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Geometric properties and neighborhood results for a subclass of analytic functions involving Komatu integral

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Abstract. In this paper, a subclass of analytic function is defined using Komatu integral. Coefficient inequalities, Fekete-Szegö inequality, extreme points, radii of starlikeness and convexity and integral means inequality for this class are obtained. Distortion theorem for the generalized fractional integration introduced by Saigo are also obtained. The inclusion relations associated with the (n,μ)-neighborhood also have been found for this class.

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1. Introduction

Let H denote the class of analytic function in the unit disk

$$\Delta = \{z : z \in C, |z| < 1\}$$

on the complex plane C. Let A denote the subclass of H consisting of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in C, |z| < 1\}$.

Also let S be the subclass of A consisting of all univalent functions in Δ normalized by $f(0) = f'(0) - 1 = 0$.

Denote by T the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \;\; a_n \geq 0, \;\; z \in \Delta. \tag{1.2}$$

studied extensively by Silverman [15].

Let f and g are analytic functions defined in Δ. The function f is said to be *subordinate* to g if there exists a Schwarz function w, analytic in Δ with $w(0) = 0$, $|w(z)| < 1$, $z \in \Delta$ such that

$$f(z) = g(w(z)), (z \in \Delta). \tag{1.3}$$

We denote this subordination by $f \prec g$ or $f(z) \prec g(z), (z \in \Delta)$.

In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$, $(z \in \Delta)$.

The convolution or Hadamard product of two functions $f(z)$ given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.4}$$

is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.5}$$

A function $f(z)$ in A is said to be in class $S^*(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$ in Δ, if $\mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} > \alpha$ for $z \in \Delta$. Let $K(\alpha)$ denote the class of all functions $f \in A$ that are convex functions of order α $(0 \leq \alpha < 1)$ in Δ, if $\mathrm{Re}\left\{1 + \dfrac{zf''(z)}{f'(z)}\right\} > \alpha$ for $z \in \Delta$. If $\alpha = 0$, the class $S^*(\alpha)$ reduces to the class S^* of starlike functions and class $K(\alpha)$ reduces to the class of convex functions K. Further, f is convex if and only if $zf'(z)$ is starlike.

Let $\phi(z)$ be an analytic function in Δ with

$$\phi(0) = 1, \ \ \phi'(0) > 0 \ \text{ and } \ \mathrm{Re}(\phi(z)) > 0, \ \ (z \in \Delta) \tag{1.6}$$

which maps the open unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to real axis. Then $S^*(\phi)$ and $K(\phi)$, respectively, be the subclasses of the normalized analytic functions f in class A, which satisfy the following subordination relations:

$$\frac{zf'(z)}{f(z)} \prec \phi(z), (z \in \Delta) \ \text{ and } \ 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), (z \in \Delta)$$

These classes are introduced by Ma and Minda [8]. In their particular case when

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (z \in \Delta; \ 0 \leq \alpha < 1), \tag{1.7}$$

these function classes would reduce, respectively, to the well known classes $S^*(\alpha)$ $(0 \leq \alpha < 1)$ of starlike function of order α in Δ and $K(\alpha)(0 \leq \alpha < 1)$ of convex functions of order α in Δ.

Definition 1.1. [4] The generalized Komatu integral operator $K_c^\delta : A \to A$ is defined for $\delta > 0$ and $c > -1$ as

$$\left(K_c^\delta f\right)(z) = \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta - 1} f(t) dt \tag{1.8}$$

and

$$K_c^0 f(z) = f(z).$$

For $f \in A$, it can be easily verified that

$$\left(K_c^\delta f\right)(z) = z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right)^\delta a_k z^k. \tag{1.9}$$

Based on the earlier works by the authors [1], we introduce the following class.

Definition 1.2. Let $0 \leq \gamma < 1$, $0 \leq \rho < 1$, $\tau \in C \setminus \{0\}$, $\delta > 0$ and $c > -1$. A function $f \in S$ is in the class $R^\tau_{\delta,\gamma,\rho,c}(\phi)$ if

$$1 + \frac{1}{\tau}\left(\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho\right) \prec \phi(z), \quad z \in \Delta, \tag{1.10}$$

where $\phi(z)$ is analytic function in Δ with

$$\phi(0) = 1,\ \phi'(0) > 0 \text{ and } \operatorname{Re}(\phi(z)) > 0. \tag{1.11}$$

If we set $\phi(z) = \frac{1+Az}{1+Bz}$, $(-1 \leq B < A \leq 1, z \in \Delta)$, in (1.10), we get

$$\begin{aligned} &R^\tau_{\delta,\gamma,\rho,c}\left(\frac{1+Az}{1+Bz}\right) = R^\tau_{\delta,\gamma,\rho,c}(A,B) \\ &= \left\{f \in A : \left|\frac{\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho}{\tau(A-B) - B\left(\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho\right)}\right| < 1\right\}, \end{aligned} \tag{1.12}$$

which is again a new class.

