Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 4, 537–542
DOI: 10.24193/subbmath.2017.4.10

On Fryszkowski's problem

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Abstract. In this paper we give two partial answers to Fryszkowski's problem which can be stated as follows: given $\alpha \in (0,1)$, an arbitrary non-empty set Ω and a set-valued mapping $F : \Omega \to 2^{\Omega}$, find necessary and (or) sufficient conditions for the existence of a (complete) metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d. More precisely, on the one hand, we provide necessary and sufficient conditions for the existence of a complete and bounded metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d, in the case that $\alpha \in (0, \frac{1}{2})$ and there exists $z \in \Omega$ such that $F(z) = \{z\}$ and, on the other hand, we give a sufficient condition for the existence of a complete metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d, in the case that Ω is finite.

Mathematics Subject Classification (2010): 54C60, 54H25.

Keywords: Fixed point of a multi-valued map, Hausdorff-Pompeiu distance, α-contractions.

1. Introduction

The first version of a converse of the Banach-Caccioppoli-Picard principle is due to C. Bessaga (see [2]). For an application of Bessaga's converse see [20] and for some other converses of the contraction principle see [3], [7], [9], [12] and [17]. For more results along this line of research one can consult [1], [8], [13], [14], [15] and [23].

An extension of the contraction principle to set-valued mappings is due to J. T. Markin and S. B. Nadler Jr. (see [11] and [16]). For more information on this topic see [4], [5], [10], [18], [19], [21], and [22].

The last section of [6] consists of the following problem formulated by Professor Andrzej Fryszkowski at the 2nd Symposium on Nonlinear Analysis in Toruń, September 13-17, 1999, which asks for a converse of the contraction principle for set-valued mappings: *Given $\alpha \in (0,1)$, an arbitrary non-empty set Ω and a set-valued mapping*

$F : \Omega \to 2^{\Omega}$, find necessary and (or) sufficient conditions for the existence of a (complete) metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d.

In this paper we give two partial answers to the above mentioned problem.

Our first result provides necessary and sufficient conditions for the existence of a complete and bounded metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d, in the case that $\alpha \in (0, \frac{1}{2})$ and there exists $z \in \Omega$ such that $F(z) = \{z\}$.

Our second result gives a sufficient condition for the existence of a complete metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d, in the case that Ω is finite.

2. Preliminaries

Definition 2.1. For a metric space (X, d), we consider the generalized Hausdorff-Pompeiu metric $H : 2^X \times 2^X \to [0, +\infty]$ described by

$$H(A, B) = \max\{\sup_{x \in A}(\inf_{y \in B} d(x, y)), \sup_{x \in B}(\inf_{y \in A} d(x, y))\},$$

for every $A, B \in 2^X$.

Definition 2.2. Given $\alpha \in (0, 1)$, an arbitrary non-empty set Ω and a metric d on Ω, a set-valued function $F : \Omega \to 2^{\Omega}$ is called Nadler set-valued α-contraction with respect to d if $H(F(x), F(y)) \leq \alpha d(x, y)$ for all $x, y \in \Omega$.

Definition 2.3. Given an arbitrary non-empty set Ω and a set-valued function $F : \Omega \to 2^{\Omega}$, $z \in \Omega$ is called a fixed point of F if $z \in F(z)$.

Definition 2.4. Given an arbitrary non-empty set Ω and a set-valued function $F : \Omega \to 2^{\Omega}$, one can consider the function $\widehat{F} : 2^{\Omega} \to 2^{\Omega}$ *given by*

$$\widehat{F}(P) = \bigcup_{x \in P} F(x)$$

for every $P \in 2^{\Omega}$.

Definition 2.5. Given an arbitrary non-empty set Ω, a function $f : \Omega \to \Omega$ and $n \in \mathbb{N}$, by f^n we mean the composition of f by itself n times, with the convention that $f^0 = \mathrm{Id}_{\Omega}$.

3. Main results

Lemma 3.1. *Given $\alpha \in (0, 1)$, an arbitrary non-empty set Ω and a set-valued function $F : \Omega \to 2^{\Omega}$ having a fixed point z such that $F(z) = \{z\}$, the following statements are equivalent:*

a) there exists a complete metric d on Ω such that F is a Nadler set-valued α-contraction with respect to d;

b) there exists a function $\varphi : \Omega \to [0, \infty)$ such that $\varphi^{-1}(\{0\}) = \{z\}$ and $\sup_{t \in F(x)} \varphi(t) \leq \alpha \varphi(x)$ for all $x \in \Omega$.

