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# A new proof of Ackermann's formula from control theory

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**Abstract.** This paper presents a novel proof for the well known Ackermann's formula, related to pole placement in linear time invariant systems. The proof uses a lemma [3], concerning rank one updates for matrices, often used to efficiently compute the determinants. The proof is given in great detail, but it can be summarised to few lines.

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## 1. Introduction

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $B \in \mathbb{R}^{n\times 1}$, it is known, see [1] that if the marix $Co(A,B) = [B|A \cdot B|\dots|A^{n-1} \cdot B]$ is invertible then there exists a unique $K \in \mathbb{R}^{n\times 1}$ such that $\hat{A} = A + B \cdot K^T$ has any desired set of eigenvalues $\mathcal{S} = \{\lambda_1^*, \dots, \lambda_n^*\}$, closed under complex conjugation, that is if $\lambda \in \mathcal{S}$ then $\bar{\lambda} \in \mathcal{S}$. Algorithms for finding $K$ are well known in literature among which the algorithm of Bass-Gura (see [2]) and Ackerman (see [1]) are mentioned.

In the following a new demonstration to Ackermann's result is given, using a well known lemma often used for computing the determinant of a certain invertible matrix, see [3]. This lemma relates the determinant of a rank-one update to the determinant of the initial matrix. For an elegant proof of this result we point the reader to [3].

**Lemma 1.1 (Matrix determinant lemma,** [3]**).** *Suppose that $A$ is an invertible square matrix and $u$ and $v$ are column vectors. Then:*

$$\det(A + uv^T) = \left(1 + v^T A^{-1} u\right) \det(A) \tag{1.1}$$

## 2. The novel proof for Ackermann's formula

**Theorem 2.1 (Ackermann).** *Let $\dot{X} = A \cdot X + B \cdot u$ be a linear time invariant dynamical system, with $X, B \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. If $Co(A,B) = [B|A \cdot B| \dots |A^{n-1} \cdot B]$ is invertible, then the matrix $\hat{A} = A - B \cdot K_x^T$ has the user-defined eigenvalues $\{\lambda_1^*, \dots, \lambda_p^*\}$, with algebraic multiplicities $q_1, \dots, q_p$, where*

$$\begin{aligned} K_x &= \left(\prod_{i=1}^{p}(A - \lambda_i^* I)^{q_i}\right)^T \cdot Co(A,B)^{-T} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= P^*(A)^T \cdot Co(A,B)^{-T} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \end{aligned}$$

*Proof.* Let $P^*(\lambda) = \prod_{i=1}^{p}(\lambda - \lambda_i^*)^{q_i} = \det(\lambda I - \hat{A})$ denote the characteristic polynomial of $\hat{A}$ and $P(\lambda) = \det(\lambda I - A)$ the characteristic polynomial of $A$. Suppose, for start, that the desired eigenvalues are not already eigenvalues for the system matrix, $A$. Therefore $\det(\lambda_i^* I - A) \neq 0$ for all $i \in \{1, \dots, p\}$. Then, from Lemma 1.1:

$$\begin{aligned} P^*(\lambda) &= \det(\lambda I - \hat{A}) \\ &= \det(\lambda I - (A - BK_x^T)) \\ &= \det((\lambda I - A) + BK_x^T) \\ &= \left(1 + K_x^T(\lambda I - A)^{-1}B\right)\det(\lambda I - A) \\ &= \left(1 + K_x^T(\lambda I - A)^{-1}B\right) \cdot P(\lambda) \end{aligned} \tag{2.1}$$

We are interested in finding $K_x$ such that Equation (2.1) holds. Equation (2.1) is a monic polynomial equality, so it is enough to hold for the roots. Let $\lambda = \lambda_i^*$ in Equation (2.1).

Because $\lambda_i^*$ has multiplicity $q_i$, then the folowing relations are obtained:

$$\begin{cases} K_x^T \cdot (\lambda_i^* I - A)^{-1} \cdot B = -1 \\ K_x^T \cdot (\lambda_i^* I - A)^{-2} \cdot B = 0 \\ \vdots \\ K_x^T \cdot (\lambda_i^* I - A)^{-q_i} \cdot B = 0 \end{cases} \quad \forall i \in \{1, \dots, p\} \tag{2.2}$$

Hence

$$\begin{bmatrix} B^T \cdot (\lambda_1^* I - A^T)^{-1} \\ B^T \cdot (\lambda_1^* I - A^T)^{-2} \\ \vdots \\ B^T \cdot (\lambda_1^* I - A^T)^{-q_1} \\ \vdots \\ B^T \cdot (\lambda_p^* I - A^T)^{-1} \\ B^T \cdot (\lambda_p^* I - A^T)^{-2} \\ \vdots \\ B^T \cdot (\lambda_1^* I - A^T)^{-q_p} \end{bmatrix} \cdot K_x = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.3}$$