Some particular cases of this class discussed in the literature as:

(1) For $\delta = 0, \rho = 1$, the above class reduce to the class $R^\tau_\gamma(A,B)$ introduced by Bansal [3].

(2) For $\delta = 0, \rho = 1$, the class $R^\tau_\gamma(1-2\beta,-1) = R^\tau_\gamma(\beta)$ for $0 \leq \beta < 1, \tau = C\setminus\{0\}$ was discussed recently by Swaminathan [20].

(3) $R^\tau_{0,\gamma,1,c}(1-2\beta,-1)$ with $\tau = e^{i\eta}\cos\eta$ where $-\pi/2 < \eta < \pi/2$ is considered in [11] (see also [10]).

(4) The class $R^\tau_{0,1,1,c}(0,-1)$ with $\tau = e^{i\eta}\cos\eta$ was considered in [5] with reference to the univalency of partial sums.

We denote by $P(\phi)$ the class of normalized functions defined as

$$P(\phi) = \{f \in H : f(0) = 1, f \prec \phi \in \Delta\}.$$

The problem on subordination and convolution were studied by Ruscheweyh in [12] and have found many applications in various fields. One of them is the following theorem due to Ruscheweyh and Stankiewicz [13] which will be useful in this paper.

Theorem 1.3. *Let $F, G \in A$ be any convex univalent functions in Δ. If $f \prec F$ and $g \prec G$, then $f * g \prec F * G$ in Δ.*

Observe that, in Theorem 1.3, nothing is said about the normalization of F and G.

2. Main results

Theorem 2.1. *If $f \in P(\phi) \cap S$, $n \in N$ then $(K_c^\delta)^n f(z) \prec (K_c^\delta)^n \phi(z)$, where K_c^δ is Komatu integral operator.*

Proof. If $f \in P(\phi) \cap S$, then $f(z) \prec \phi(z)$ where $\phi(z)$ is convex univalent function. It is well known that the function

$$h_1(z) = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^\delta z^n, \ (\delta > 0), \tag{2.1}$$

belongs to the class K of convex univalent and normalized function and for $f \in A$

$$\begin{aligned}(f * h_1)(z) &= z + a_2 \left(\frac{c+1}{c+2}\right)^\delta z^2 + a_3 \left(\frac{c+1}{c+3}\right)^\delta z^3 + ... \\ &= \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t) dt \\ &= K_c^\delta f(z).\end{aligned}$$

Therefore the function $h_2(z) = 1 + h_1(z)$ $(z \in \Delta)$ is convex univalent in Δ and for

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$$

$$\begin{aligned}(p * h_2)(z) &= 1 + p_1 z + p_2 \left(\frac{c+1}{c+2}\right)^\delta z^2 + p_3 \left(\frac{c+1}{c+3}\right)^\delta z^3 + ... \\ &= 1 - \left(\frac{c+1}{c}\right)^\delta + \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \left[\frac{\Gamma(\delta) z^c}{c^\delta} + \frac{p_1 z^{c+1} \Gamma(\delta)}{(c+1)^\delta} + \frac{p_2 z^{c+2} \Gamma(\delta)}{(c+2)^\delta} + ...\right] \\ &= 1 - \left(\frac{c+1}{c}\right)^\delta + \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} p(t) dt\end{aligned}$$

Thus, $f \prec \phi$. Applying Theorem 1.3, we obtain

$$\begin{aligned}& f * h_2 \prec \phi * h_2 \\ \Rightarrow & 1 - \left(\frac{c+1}{c}\right)^\delta + \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t) dt \\ & \prec 1 - \left(\frac{c+1}{c}\right)^\delta + \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} \phi(t) dt \\ \Rightarrow & K_c^\delta f \prec K_c^\delta \phi, \ \delta > 0.\end{aligned}$$

