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Applications of generalized fractional integral operator to unified subclass of prestarlike functions with negative coefficients

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Abstract. In this paper, we have introduced and studied various properties of unified class of prestarlike functions with negative coefficients in the unit disc U. Also distortion theorem involving a generalized fractional integral operator for functions in this class is established.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n \tag{1.1}$$

which are analytic in the unit disc U= $\{z : |z| < 1\}$. Let S denote the subclass of A, which consists of functions of the form (1.1) that are univalent in U.

A function $f \in S$ is said to be starlike of order $\mu (0 \leq \mu < 1)$ if and only if

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \mu, \; z \in U$$

and convex of order $\mu (0 \leq \mu < 1)$ if and only if

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \mu, \; z \in U.$$

Denote these classes respectively by $S^*(\mu)$ and $K(\mu)$.

Let T denote subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n \, z^n, \;\; a_n \geq 0. \tag{1.2}$$

The classes obtained by taking intersections of the classes $S^*(\mu)$ and $K(\mu)$ with T are denoted by $T^*(\mu)$ and $K^*(\mu)$ respectively. The classes $T^*(\mu),\; K^*(\mu)$ were studied by Silverman [9].

The function

$$S_\mu(z) = z(1-z)^{-2(1-\mu)} \;, \qquad 0 \leq \mu < 1 \;, \tag{1.3}$$

is the familiar extremal function for the class $S^*(\mu)$, setting

$$C(\mu, n) = \frac{\prod_{i=2}^{n} (i - 2\mu)}{(n-1)!} \;, \quad n \in \mathbb{N}\backslash\{1\}, \;\; \mathbb{N} = \{1, 2, 3, \cdots\} \;, \tag{1.4}$$

then

$$S_\mu(z) = z + \sum_{n=2}^{\infty} C(\mu, n) \; z^n \;. \tag{1.5}$$

We note that $C(\mu, n)$ is a decreasing function in μ, and that

$$\lim_{n\to\infty} C(\mu, n) = \begin{cases} \infty \;, & \mu < \frac{1}{2} \\ 1 \;, & \mu = \frac{1}{2} \\ 0 \;, & \mu > \frac{1}{2} \end{cases} \;.$$

If $f(z)$ is given by (1.2) and $g(z)$ defined by

$$g(z) = z - \sum_{n=2}^{\infty} b_n \; z^n, \quad b_n \geq 0 \;,$$

belonging to T, then convolution or Hadamard product of f and g is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n \; b_n \; z^n.$$

Let $R_\mu(\alpha, \beta, \gamma)$ be the subclass of A consisting functions $f(z)$ such that

$$\left| \frac{\frac{zh'(z)}{h(z)} - 1}{\gamma \frac{zh'(z)}{h(z)} + 1 - (1+\gamma)\alpha} \right| < \beta \;,$$

where, $h(z) = (f * S_\mu(z)), \; 0 \leq \alpha < 1, \; 0 < \beta \leq 1, \; 0 \leq \gamma \leq 1, \; 0 \leq \mu < 1.$

Also let $C_\mu(\alpha, \beta, \gamma)$ be the subclass of A consisting of functions $f(z)$, which satisfy the condition

$$zf'(z) \in R_\mu(\alpha, \beta, \gamma).$$

The classes $R_\mu(\alpha,\beta,\gamma)$ and $C_\mu(\alpha,\beta,\gamma)$ of prestarlike functions was investigated by Joshi [1]. In particular, the subclasses

$$R_\mu[\alpha,\beta,\gamma]=R_\mu(\alpha,\beta,\gamma)\cap T, \quad C_\mu[\alpha,\beta,\gamma]=C_\mu(\alpha,\beta,\gamma)\cap T\ ,$$

were also studied by Joshi [1].

The following results will be required for our investigation.

