

Bounds on third Hankel determinant for certain classes of analytic functions

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Abstract. In this paper, the estimate for the third Hankel determinant $H_{3,1}(f)$ of Taylor coefficients of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, belonging to certain classes of analytic functions in the open unit disk \mathbb{D} , are investigated.

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1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and \mathcal{A} be the class of functions $f \in \mathcal{H}(\mathbb{D})$, having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \quad (1.1)$$

with the standard normalization $f(0) = 0$, $f'(0) = 1$. We denote by \mathcal{S} , the subclass of \mathcal{A} consisting of functions which are also univalent in \mathbb{D} , and \mathcal{P} denotes the class of functions $p \in \mathcal{H}(\mathbb{D})$ with $\Re(p(z)) > 0$, $z \in \mathbb{D}$.

A function $f \in \mathcal{A}$ is called starlike (with respect to origin 0), if f is univalent in \mathbb{D} and $f(\mathbb{D})$ is a starlike domain. We denote this class of starlike functions by \mathcal{S}^* . A function $f \in \mathcal{S}$ maps the unit disk \mathbb{D} onto a convex domain is called convex function, and this class of functions is denoted by \mathcal{K} . Let $\mathcal{M}(\lambda)$ be the subclass of \mathcal{A} consisting of functions $f(z)$ which satisfy the inequality

$$\Re \left(\frac{z f'(z)}{f(z)} \right) < \lambda, \quad z \in \mathbb{D}, \quad (1.2)$$

for some $\lambda (\lambda > 1)$. And let $\mathcal{N}(\lambda)$ be the subclass of \mathcal{A} consisting of functions $f(z)$ if and only if $zf'(z) \in \mathcal{M}(\lambda)$, i.e. $f(z)$ satisfy the inequality

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < \lambda, \quad z \in \mathbb{D}, \tag{1.3}$$

for some $\lambda (\lambda > 1)$. These classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated recently by Nishiwaki and Owa [19] (see also [23]). For $1 < \lambda \leq 4/3$, the classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated by Uralegaddi *et al.* [32].

Throughout the present paper, by \mathcal{M} we always mean the class of functions $\mathcal{M}(3/2)$, and by \mathcal{N} we always mean the class of functions $\mathcal{N}(3/2)$. Ozaki [24] proved that functions in \mathcal{N} are univalent in \mathbb{D} . Moreover, if $f \in \mathcal{N}$, then (see e.g. [11, Theorem 1] and [21, p. 196]) one have

$$\frac{zf'(z)}{f(z)} \prec g(z) = \frac{2(1-z)}{2-z}, \quad z \in \mathbb{D},$$

where \prec denotes the subordination [18]. We see that g above is univalent in \mathbb{D} and maps \mathbb{D} onto the disk $|w - (2/3)| < 2/3$. Thus, functions in \mathcal{M} are starlike in \mathbb{D} .

For $f \in \mathcal{A}$ of the form (1.1), a classical problem settled by Fekete and Szegö [9] is to find the maximum value of the coefficient functional $\Phi_\lambda(f) := a_3 - \lambda a_2^2$ for each $\lambda \in [0, 1]$, over the function $f \in \mathcal{S}$. By applying the Löwner method they proved that

$$\max_{f \in \mathcal{S}} |\Phi_\lambda(f)| = \begin{cases} 1 + 2 \exp\left(\frac{-2\lambda}{1-\lambda}\right), & \lambda \in [0, 1), \\ 1, & \lambda = 1. \end{cases}$$

The problem of calculating the maximum of the coefficient functional $\Phi_\lambda(f)$ for various compact subfamilies of \mathcal{A} , as well as λ being an arbitrary real or complex number, has been studied by many authors (see e.g. [1, 12, 13, 17, 30, 31]).