Hence, the theorem is true for $n = 1$.
Again by Theorem 1.3,

$$K_c^\delta f * h_2 \prec K_c^\delta \phi * h_2$$

$$\begin{aligned}
&\Rightarrow 1-\left(\frac{c+1}{c}\right)^{\delta}+\frac{(c+1)^{\delta}}{\Gamma(\delta)z^{c}}\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}K_c^{\delta}f(t)dt\\
&\prec 1-\left(\frac{c+1}{c}\right)^{\delta}+\frac{(c+1)^{\delta}}{\Gamma(\delta)z^{c}}\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}K_c^{\delta}\phi(t)dt\\
&\Rightarrow K_c^{\delta}(K_c^{\delta}f)\prec K_c^{\delta}(K_c^{\delta})\phi\\
&\Rightarrow (K_c^{\delta})^2 f\prec (K_c^{\delta})^2\phi.
\end{aligned}$$

Thus, the theorem is true for $n = 2$.
Further, let the theorem is true for $n = m$ i.e.

$$(K_c^{\delta})^m f\prec (K_c^{\delta})^m\phi$$

which on application of Theorem 1.3 gives

$$(K_c^{\delta})^m f * h_2(z)\prec (K_c^{\delta})^m\phi * h_2(z)$$

$$\begin{aligned}
&\Rightarrow 1-\left(\frac{c+1}{c}\right)^{\delta}+\frac{(c+1)^{\delta}}{\Gamma(\delta)z^{c}}\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}(K_c^{\delta})^m f(t)dt\\
&\prec 1-\left(\frac{c+1}{c}\right)^{\delta}+\frac{(c+1)^{\delta}}{\Gamma(\delta)z^{c}}\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}(K_c^{\delta})^m\phi(t)dt\\
&\Rightarrow K_c^{\delta}[(K_c^{\delta})^m f](z)\prec K_c^{\delta}[(K_c^{\delta})^m\phi](z)\\
&\Rightarrow (K_c^{\delta})^{m+1} f(z)\prec (K_c^{\delta})^{m+1}\phi(z).
\end{aligned}$$

The theorem follows by the principle of Mathematical induction. □

Corollary 2.2. *Let $g' \in P(\phi), \alpha < 1$. If we take $\phi(z) = \frac{1-z(2\alpha-1)}{1-z}$, $n = 1$ and*

$$h_1(z)=\sum_{n=1}^{\infty}\frac{2}{n+1}z^n,\ \ (z\in\Delta).$$

Then

$$\frac{1}{z}\int_0^z\frac{g(t)}{t}dt\prec Q(z),\ \ (z\in\Delta),$$

where

$$Q(z)=1+2(1-2\alpha)\left[\frac{z}{2^2}+\frac{z^2}{3^2}+\frac{z^3}{4^2}+...\right]$$

is convex univalent function.

This particular result is given by Janusz Sokol [18].

3. Coefficient inequality

Theorem 3.1. *Let $f \in R_{\gamma}^{\tau}(A, B)$ [3]. Then f is in the class $R^{\tau}_{\delta,\gamma,\rho,c}$ if and only if*

$$\sum_{k=2}^{\infty}(1+B)k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}a_k\le|\tau(A-B)|. \tag{3.1}$$

The result is sharp for the function $f(z)$ given by the following form

$$f(z) = z + \frac{|\tau(A-B)|}{2(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}} z^2. \tag{3.2}$$

Proof. For $|z| = 1$, we have

$$\begin{aligned}
&\left|\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho\right| \\
&- \left|\tau(A-B) - B[\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho]\right| \\
=&\left|\rho\left[1 + \sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}\right] + \gamma z\left[\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta} k(k-1) a_k z^{k-2}\right] - \rho\right| \\
&- \Bigg|\tau(A-B) - B\Bigg[\rho\left\{1 + \sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}\right\} \\
&\qquad\qquad + \gamma z \sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-2} - \rho\Bigg]\Bigg| \\
\leq&\rho\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta} k a_k + \gamma\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta} k(k-1) a_k - |\tau(A-B)| \\
&+ B\left|\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1} + \gamma z\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-2}\right| \\
\leq&\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k + \gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k - |\tau(A-B)| \\
&+ B\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k + B\gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k \\
\leq&(1+B)\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k + (1+B)\gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k - |\tau(A-B)| \\
&\leq 0. \qquad\qquad \text{(By hypothesis)}
\end{aligned}$$

Thus, by maximum modulus theorem, $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$.
Conversely, assume that

$$\left|\frac{\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho}{\tau(A-B) - B\{\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho\}}\right| < 1$$

$$\Rightarrow\left|\frac{\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1} + \gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}}{\tau(A-B) - B\left\{\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1} + \gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}\right\}}\right| < 1.$$