Lemma 1.1. [1]. *A function f defined by (1.2) is in the class $R_\mu[\alpha,\beta,\gamma]$ if and only if*

$$\sum_{n=2}^{\infty} C(\mu,n)\ \{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\ a_n \le \beta(1+\gamma)(1-\alpha). \tag{1.6}$$

The result (1.6) *is sharp.*

Lemma 1.2. [1]. *A function f defined by (1.2) is in the class $C_\mu[\alpha,\beta,\gamma]$ if and only if*

$$\sum_{n=2}^{\infty} C(\mu,n)\ n\,\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\ a_n \le \beta(1+\gamma)(1-\alpha). \tag{1.7}$$

The result (1.7) *is sharp.*

Further we note that such type of classes were extensively studied by Sheil-Small *et al.* [8], Owa and Uralegaddi [4], Srivastava and Aouf [10] and Raina and Srivastava [7].

In view of Lemma 1.1 and Lemma 1.2, we present here a unified study of the classes $R_\mu[\alpha,\beta,\gamma]$ and $C_\mu[\alpha,\beta,\gamma]$ by introducing a new subclass $P_\mu(\alpha,\beta,\gamma,\sigma)$. Indeed, we say that a function $f(z)$ defined by (1.2) is in the class $P_\mu(\alpha,\beta,\gamma,\sigma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)}{\beta(1+\gamma)(1-\alpha)}\ C(\mu,n)a_n \le 1\ , \tag{1.8}$$

where, $0\le\alpha<1,\ 0<\beta\le1,\ 0\le\gamma\le1,\ 0\le\mu<1,\ 0\le\sigma\le1.$
Then clearly we have,

$$P_\mu(\alpha,\beta,\gamma,\sigma)=(1-\sigma)R_\mu[\alpha,\beta,\gamma]+\sigma C_\mu[\alpha,\beta,\gamma]\ , \tag{1.9}$$

where, $0\le\sigma\le1$. So that

$$P_\mu(\alpha,\beta,\gamma,0)=R_\mu[\alpha,\beta,\gamma], \quad P_\mu(\alpha,\beta,\gamma,1)=C_\mu[\alpha,\beta,\gamma]. \tag{1.10}$$

The main object of this paper is to investigate various interesting properties and characterization of the general class $P_\mu(\alpha,\beta,\gamma,\sigma)$. Also distortion theorem involving a generalized fractional integral operator for functions in this class are obtained.

2. Main results

Theorem 2.1. *A function f defined by (1.2) is in the class $P_\mu(\alpha,\beta,\gamma,\sigma)$ then*

$$a_n \le \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\,\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)},\ n\in\mathbb{N}\backslash\{1\}. \tag{2.1}$$

Equality holds true for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\left\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\right\}(1-\sigma+\sigma n)} z^n, \;\; n \in \mathbb{N}\backslash\{1\}. \tag{2.2}$$

Proof. The proof of Theorem 2.1 is straightforward and hence details are omitted. □

A distortion theorem for function f in the class $P_\mu(\alpha,\beta,\gamma,\sigma)$ is given as follows:

Theorem 2.2. *If the function f defined by (1.2) is in the class $P_\mu(\alpha,\beta,\gamma,\sigma)$ then*

$$|z| - \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|^2 \le |f(z)|$$
$$\le |z| + \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|^2 \;, \tag{2.3}$$

and

$$1 - \frac{\beta(1+\gamma)(1-\alpha)}{\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z| \le |f'(z)|$$
$$\le 1 + \frac{\beta(1+\gamma)(1-\alpha)}{\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z| \;. \tag{2.4}$$

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

Since $f(z) \in P_\mu(\alpha,\beta,\gamma,\sigma)$ and clearly $C(\mu,n)$ defined by (1.4) is non-decreasing for $0 \le \mu \le \frac{1}{2}$ and using (1.8) we get

$$\sum_{n=2}^{\infty} a_n \le \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}, \;\; n \in \mathbb{N}\backslash\{1\}. \tag{2.5}$$

Then using (1.2) and (2.5) we get (for $z \in U$),

$$\begin{aligned} |f(z)| &\le |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\le |z| + \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|^2 \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\ge |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\ge |z| - \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|^2. \end{aligned}$$

which proves the assertion (2.3) of Theorem 2.2.

