alpha_1^*)} \left(\frac{s}{r} + 1, \lambda q + \rho k + 1 \right) \right] \right]^{\frac{1}{rq}} \Bigg\}. \tag{3.3}
 \end{aligned}$$

Letting $M(\varphi(a), \varphi(b)) = A, G, H, P_r, I, L, L_p, M_p$ in (3.1), (3.2) and (3.3), we get the inequalities involving means for a particular choice of a differentiable generalized $(r; g, s, 1, \varphi)$ -preinvex functions f . The details are left to the interested reader.

4. Conclusion

In the present paper, the notion of generalized $(r; g, s, m, \varphi)$ -preinvex function was applied for established some new generalizations of Ostrowski type inequalities via fractional integral operators. These results not only extended the results appeared in the literature (see [38]) but also provided new estimates on these type. At the end, some applications to special means are obtained. Motivated by this new interesting class of generalized $(r; g, s, m, \varphi)$ -preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Ostrowski type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k -fractional integrals, local fractional integrals, fractional integral operators, q -calculus, (p, q) -calculus, time scale calculus and conformable fractional integrals.

References

- [1] Abdeljawad, T., *On conformable fractional calculus*, J. Comput. Appl. Math., **279**(2015), 57-66.
- [2] Agarwal, R.P., Luo, M.J., Raina, R.K., *On Ostrowski type inequalities*, Fasc. Math., **204**(2016), 5-27.
- [3] Ahmadmir, M., Ullah, R., *Some inequalities of Ostrowski and Grüss type for triple integrals on time scales*, Tamkang J. Math., **42**(4)(2011), 415-426.
- [4] Alomari, M., Darus, M., Dragomir, S.S., Cerone, P., *Ostrowski type inequalities for functions whose derivatives are s -convex in the second sense*, Appl. Math. Lett., **23**(2010), 1071-1076.
- [5] Antczak, T., *Mean value in invexity analysis*, Nonlinear Anal., **60**(2005), 1473-1484.
- [6] Bullen, P.S., *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, 2003.
- [7] Dahmani, Z., *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal., **1**(1)(2010), 51-58.
- [8] Dahmani, Z., *New inequalities in fractional integrals*, Int. J. Nonlinear Sci., **9**(4)(2010), 493-497.
- [9] Dahmani, Z., Tabharit, L., Taf, S., *New generalizations of Grüss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., **2**(3)(2010), 93-99.
- [10] Dahmani, Z., Tabharit, L., Taf, S., *Some fractional integral inequalities*, Nonlinear. Sci. Lett. A, **1**(2)(2010), 155-160.
- [11] Dragomir, S.S., *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Ineq. & Appl., **1**(2)(1998).
- [12] Dragomir, S.S., *The Ostrowski integral inequality for Lipschitzian mappings and applications*, Comput. Math. Appl., **38**(1999), 33-37.
- [13] Dragomir, S.S. Wang, S., *An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl., **13**(11)(1997), 15-20.

- [14] Dragomir, S.S., Wang, S., *A new inequality of Ostrowski's type in L_1 -norm and applications to some special means and to some numerical quadrature rules*, Tamkang J. Math., **28**(1997), 239-244.
- [15] Du, T.S., Liao, J.G., Li, Y.J., *Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m) -preinvex functions*, J. Nonlinear Sci. Appl., **9**(2016), 3112-3126.
- [16] Hudzik, H., Maligranda, L., *Some remarks on s -convex functions*, Aequationes Math., **48**(1994), 100-111.
- [17] Kashuri, A., Liko, R., *Ostrowski type fractional integral inequalities for generalized (s, m, φ) -preinvex functions*, Aust. J. Math. Anal. Appl., **13**(1)(2016), Article 16, 1-11.
- [18] Kashuri, A., Liko, R., *Hermite-Hadamard type fractional integral inequalities for generalized $(r; s, m, \varphi)$ -preinvex functions*, Eur. J. Pure Appl. Math., **10**(3)(2017), 495-505.
- [19] Katugampola, U.N., *A new approach to generalized fractional derivatives*, Bulletin Math. Anal. Appl., **6**(4)(2014), 1-15.
- [20] Khalil, R., Horani, M.A., Yousef, A., Sababheh, M., *A new definition of fractional derivative*, J. Comput. Appl. Math., **264**(2014), 65-70.
- [21] Liu, Z., *Some Ostrowski-Grüss type inequalities and applications*, Comput. Math. Appl., **53**(2007), 73-79.
- [22] Liu, Z., *Some companions of an Ostrowski type inequality and applications*, J. Inequal. in Pure and Appl. Math, **10**(2)(2009), Art. 52, 12 pp.
- [23] Liu, W., Wen, W., Park, J., *Ostrowski type fractional integral inequalities for MT-convex functions*, Miskolc Math. Notes, **16**(1)(2015), 249-256.
- [24] Liu, W., Wen, W., Park, J., *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl., **9**(2016), 766-777.
- [25] Mitrinovic, D.S., Pečarić, J.E., Fink, A.M., *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
- [26] Özdemir, M.E., Kavurmac, H., Set, E., *Ostrowski's type inequalities for (α, m) -convex functions*, Kyungpook Math. J., **50**(2010), 371-378.
- [27] Pachpatte, B.G., *On an inequality of Ostrowski type in three independent variables*, J. Math. Anal. Appl., **249**(2000), 583-591.
- [28] Pachpatte, B.G., *On a new Ostrowski type inequality in two independent variables*, Tamkang J. Math., **32**(1)(2001), 45-49.
- [29] Pini, R., *Invexity and generalized convexity*, Optimization, **22**(1991), 513-525.
- [30] Purohit, S.D., Kalla, S.L., *Certain inequalities related to the Chebyshev's functional involving Erdelyi-Kober operators*, Scientia Series A: Math. Sci., **25**(2014), 53-63.
- [31] Rafiq, A., Mir, N. A., Ahmad, F., *Weighted Čebyšev-Ostrowski type inequalities*, Applied Math. Mechanics (English Edition), **28**(7)(2007), 901-906.
- [32] Raina, R.K., *On generalized Wright's hypergeometric functions and fractional calculus operators*, East Asian Math. J., **21**(2)(2005), 191-203.
- [33] Sarikaya, M.Z., *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenianae, **79**(1)(2010), 129-134.
- [34] Set, E., Gözpinar, A., Choi, J., *Hermite-Hadamard type inequalities for twice differentiable m -convex functions via conformable fractional integrals*, Far East J. Math. Sci., **101**(4)(2017), 873-891.

- [35] Tunç, M., *Ostrowski type inequalities for functions whose derivatives are MT-convex*, J. Comput. Anal. Appl., **17**(4)(2014), 691-696.
- [36] Ujević, N., *Sharp inequalities of Simpson type and Ostrowski type*, Comput. Math. Appl., **48**(2004), 145-151.
- [37] Yang, X.M., Yang, X.Q., Teo, K.L., *Generalized invexity and generalized invariant monotonicity*, J. Optim. Theory Appl., **117**(2003), 607-625.
- [38] Yildiz, Ç., Özdemir, M.E., Sarikaya, M.Z., *New generalizations of Ostrowski-like type inequalities for fractional integrals*, Kyungpook Math. J., **56**(2016), 161-172.
- [39] Zhongxue, L., *On sharp inequalities of Simpson type and Ostrowski type in two independent variables*, Comput. Math. Appl., **56**(2008), 2043-2047.

Artion Kashuri

Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali"
Vlora, Albania

e-mail: artionkashuri@gmail.com

Rozana Liko

Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali"
Vlora, Albania

e-mail: rozanaliko86@gmail.com