Since $|Re(z)| < |z|$,

$$Re\left[\frac{\sum_{k=2}^{\infty} k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}}{|\tau(A-B)| - B\sum_{k=2}^{\infty} k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}}\right] < 1.$$

By choosing the value of z on the real axis so that $K_c^{\delta} f(z)$ is real. Let $z \to 1^-$ through real values. So we can write as

$$\begin{aligned}\sum_{k=2}^{\infty} k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k &\leq |\tau(A-B)| - B\sum_{k=2}^{\infty} k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k \\ &\leq |\tau(A-B)|\end{aligned}$$

□

Corollary 3.2. *Let $f(z) \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, then*

$$a_k \leq \frac{|\tau(A-B)|}{(1+B)k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}}; \quad k \geq 2.$$

4. Fekete-Szegö inequality

We recall the following lemma to prove our results:

Lemma 4.1. [6] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...(z \in \Delta)$ is a function with positive real part, then for any complex number ε,*

$$|c_3 - \varepsilon c_2^2| \leq 2 \max\{1, |2\varepsilon - 1|\}$$

and the result is sharp for the functions given by

$$p_1(z) = \frac{1+z^2}{1-z^2} \quad or \quad p_1(z) = \frac{1+z}{1-z}.$$

Theorem 4.2. *Let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \tag{4.1}$$

where $\phi(z) \in A$ with $\phi'(0) > 0$.
If $f(z)$ given by (1.1) belongs to $R^{\tau}_{\delta,\gamma,\rho,c}(\phi)$ ($\gamma, \rho \in [0,1)$; $\tau \in C \setminus \{0\}$; $\delta > 0$; $c > -1$), $z \in \Delta$, then for any complex number ν

$$|a_3 - \nu a_2^2| \leq \frac{|\tau|\, B_1}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3\nu B_1 \tau(\rho+2\gamma)(c+2)^{2\delta}}{(\rho+\gamma)^2(c+3)^{\delta}(c+1)^{\delta}}\right|\right\}. \tag{4.2}$$

The result is sharp for the functions $\frac{1+z^2}{1-z^2}$ or $\frac{1+z}{1-z}$.

Proof. If $f(z) \in R^{\tau}_{\delta,\gamma,\rho,c}(\phi)$, then there exists a Schwarz function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \Delta)$ such that

$$1 + \frac{1}{\tau}(\rho\{K_c^{\delta} f(z)\}' + \gamma z\{K_c^{\delta} f(z)\}'' - \rho) = \phi(w(z)). \tag{4.3}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \tag{4.4}$$

Since $w(z)$ is a Schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1$.
Define the function $p(z)$ by

$$p(z) = 1 + \frac{1}{\tau}[\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho] = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \tag{4.5}$$

In view of (4.3), (4.4), (4.5)

$$p(z) = \phi\left[\frac{p_1(z)-1}{p_1(z)+1}\right] = \phi\left[\frac{c_1 z + c_2 z^2 + c_3 z^3 + ...}{2 + c_1 z + c_2 z^2 + c_3 z^3 + ...}\right]$$
$$= \phi\left[\frac{c_1 z}{2} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + ...\right].$$

From equation (4.1)

$$p(z) = 1 + \frac{B_1 c_1 z}{2} + \frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \frac{B_2 c_1^2}{4} z^2 + \dots. \tag{4.6}$$

Now, from (1.9)

$$K_c^\delta f(z) = z + \left(\frac{c+1}{c+2}\right)^\delta a_2 z^2 + \left(\frac{c+1}{c+3}\right)^\delta a_3 z^3 + ...,$$

$$\{K_c^\delta f(z)\}' = 1 + 2\left(\frac{c+1}{c+2}\right)^\delta a_2 z + 3\left(\frac{c+1}{c+3}\right)^\delta a_3 z^2 + ...,$$

and

$$\{K_c^\delta f(z)\}'' = 2\left(\frac{c+1}{c+2}\right)^\delta a_2 + 6\left(\frac{c+1}{c+3}\right)^\delta a_3 z + \dots.$$