Also for $z \in U$, we find that

$$\begin{aligned} |f'(z)| &\leq 1 + |z| \sum_{n=2}^{\infty} n\ |a_n| \\ &\leq 1 + \frac{\beta(1+\gamma)(1-\alpha)}{\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\}(1-\mu)(1+\sigma)}|z| \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq 1 - |z| \sum_{n=2}^{\infty} n\ |a_n| \\ &\geq 1 - \frac{\beta(1+\gamma)(1-\alpha)}{\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\}(1-\mu)(1+\sigma)}|z|. \end{aligned}$$

which proves the assertion (2.4) of Theorem 2.2
This completes the proof. □

We note that results (2.3) and (2.4) is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{2\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\}(1-\mu)(1+\sigma)} z^2. \tag{2.6}$$

3. Closure theorems

In this section, we shall prove that the class $P_\mu(\alpha, \beta, \gamma, \sigma)$ is closed under linear combination.

Theorem 3.1. *The class $P_\mu(\alpha, \beta, \gamma, \sigma)$ is closed under linear combination.*

Proof. Suppose $f(z),\ g(z) \in P_\mu(\alpha, \beta, \gamma, \sigma)$ and

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n\ .$$

It is sufficient to prove that the function H defined by

$$H(z) = \lambda f(z) + (1-\lambda) g(z)\ , \quad (0 \leq \lambda \leq 1)$$

is also in the class $P_\mu(\alpha, \beta, \gamma, \sigma)$. Since

$$H(z) = z - \sum_{n=2}^{\infty} [\lambda a_n + (1-\lambda) b_n] z^n\ .$$

We observe that

$$\sum_{n=2}^{\infty} \frac{\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\}(1-\sigma+\sigma n)}{\beta(1+\gamma)(1-\alpha)}\ C(\mu, n)[\lambda a_n + (1-\lambda) b_n] \leq 1.$$

Thus $H \in P_\mu(\alpha, \beta, \gamma, \sigma)$. This completes the proof. □

Theorem 3.2. *If*

$$f_1(z) = z$$

and

$$f_n(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)C(\mu,n)}\ z^n\ , \quad (n \geq 2).$$

Then $f \in P_\mu(\alpha,\beta,\gamma,\sigma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)C(\mu,n)}\ \lambda_n z^n \\ &= z - \sum_{n=2}^{\infty} a_n\ z^n\ , \end{aligned}$$

where

$$a_n = \frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)C(\mu,n)}\ \lambda_n \geq 0,\ (n \geq 2).$$

Since,

$$\sum_{n=2}^{\infty} \left[\frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)C(\mu,n)} \right. \\ \left. \frac{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)C(\mu,n)}{\beta(1+\gamma)(1-\alpha)} \right] \lambda_n$$

$$= \sum_{n=2}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n - \lambda_1 = 1 - \lambda_1 \leq 1\ .$$

Therefore $f(z) \in P_\mu(\alpha,\beta,\gamma,\sigma)$.
Conversely, suppose that $f \in P_\mu(\alpha,\beta,\gamma,\sigma)$ and since

$$a_n = \frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)C(\mu,n)}\ \lambda_n \geq 0,\ (n \geq 2).$$

Setting

$$\lambda_n = \frac{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)C(\mu,n)}{\beta(1+\gamma)(1-\alpha)}\ ,\ (n \geq 2)$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n \ .$$

We get

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \ .$$

This completes the proof. □

4. Generalized fractional integral operator

In recent years the theory of fractional calculus operator have been fruitfully applied to analytic functions. Moreover generalized operator of fractional integrals (or derivatives) having kernels of different types of special functions (including Fox's H-function) have generated keen interest in this area. For details one may refer to Kiryakova [2], Raina and Saigo [6], Srivastava and Owa [11] and Raina and Bolia [5]. Further we note that Riemann-Liouville fractional calculus operators have been used to obtain basic results which include coefficient estimates, boundedness properties for various subclasses of analytic and univalent functions.

A generalized fractional integral operator involving the celebrated Fox's H-function [2, 3] defined below.