Proof. a)⇒b) We consider the function $\varphi : \Omega \to [0, \infty)$ given by $\varphi(x) = d(x, z)$ for all $x \in \Omega$. It is clear that $\varphi^{-1}(\{0\}) = \{z\}$. Moreover, we have

$$\sup_{t\in F(x)} \varphi(t) = \sup_{t\in F(x)} d(t, z) \le H(F(x), \{z\}) = H(F(x), F(z)) \le \alpha d(x, z) = \alpha\varphi(x)$$

for all $x \in \Omega$.

b)⇒a) Considering the metric $d : \Omega \times \Omega \to [0, \infty)$, given by

$$d(x, y) = \begin{cases} \varphi(x) + \varphi(y), & \text{if } x \ne y \\ 0, & \text{if } x = y \end{cases},$$

we have

$$\begin{aligned} \sup_{t\in F(x)} d(t, F(y)) &= \sup_{t\in F(x)} \inf_{u\in F(y)} d(t, u) \le \sup_{t\in F(x)} \inf_{u\in F(y)} (\varphi(t) + \varphi(u)) \\ &= \sup_{t\in F(x)} (\varphi(t) + \inf_{u\in F(y)} \varphi(u)) = \sup_{t\in F(x)} \varphi(t) + \inf_{u\in F(y)} \varphi(u) \\ &\le \alpha(\varphi(x) + \varphi(y)) = \alpha d(x, y) \end{aligned}$$

for all $x, y \in \Omega$, $x \ne y$. In a similar way we get $\sup_{t\in F(y)} d(t, F(x)) \le \alpha d(x, y)$ for all $x, y \in \Omega$, $x \ne y$. Consequently we infer that

$$H(F(x), F(y)) = \max\{\sup_{t\in F(x)} d(t, F(y)), \sup_{t\in F(y)} d(t, F(x))\} \le \alpha d(x, y)$$

for all $x, y \in \Omega$, $x \ne y$. Note that the last inequality is true for $x = y$. The proof of the fact that d is complete is identical to the one presented in Lemma 1 from [6]. □

Corollary 3.2. *If $\alpha \in (0, 1)$, (Ω, d) is a complete metric space and $F : \Omega \to 2^\Omega$ is a Nadler set-valued α-contraction with respect to d having a fixed point z such that $F(z) = \{z\}$, then z is the unique fixed point of F.*

Proof. Let us suppose that y is another fixed point of F. Then, from Lemma 3.1, we obtain $\varphi(y) \le \sup_{x\in F(y)} \varphi(x) \le \alpha\varphi(y)$, so $\varphi(y) = 0$, i.e. $y \in \varphi^{-1}(\{0\}) = \{z\}$. Hence $y = z$. □

Theorem 3.3. *Given $\alpha \in (0, \frac{1}{2})$, an arbitrary non-empty set Ω and a set-valued function $F : \Omega \to 2^\Omega$ having a fixed point z such that $F(z) = \{z\}$, the following statements are equivalent:*

a) $\bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega) = \{z\}$;

b) there exists a bounded function $\varphi : \Omega \to [0, \infty)$ such that $\varphi^{-1}(\{0\}) = \{z\}$ and $\sup_{t\in F(x)} \varphi(t) \le \alpha\varphi(x)$ for all $x \in \Omega$;

c) there exists a complete and bounded metric d on Ω such that F is a Nadler set-valued α-contraction with respect to d.

Proof. a)⇒b) Let us consider the bounded function $\varphi : \Omega \to [0, \infty)$ given by $\varphi(x) = \alpha^{n_x}$ for every $x \in \Omega$, where $n_x = \sup\{n \in \mathbb{N} \mid x \in \widehat{F}^n(\Omega)\}$ and we use the convention $\alpha^\infty = 0$. In the view of the hypothesis, $n_x \in \mathbb{N}$ for $x \neq z$ and $n_z = \infty$, so $\varphi^{-1}(\{0\}) = \{z\}$. Moreover, since, for $t \in F(x)$, we have $t \in \widehat{F}(\widehat{F}^{n_x}(\Omega)) = \widehat{F}^{n_x+1}(\Omega)$, so $n_t \geq n_x+1$, we infer that

$$\sup_{t\in F(x)} \varphi(t) = \sup_{t\in F(x)} \alpha^{n_t} \leq \sup_{t\in F(x)} \alpha^{n_x+1} = \alpha \cdot \alpha^{n_x} = \alpha\varphi(x)$$

for all $x \in \Omega$.

b)⇒c) The proof is the same with the one of b)⇒a) from Lemma 3.1, with the remark that

$$\operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} d(x,y) \leq \sup_{x,y\in\Omega} (\varphi(x) + \varphi(y)) \leq 2 \sup_{x\in\Omega} \varphi(x).$$

c)⇒a) According to our hypothesis, we have $\{z\} \subseteq \bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega)$.