From equation (4.5)

$$\begin{aligned} p(z) = 1 + \frac{1}{\tau}\Bigg[&\left\{2\rho\left(\frac{c+1}{c+2}\right)^\delta a_2 + 2\gamma\left(\frac{c+1}{c+2}\right)^\delta a_2\right\} z \\ &+ \left\{3\rho\left(\frac{c+1}{c+3}\right)^\delta a_3 + 6\gamma\left(\frac{c+1}{c+3}\right)^\delta a_3\right\} z^2 + ...\Bigg] \end{aligned} \tag{4.7}$$

Thus from (4.6) and (4.7)

$$\frac{B_1 c_1}{2} = \frac{2(\rho+\gamma)}{\tau}\left(\frac{c+1}{c+2}\right)^\delta a_2 \;\Rightarrow\; a_2 = \frac{B_1 c_1 \tau}{4(\rho+\gamma)}\left(\frac{c+2}{c+1}\right)^\delta$$

$$\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4} = \frac{3a_3(\rho+2\gamma)}{\tau}\left(\frac{c+1}{c+3}\right)^\delta$$

$$\Rightarrow a_3 = \frac{\tau}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^\delta\left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right]$$

Therefore, we have

$$a_3-\nu a_2^2=\frac{\tau}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\right]-\nu\frac{B_1^2c_1^2\tau^2}{4(\rho+\gamma)^2}\left(\frac{c+2}{c+1}\right)^{2\delta}$$

Simplifying, we get

$$a_3-\nu a_2^2=\frac{\tau B_1}{6(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}(c_2-\varepsilon c_1^2),$$

where

$$\varepsilon=\frac{1}{2}\left\{1-\frac{B_2}{B_1}+\frac{3\nu B_1\tau(\rho+2\gamma)(c+2)^{2\delta}}{(\rho+\gamma)^2(c+3)^{\delta}(c+1)^{\delta}}\right\}.$$

Thus

$$|a_3-\nu a_2^2|=\frac{|\tau|B_1}{6(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}|c_2-\varepsilon c_1^2|$$

By application of the Lemma (4.1), we obtain

$$|a_3-\nu a_2^2|\le\frac{2|\tau|B_1}{6(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}\max\{1,|2\varepsilon-1|\}$$

$$|a_3-\nu a_2^2|\le\frac{|\tau|B_1}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}\max\left\{1,\left|\frac{B_2}{B_1}+\frac{3\nu B_1\tau(\rho+2\gamma)(c+2)^{2\delta}}{(\rho+\gamma)^2(c+3)^{\delta}(c+1)^{\delta}}\right|\right\}$$

Equality in (4.2) is obtained when

$$p_1(z)=\frac{1+z^2}{1-z^2}\quad or\quad p_1(z)=\frac{1+z}{1-z}.\qquad\square$$

For class $R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$

$$\phi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-(AB-B^2)z^2+...$$

Thus writing $B_1=A-B$ and $B_2=-B(A-B)$ in the Theorem 3.1, we get the following corollary:

Corollary 4.3. *If $f(z)$ given by (1.1) belongs to $R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, then*

$$|a_3-\nu a_2^2|\le\frac{|\tau|(A-B)}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}\max\left\{1,\left|B-\frac{3\nu(A-B)\tau(\rho+2\gamma)(c+2)^{2\delta}}{(\rho+\gamma)^2(c+3)^{\delta}(c+1)^{\delta}}\right|\right\}.$$

5. Distortion theorem

Saigo's fractional calculus operator $I^{\alpha,\beta,\eta}_{0,z}f(z)$ of $f(z)\in A$ is defined by Srivastava et al. [19] (see also, Saigo [14]) as follows:

Definition 5.1. For real numbers $\alpha>0,\beta$ and η , the fractional integral operator $I^{\alpha\beta,\eta}_{0,z}f(z)$ of $f(z)$ is defined by

$$I^{\alpha\beta,\eta}_{0,z}f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\,{}_2F_1\left[\alpha+\beta,-\eta;\alpha;1-\frac{\zeta}{z}\right]f(\zeta)d\zeta,$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order $f(z) = O(|z|^\epsilon)$ $(z \to 0)$, $\epsilon > \max\{0, \beta - \eta\} - 1$, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

In order to derive the inequalities involving Saigo's fractional operators, we need the following lemma due to Srivastava, Saigo and Owa [19].