Definition 4.1. *Let* $m \in \mathbb{N}$, $\beta_k \in \mathbb{R}$ *and* $\gamma_k, \ \delta_k \in \mathbb{C}$, $\forall \ k = 1, 2, \cdots, m$. *Then the integral operator*

$$\begin{aligned} I_{(\beta_m);m}^{(\gamma_m),(\delta_m)} f(z) &= I_{(\beta_1,\cdots,\beta_m);m}^{(\gamma_1,\cdots,\gamma_m),(\delta_1,\cdots,\delta_m)} f(z) \\ &= \frac{1}{z} \int_0^z H_{m,m}^{m,0} \left[\frac{t}{z} \left| \begin{matrix} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_{1,m} \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_{1,m} \end{matrix} \right. \right] f(t)dt, \\ &\qquad \text{for } \sum_{i}^{m} Re(\delta_k) > 0, \qquad (4.1) \\ &= f(z), \quad \text{for } \delta_1 = \cdots = \delta_m = 0 \ , \end{aligned}$$

is said to be a multiple fractional integral operator of Riemann-Liouville type of multiorder $\delta = (\delta_1, \cdots, \delta_m)$.

Following [2], let Δ denote a complex domain starlike with respect to the origin $z = 0$, and $A(\Delta)$ denote the space of functions analytic in Δ. If $A_\rho(\Delta)$ denote the class of functions

$$A_\rho(\Delta) = \{f(z) = z^\rho \bar{f}(z) : \bar{f}(z) \in A(\Delta)\}, \qquad \rho \geq 0; \tag{4.2}$$

then clearly $A_\rho(\Delta) \subseteq A_v(\Delta) \subseteq A(\Delta)$ for $\rho \geq v \geq 0$.

The fractional integral operator (4.1) includes various useful and important fractional integral operators as special cases. For more details of these special cases, one may refer to Raina and Saigo [6]. Throughout this paper $(\lambda)_k$ stands for $\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$.

The following results will be required for our investigation.

Lemma 4.2. [2]. *Let* $\gamma_k > -\frac{p}{\beta_k} - 1,\ \delta_k \leq 0\ (\forall\ k = 1, \cdots, m)$. *Then the operator* $I^{(\gamma_m),(\delta_m)}_{(\beta_m);m}$ *maps the class* $\Delta_p(G)$ *into itself preserving the power functions* $f(z) = z^p$ *(up to a constant multiplier):*

$$I^{(\gamma_m),(\delta_m)}_{(\beta_m);m}\{z^p\} = \prod_{k=1}^{m} \left\{ \frac{\Gamma\left(\frac{p}{\beta_k} + \gamma_k + 1\right)}{\Gamma\left(\frac{p}{\beta_k} + \gamma_k + \delta_k + 1\right)} \right\} z^p \ . \tag{4.3}$$

Theorem 4.3. *Let* $m \in \mathbb{N},\ h_k \in \mathbb{R}_+$, *and* $\gamma_k,\ \delta_k \in \mathbb{R}$ *such that* $1 + \gamma_k + \delta_k > 0$ $(k = 1, \cdots, m)$, *and*

$$\prod_{k=1}^{m} \left\{ \frac{(1 + \gamma_k + 2h_k)_{h_k}}{(1 + \gamma_k + \delta_k + 2h_k)_{h_k}} \right\} \leq 1 \tag{4.4}$$

and $f(z)$ *defined by* (1.2) *be in the class* $P_\mu(\alpha, \beta, \gamma, \sigma)$ *with* $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ 0 \leq \gamma \leq 1,\ 0 \leq \mu \leq \frac{1}{2},\ 0 \leq \sigma \leq 1$. *Then*