Claim. $d(x,y) \leq (2\alpha)^n \operatorname{diam}(\Omega)$ for all $n \in \mathbb{N}^*$, $x, y \in \widehat{F}^n(\Omega)$.

Justification of the claim. We are going to prove the claim by using the method of mathematical induction. If $x, y \in \widehat{F}(\Omega)$, then there exist $u, v \in \Omega$ such that $x \in F(u)$ and $y \in F(v)$, so

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) = d(x, F(z)) + d(y, F(z)) \\ &\leq H(F(u), F(z)) + H(F(v), F(z)) \\ &\leq \alpha d(u,z) + \alpha d(z,y) \leq 2\alpha \operatorname{diam}(\Omega). \end{aligned}$$

Thus the statement is valid for $n = 1$. Now, given $n \in \mathbb{N}^*$, we suppose that the statement is valid for $n-1$ and prove that it is true also for n. Indeed, if $x, y \in \widehat{F}^n(\Omega)$, then there exist $u, v \in \widehat{F}^{n-1}(\Omega)$ such that $x \in F(u)$ and $y \in F(v)$, so

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) = d(x, F(z)) + d(y, F(z)) \\ &\leq H(F(u), F(z)) + H(F(v), F(z)) \\ &\leq \alpha d(u,z) + \alpha d(v,z). \end{aligned}$$

Because $u, v, z \in \widehat{F}^{n-1}(\Omega)$, we get

$$d(u,z) \leq (2\alpha)^{n-1} \operatorname{diam}(\Omega) \text{ and } d(v,z) \leq (2\alpha)^{n-1} \operatorname{diam}(\Omega).$$

So $d(x,y) \leq \alpha d(u,z) + \alpha d(v,z) \leq (2\alpha)^n \operatorname{diam}(\Omega)$. Consequently, the statement is valid for n. The proof of the claim is done.

Based on the claim, we conclude that $\lim_{n\to\infty} \operatorname{diam}(\widehat{F}^n(\Omega)) = 0$, so $\bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega)$ is a singleton, namely $\bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega) = \{z\}$. □

Theorem 3.4. *Let $\alpha \in (0,1)$, an arbitrary non-empty finite set Ω, $F : \Omega \to 2^\Omega$ a set-valued function and $z \in \Omega$ such that $\{z\}$ is the unique fixed point for $\widehat{F}$. Then there exists a complete metric d on Ω such that F is a Nadler set-valued α-contraction with respect to d.*

Proof. We have the following chain of inclusions:

$$\Omega = \widehat{F}^0(\Omega) \supseteq \widehat{F}^1(\Omega) = \widehat{F}(\Omega) \supseteq \widehat{F}^2(\Omega) \supseteq ... \supseteq \widehat{F}^n(\Omega) \supseteq ...,$$

where $n \in \mathbb{N}$ and $z \in \bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega)$. Note that $\widehat{F}^n(\Omega) = \widehat{F}^{n+1}(\Omega)$ if and only if $\widehat{F}^n(\Omega) = \{z\}$. There exists $n \in \mathbb{N}$ such that $\widehat{F}^n(\Omega) = \{z\}$ otherwise we would get the following strictly decreasing sequence of non-negative integers:

$$|\Omega| > \left|\widehat{F}(\Omega)\right| > \left|\widehat{F}^2(\Omega)\right| > ... > \left|\widehat{F}^n(\Omega)\right| > ...$$

where $n \in \mathbb{N}$. This yields a contradiction with the fact that $\mathbb{N}$ is well-ordered. Thus we can consider the smallest $p \in \mathbb{N}$ having the property that $\widehat{F}^p(\Omega) = \{z\}$. To every $x \in \Omega \setminus \{z\}$ we associate $n_x = \max\{n \in \mathbb{N} \mid x \in \widehat{F}^n(\Omega)\} < p$. Moreover, we define $n_z = \infty$. Note that for $t \in F(x)$, we have $t \in \widehat{F}(\widehat{F}^{n_x}(\Omega)) = \widehat{F}^{n_x+1}(\Omega)$, so $n_t \geq n_x + 1$. Considering the function $\varphi : \Omega \to [0, \infty)$ given by $\varphi(x) = \alpha^{n_x}$ for every $x \in \Omega$, with the convention $\alpha^\infty = 0$, we have

$$\sup_{t\in F(x)} \varphi(t) = \sup_{t\in F(x)} \alpha^{n_t} \leq \sup_{t\in F(x)} \alpha^{n_x+1} = \alpha \cdot \alpha^{n_x} = \alpha\varphi(x)$$

for all $x \in \Omega$ and $\varphi^{-1}(\{0\}) = \{z\}$. Hence, the conclusion follows using Lemma 3.1. □

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