Lemma 5.2. *Let $\alpha > 0, \beta$ and η be real. Then, for $k > \max\{0, \beta - \eta\} - 1$,*

$$I_{0,z}^{\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} z^{k-\beta}. \tag{5.1}$$

Theorem 5.3. *Let $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A, B)$, then*

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \le \frac{\Gamma(2-\beta+\eta)|z|^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} \left[1 + \frac{(2-\beta+\eta)|\tau(A-B)||z|}{(2-\beta)(2+\alpha+\eta)(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}}\right] \tag{5.2}$$

and

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \ge \frac{\Gamma(2-\beta+\eta)|z|^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} \left[1 - \frac{(2-\beta+\eta)|\tau(A-B)||z|}{(2-\beta)(2+\alpha+\eta)(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}}\right] \tag{5.3}$$

The equalities in (5.2) and (5.3) are attained for the function $f(z)$ given by (3.2)

Proof. The generalized Saigo [19] fractional integration of $f \in A$ for real numbers $\alpha > 0$, β and η, is given by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} a_k z^{k-\beta}, \qquad (a_1 = 1)$$

$$\Rightarrow \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I_{0,z}^{\alpha,\beta,\eta} f(z) = z + \sum_{k=2}^{\infty} B^{\alpha,\beta,\eta}(k) a_k z^k,$$

where

$$B^{\alpha,\beta,\eta}(k) = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)\Gamma(2-\beta+\eta)}.$$

Therefore,

$$\frac{B^{\alpha,\beta,\eta}(k)}{B^{\alpha,\beta,\eta}(k+1)} = \frac{(k-\beta+1)(k+\alpha+\eta+1)}{(k+1)(k-\beta+\eta+1)} = \frac{1 + \left(\frac{\alpha+\eta}{k+1}\right)}{1 + \left(\frac{\eta}{k-\beta+1}\right)}.$$

Now, $(\alpha+\eta) > \eta$ and $\frac{1}{k+1} > \frac{1}{k-\beta+1}$ for $\beta < 0$. Therefore,

$$\frac{\alpha+\eta}{k+1} > \frac{\eta}{k-\beta+1},$$

and hence

$$B^{\alpha,\beta,\eta}(k) > B^{\alpha,\beta,\eta}(k+1)$$

Therefore, $B^{\alpha,\beta,\eta}(k)$, $\beta < 0$ is decreasing for k, Then

$$B^{\alpha,\beta,\eta}(k) \leq B^{\alpha,\beta,\eta}(2) = \frac{2(2-\beta+\eta)}{(2-\beta)(2+\alpha+\eta)}.$$

By using Theorem 3.1, we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{|\tau(A-B)|}{2(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}}; \quad k \geq 2.$$

Thus

$$\left|\frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I_{0,z}^{\alpha,\beta,\eta} f(z)\right| \leq |z| + B^{\alpha,\beta,\eta}(2)|z|^2 \sum_{k=2}^{\infty} a_k$$

$$\Rightarrow \left|I_{0,z}^{\alpha,\beta,\eta} f(z)\right| \leq \frac{\Gamma(2-\beta+\eta)|z|^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} \left[1+ \frac{(2-\beta+\eta)|\tau(A-B)||z|}{(2-\beta)(2+\alpha+\eta)(1+B)(\rho+\gamma)(\frac{c+1}{c+2})^{\delta}}\right].$$

Following the similar steps as above, we obtain

$$\left|I_{0,z}^{\alpha,\beta,\eta} f(z)\right| \geq \frac{\Gamma(2-\beta+\eta)|z|^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} \left[1- \frac{(2-\beta+\eta)|\tau(A-B)||z|}{(2-\beta)(2+\alpha+\eta)(1+B)(\rho+\gamma)(\frac{c+1}{c+2})^{\delta}}\right].$$

□

6. Extreme points

Theorem 6.1. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{|\tau(A-B)|}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}} z^k.$$

Then $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ *if and only if* $f(z)$ *can be expressed in the form*

$$f(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k f_k(z) \tag{6.1}$$

where

$$\lambda_1 + \sum_{k=2}^{\infty} \lambda_k = 1, \quad (\lambda_1 \geq 0,\ \lambda_k \geq 0).$$

Proof. Let $f(z)$ is given by (6.1). Then

$$f(z) = \lambda_1 z + \sum_{k=2}^{\infty} \lambda_k z + \frac{|\tau(A-B)|}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}} \lambda_k z^k = z + \sum_{k=2}^{\infty} t_k z^k,$$

where

$$t_k = \frac{|\tau(A-B)|\lambda_k}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}}.$$

Now,

$$\sum_{k=2}^{\infty} \frac{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}}{|\tau(A-B)|} t_k = \sum_{k=2}^{\infty} \lambda_k = 1-\lambda_1 < 1.$$

Therefore, $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$.