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \right| \geq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1 + \gamma_k + h_k)}{\Gamma(1 + \gamma_k + \delta_k + h_k)} \right) \cdot \left[|z| - \frac{A^*\beta(1+\gamma)(1-\alpha)}{2\{1 + \beta[2\gamma + 1 - (1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\}, \tag{4.5}$$

and

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \right| \leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1 + \gamma_k + h_k)}{\Gamma(1 + \gamma_k + \delta_k + h_k)} \right) \cdot \left[|z| + \frac{A^*\beta(1+\gamma)(1-\alpha)}{2\{1 + \beta[2\gamma + 1 - (1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\}, \tag{4.6}$$

for $z \in U$. *The inequalities in* (4.5) *and* (4.6) *are attained by the function* $f(z)$ *given by* (2.6), *where*

$$A^* = \prod_{k=1}^{m} \left\{ \frac{(1 + \gamma_k + h_k)_{h_k}}{(1 + \gamma_k + \delta_k + h_k)_{h_k}} \right\}. \tag{4.7}$$

Proof. By using lemma 4.2, we get

$$\begin{aligned} I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) &= \prod_{k=1}^{m} \left\{ \frac{\Gamma(1 + \gamma_k + h_k)}{\Gamma(1 + \gamma_k + \delta_k + h_k)} \right\} z \\ &\quad - \sum_{n=2}^{\infty} \prod_{k=1}^{m} \left\{ \frac{\Gamma(1 + \gamma_k + n\ h_k)}{\Gamma(1 + \gamma_k + \delta_k + n\ h_k)} \right\} a_n z^n \ . \end{aligned} \tag{4.8}$$

Letting

$$G(z) = \prod_{k=1}^{m} \left\{ \frac{\Gamma(1+\gamma_k+\delta_k+h_k)}{\Gamma(1+\gamma_k+h_k)} \right\} I_{(h_m^{-1});m}^{(\gamma_m),(\delta_m)} f(z) = z - \sum_{n=2}^{\infty} \phi(n)\ a_n\ z^n \ , \tag{4.9}$$

where,

$$\phi(n) = \prod_{k=1}^{m} \left\{ \frac{(1+\gamma_k+h_k)_{h_k(n-1)}}{(1+\gamma_k+\delta_k+h_k)_{h_k(n-1)}} \right\} \ , \quad (n \in \mathbb{N}\backslash\{1\}) \, . \tag{4.10}$$

Under the hypothesis of Theorem 4.3 (along with the conditions (4.4)), we can see that $\phi(n)$ is non-increasing for integers n ($n \geq 2$), and we have

$$0 < \phi(n) \leq \phi(2) = \prod_{k=1}^{m} \left\{ \frac{(1+\gamma_k+h_k)_{h_k}}{(1+\gamma_k+\delta_k+h_k)_{h_k}} \right\} = A^* \ , \ (n \in \mathbb{N}\backslash\{1\}) \, . \tag{4.11}$$

Now in view equation (1.8) and (4.11), we have

$$\begin{aligned} |G(z)| &\geq |z| - \phi(2)\ |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \right. \\ &\quad \left. . \left[|z| - \frac{A^* \beta(1+\gamma)(1-\alpha)}{2\{1+\beta[2\gamma+1-(1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} \ . \end{aligned}$$

and

$$\begin{aligned} |G(z)| &\leq |z| + \phi(2)\ |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \right. \\ &\quad \left. . \left[|z| + \frac{A^* \beta(1+\gamma)(1-\alpha)}{2\{1+\beta[2\gamma+1-(1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} \ . \end{aligned}$$

It can be easily verified that the following inequalities are attained by the function $f(z)$ given by (2.6).

$$\begin{aligned} \left| I_{(h_m^{-1});m}^{(\gamma_m),(\delta_m)} f(z) \right| \geq &\left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \right. \\ &\left. . \left[|z| - \frac{A^* \beta(1+\gamma)(1-\alpha)}{2\{1+\beta[2\gamma+1-(1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} \ , \end{aligned}$$

and

$$\left| I_{(h_m^{-1});m}^{(\gamma_m),(\delta_m)} f(z) \right| \leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \cdot \left[|z| + \frac{A^*\beta(1+\gamma)(1-\alpha)}{2\{1+\beta[2\gamma+1-(1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} .$$

Which are as desired in (4.5) and (4.6). This completes the proof of Theorem 4.3. □

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