Denote

$$C = \begin{bmatrix} (\lambda_1^* I - A)^{-1} \cdot B | & \ldots & | (\lambda_1^* I - A)^{-q_1} \cdot B | & \ldots \end{bmatrix}$$

and

$$N = \begin{bmatrix} -1 & 0 & \ldots & 0 & \ldots & -1 & 0 & \ldots & 0 \end{bmatrix}^T$$

then

$$C^T \cdot K_x = N$$

Looking closely at $C$ one can see:

$$\begin{aligned} \prod_{i=1}^{p} (\lambda_i^* I - A)^{q_i} \cdot C &= \begin{bmatrix} P_1\{\lambda_1^*\}(A) \cdot B | & \ldots & | P_{q_1}\{\lambda_1^*\}(A) \cdot B | & \ldots \end{bmatrix} \\ &= \bar{C} \end{aligned} \tag{2.4}$$

where $P_j\{\lambda_k^*\}(A) = \left( \prod_{i=1, i \neq k}^{p} (\lambda_i^* I - A)^{q_i} \right) \cdot (\lambda_k^* I - A)^{q_k - j}$ with $k \in \overline{1,p}$ and $j \in \overline{1,q_k}$. If seen as a polynomial over $\mathbb{R}$, then it's roots are $\{\lambda_1^*, \ldots, \lambda_k^*, \ldots, \lambda_p^*\}$, with the multiplicity $q_1, \ldots, q_k - j, \ldots, q_p$. The order of the polynomial is $n - j$. Stacking the polynomial's coefficients in a vector, with the coefficient of the smallest power in the first position, and leaving the same name for the vector, one has:

$$\begin{aligned} \bar{C} &= \begin{bmatrix} B | & A \cdot B | & \ldots | & A^{n-1} \cdot B \end{bmatrix} \cdot \\ &\cdot \begin{bmatrix} P_1\{\lambda_1^*\} | & \ldots | & P_{q_1}\{\lambda_1^*\} | & \ldots | & P_1\{\lambda_p^*\} | & \ldots | & P_{q_p}\{\lambda_p^*\} \end{bmatrix} \\ &= Co(A,B) \cdot \mathcal{P} \end{aligned} \tag{2.5}$$

Of course, $\mathcal{P}$ is invertible, since it has linearly independent columns. Indeed let

$$\alpha_1^1 \cdot P_1\{\lambda_1^*\} + \ldots + \alpha_1^p \cdot P_1\{\lambda_p^*\} + \ldots = 0$$

be a null linear combination of the columns of $\mathcal{P}$. Suppose the polynomial's variable is $X$. Let $k \in \overline{1,p}$ and let $\alpha_j^k$ be the the coefficient of the polynomial having $\lambda_k^*$ as a root with the smallest multiplicity $m_k$. Differentiating the above linear combination, $m_k$ times, with respect to $X$, then replacing $X$ with $\lambda_k^*$, will yield $\alpha_{q_k}^k = 0$. Repeating the process will conclude that the polynomials are linear independent. Hence:

$$C^{-T} = \left( \prod_{i=1}^{p} (\lambda_i^* I - A)^{q_i} \right)^T \cdot Co(A,B)^{-T} \cdot \mathcal{P}^{-T} \tag{2.6}$$

therefore

$$\begin{aligned} K_x &= \left(\prod_{i=1}^{p}(A-\lambda_i^* I)^{q_i}\right)^T \cdot Co(A,B)^{-T}\cdot(-1)^n\cdot\mathcal{P}^{-T}\cdot N \\ &= P^*(A)^T\cdot Co(A,B)^{-T}\cdot(-1)^n\cdot\mathcal{P}^{-T}\cdot N \end{aligned} \tag{2.7}$$

Denote $V=(-1)^n\cdot\mathcal{P}^{-T}\cdot N$ therefore $(-1)^n\cdot\mathcal{P}^{T}\cdot V=N$. Because $\mathcal{P}$ is invertible, $V$ is unique.

$$(-1)^n\cdot\begin{bmatrix} P_1\{\lambda_1^*\}^T \\ P_2\{\lambda_1^*\}^T \\ \vdots \\ P_{q_1}\{\lambda_1^*\}^T \\ \vdots \\ P_1\{\lambda_p^*\}^T \\ P_2\{\lambda_p^*\}^T \\ \vdots \\ P_{q_p}\{\lambda_p^*\}^T \end{bmatrix}\cdot\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.8}$$

Because $P_j\{\lambda_k^*\}$ has the order $n-j$, and the coefficient of the smallest power is on the first position in vector, that is the coefficient of the greatest power is on the last position, follows:

$$(-1)^n\cdot\begin{bmatrix} \dots & (-1)^{n-1} \\ \dots & 0 \\ \vdots & \vdots \\ \dots & 0 \\ \vdots & \vdots \\ \dots & (-1)^{n-1} \\ \dots & 0 \\ \vdots & \vdots \\ \dots & 0 \end{bmatrix}\cdot\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.9}$$

It is easy to see that $V=[0,\dots,0,1]^T$ is a solution. Therefore

$$K_x=P^*(A)^T\cdot Co(A,B)^{-T}\cdot V \tag{2.10}$$

If $\lambda_i^*=\lambda_i$, for some $i\in\overline{1,p}$, then take $\lambda_i^*(\epsilon)=\epsilon+\lambda_i^*$ to obtain

$$\det(\lambda I-(A-B\cdot K_x(\epsilon)^T))=P^*\{\epsilon\}(\lambda).$$

Letting $\epsilon\longrightarrow 0$, one has $\det(\lambda I-(A-B\cdot K_x^T))=P^*(\lambda)$. $\square$

## 3. Conclusions

A new proof for the well known Akermann's formula was presented. The proof uses a matrix lemma, giving an in depth look at the mechanics of eigenvalues change using rank one updates. The state feedback matrix $K_x$ is shown to be the unique solution to a system of equations, obtained using a well known matrix lemma. The proof can be summarised as follows:

1. Use Equation (2.1) to obtaing Equation (2.3)
2. Use Equations (2.4) and (2.5) to obtain Equation (2.6) regardind the resolvent matrix
3. Use Equation (2.8) and (2.9) in Equation (2.7) to obtain $K_x$

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