Conversely, suppose that, $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, then by (3.1)

$$a_k < \frac{|\tau(A-B)|}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}}, \quad k \geq 2.$$

So, if we set

$$\lambda_k = \frac{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k}{|\tau(A-B)|} < 1, \quad k \geq 2$$

and $\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$, then,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k = z + \sum_{k=2}^{\infty} \frac{|\tau(A-B)|}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}} z^k,$$

$$\Rightarrow f(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k f_k(z)$$

which leads to (6.1). □

From the Theorem 6.1, it follows that:

Corollary 6.2. *The extreme points of the class $R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ are the functions $f_1(z)$ and $f_k(z)$, $(k \geq 2)$.*

7. Radii of starlikeness and convexity

Theorem 7.1. *Let $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$. Then $f(z)$ is starlike of order α $(0 \leq \alpha < 1)$ in $|z| < r_1$ where*

$$r_1 = \inf_k \left[\frac{(1-\alpha)k(1+B)\{\rho+\gamma(k-1)\}(\frac{c+1}{c+k})^{\delta}}{(k-\alpha)|\tau(A-B)|}\right]^{\frac{1}{k-1}}.$$

Proof. For $0 \leq \alpha < 1$, we require to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\alpha,$$

that is, for $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$,

$$\frac{\sum_{k=2}^{\infty} a_k (k-1)|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}} < 1 - \alpha$$

or, alternatively $\sum_{k=2}^{\infty} a_k \left(\frac{k-\alpha}{1-\alpha}\right) |z|^{k-1} < 1$, which holds if

$$|z|^{k-1} < \left[\frac{(1-\alpha)k(1+B)\{\rho+\gamma(k-1)\}(\frac{c+1}{c+k})^{\delta}}{(k-\alpha)|\tau(A-B)|}\right].$$

$$\Rightarrow r_1 = \inf_k \left[\frac{(1-\alpha)k(1+B)\{\rho+\gamma(k-1)\}(\frac{c+1}{c+k})^{\delta}}{(k-\alpha)|\tau(A-B)|}\right]^{\frac{1}{k-1}} \qquad \square$$

Noting the fact that $f(z)$ is convex iff $zf'(z)$ is starlike, we have

Theorem 7.2. *Let $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$. Then f is convex of order α $(0 \leq \alpha < 1)$ in $|z| < r_2$ where*

$$r_2 = \inf_k \left[\frac{(1-\alpha)(1+B)\{\rho+\gamma(k-1)\}(\frac{c+1}{c+k})^{\delta}}{(k-\alpha)|\tau(A-B)|}\right]^{\frac{1}{k-1}}.$$

8. Neighborhood results

Definition 8.1. For $f \in A$ of the form (1.1) and $\mu \geq 0$. We define a $(n,\mu)-$neighborhood of a function f by

$$N_{n,\mu}(f) = \left\{g : g \in A, g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \textit{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \mu\right\}. \qquad (8.1)$$

In particular, for the identity function $e(z) = z$, we immediately have

$$N_{n,\mu}(e) = \left\{g : g \in A, g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \textit{and} \sum_{k=n+1}^{\infty} k|b_k| \leq \mu\right\} \qquad (8.2)$$

where $n \in N \setminus \{1\}$.

Theorem 8.2. *If*

$$\mu = \frac{|\tau(A-B)|}{(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}}$$

then,

$$R^{\tau}_{\delta,\gamma,\rho,c}(A,B) \subset N_{n,\mu}(e)$$

Proof. For a function $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ of the form (1.1), Theorem 3.1 immediately yields,

$$\sum_{k=n+1}^{\infty} (1+B)k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k \leq |\tau(A-B)|,$$

where, $n \in N \setminus \{1\}$.

$$\Rightarrow (1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta} \sum_{k=n+1}^{\infty} ka_k \leq |\tau(A-B)|$$

$$\Rightarrow \sum_{k=n+1}^{\infty} ka_k \leq \frac{|\tau(A-B)|}{(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}} = \mu. \qquad \square$$

A function, $f \in A$ is said to be in the class $R^{\tau,\alpha}_{\delta,\gamma,\rho,c}(A,B)$, if there exists a function $g \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha, \ \ (z \in U, \ 0 < \alpha < 1). \tag{8.3}$$

Now, we determine the neighborhood for the class $R^{\tau,\alpha}_{\delta,\gamma,\rho,c}(A,B)$.

Theorem 8.3. *If $g \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ and*

$$\alpha = 1 - \frac{\mu(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}}{n(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta} - |\tau(A-B)|}. \tag{8.4}$$

Then,

$$N_{n,\mu}(g) \subset R^{\tau,\alpha}_{\delta,\gamma,\rho,c}(A,B).$$

Proof. Suppose that, $f \in N_{\mu}(g)$ we then find from the definition (8.1) that,

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \mu,$$

which implies that the coefficient inequality:

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\mu}{n+1} \quad (n \in N).$$

Next since, $g \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, we have

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{|\tau(A-B)|}{(n+1)(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}},$$

so that,

$$\left|\frac{f(z)}{g(z)}-1\right| \le \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty}b_k}$$

$$\le \frac{\mu}{(n+1)\left[1-\frac{|\tau(A-B)|}{(n+1)(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}}\right]}$$

$$\le \frac{\mu(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}}{(n+1)(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}-|\tau(A-B)|} \le 1-\alpha \tag{8.5}$$

provided that α is given precisely by (8.4). Thus by definition $f \in R^{\tau,\alpha}_{\delta,\gamma,\rho,c}(A,B)$ for α given by (8.4). This completes the proof. □

9. Integral means inequality

In 1975, Silverman[15] (see, e.g., [17]) found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T and applied this function to resolve his integral means inequality, conjectured in [16] that

$$\int_0^{2\pi} |f(re^{i\theta})|^{\eta} d\theta \le \int_0^{2\pi} |f_2(re^{i\theta})|^{\eta} d\theta, \tag{9.1}$$

for all $f \in T, \eta > 0$ and $0 < r < 1$ and settled in 1997. He also proved his conjecture for the subclasses $S^*(\alpha)$ and $K(\alpha)$ of T.

Lemma 9.1. [7] *If $f(z)$ and $g(z)$ are analytic in Δ with $f(z) \prec g(z)$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^{\eta} d\theta \le \int_0^{2\pi} |g(re^{i\theta})|^{\eta} d\theta, \tag{9.2}$$

$$\eta \ge 0,\ z = re^{i\theta} \ \text{and} \ \ 0 < r < 1.$$

Application of Lemma (9.1) to function of f in the class $R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, gives the following result.

Theorem 9.2. *Let $\eta > 0$. If $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ is given by (1.1) and $f_2(z)$ is defined by*

$$f_2(z) = z + \frac{|\tau(A-B)|}{2(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}} z^2 \tag{9.3}$$

$$= z + \frac{1}{\phi^A_B(2,\delta,\gamma,\rho,c,\tau)} z^2,$$

where,

$$\phi^A_B(2,\delta,\gamma,\rho,c,\tau) = \frac{2(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}}{|\tau(A-B)|}.$$

then, for $z = re^{i\theta}, 0 < r < 1$*, we have*

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \tag{9.4}$$

Proof. For function f of the form (1.1) is equivalent to proving that

$$\int_0^{2\pi} \left|1 + \sum_{k=2}^{\infty} a_k z^{k-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 + \frac{1}{\phi_B^A(2, \delta, \gamma, \rho, c, \tau)} z\right|^\eta d\theta$$

By Lemma (9.1), it suffices to show that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{1}{\phi_B^A(2, \delta, \gamma, \rho, c, \tau)} z.$$

Setting

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} = 1 + \frac{1}{\phi_B^A(2, \delta, \gamma, \rho, c, \tau)} w(z)$$

and using Theorem 3.1, we obtain

$$|w(z)| \leq \left|\sum_{k=2}^{\infty} \phi_B^A(2, \delta, \gamma, \rho, c, \tau) a_k z^{k-1}\right| \leq |z| \sum_{k=2}^{\infty} \phi_B^A(2, \delta, \gamma, \rho, c, \tau) a_k \leq |z|$$

which completes the proof. □

10. Conclusion

We conclude this paper in view of the function class $R^\tau_{\delta,\gamma,\rho,c}(\phi)$ defined by the subordination relation involving arbitrary coefficients and Komatu integral operator $K_c^\delta : A \to A$ defined for $\delta > 0$ and $c > -1$. The classes defined earlier by Bansal [3], Swaminathan [20], Ponnusamy [11] (see also [10]) and Li [5] follow as special cases of this class defined by the authors. The main result gives sufficient condition for coefficient inequalities. Some particular results in this paper leads to the results given earlier by Sokol [18]. A few geometric properties are obtained for this class.

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