We denote by $H_{q,n}(f)$ where $n, q \in \mathbb{N} = \{1, 2, \dots\}$, the Hankel determinant of functions $f \in \mathcal{A}$ of the form (1.1), which is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (a_1 = 1). \tag{1.4}$$

The Hankel determinant $H_{q,n}(f)$ has been studied by several authors including Cantor [6], Noonan and Thomas [20], Pommerenke [26, 25], Hayman [10], Ehrenborg [8], which are useful, in showing that a function of bounded characteristic in \mathbb{D} .

Indeed, $H_{2,1}(f) = \Phi_1(f)$ is the Fekete-Szegö coefficient functional. Many authors have studied the problem of calculating $\max_{f \in \mathcal{F}} |H_{2,2}(f)|$ for various subfamily \mathcal{F} of the class $f \in \mathcal{A}$ (see e.g. [2, 4, 14]). Recently, several authors including Babalola [3], Bansal *et al.* [5], Prajapat *et al.* [28], Raza and Malik [29] have obtained the bounds on the third Hankel determinant $H_{3,1}(f)$ for certain families of analytic functions,

which is defined by

$$\begin{aligned}
 H_{3,1}(f) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\
 &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \tag{1.5}
 \end{aligned}$$

In the present paper, we investigate the bounds on $H_{3,1}(f)$ for the functions belonging to the classes \mathcal{M} and \mathcal{N} defined above. In order to get the main results, we need the following known results.

Lemma 1.1. ([16]) *If $p \in \mathcal{P}$ be of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

and

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 1.2. ([22, Theorem 1]) *If $f \in \mathcal{N}$ be given by (1.1), then*

$$|a_n| \leq \frac{1}{n(n-1)}, \quad n \geq 2.$$

The result is sharp for the function f_n such that $f'_n(z) = (1 - z^{n-1})^{1/(n-1)}$, $n \geq 2$.

As it is known that, if $f(z) \in \mathcal{N}$ then $zf'(z) \in \mathcal{M}$, therefore from Lemma 1.2, we conclude that

Lemma 1.3. *If $f(z) \in \mathcal{M}$ be given by (1.1), then*

$$|a_n| \leq \frac{1}{n-1}, \quad n \geq 2.$$

The result is sharp for the function $g_n(z) = z(1 - z^{n-1})^{1/(n-1)}$, $n \geq 2$.

Lemma 1.4. ([22, Corollary 2]) *If $f \in \mathcal{N}$ be given by (1.1), then*

$$|a_3 - a_2^2| \leq 1/4.$$

Equality is attained for the function f such that $f'(z) = (1 - z^2 e^{i\theta})^{1/2}$, $\theta \in [0, 2\pi]$.

2. Main results

Our first main result is contained in the following theorem:

Theorem 2.1. *Let the function $f \in \mathcal{M}$ be given by (1.1), then*

$$|a_3 - a_2^2| \leq 1. \tag{2.1}$$

The result (2.1) is sharp and equality in (2.1) is attained for the function

$$e_1(z) = z - z^2.$$

Proof. If the function $f \in \mathcal{M}$ be given by (1.1), then we may write

$$\frac{zf'(z)}{f(z)} = \frac{3}{2} - \frac{1}{2}p(z), \tag{2.2}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is analytic in \mathbb{D} and $\Re(p(z)) > 0$ in \mathbb{D} . Also, we have $|c_n| \leq 2$ for all $n \geq 1$ (see [7]). In terms of power series expansion, the last identity is equivalent to

$$\sum_{n=1}^{\infty} n a_n z^n = \left(1 - \frac{1}{2} \sum_{n=1}^{\infty} c_n z^n\right) \left(\sum_{n=1}^{\infty} a_n z^n\right),$$

where $a_1 = 1$. Equating the coefficients of z^n on both sides, we deduce that

$$a_2 = -\frac{1}{2}c_1, \quad a_3 = \frac{1}{8}(c_1^2 - 2c_2), \quad a_4 = \frac{1}{48}(6c_1c_2 - 8c_3 - c_1^3). \tag{2.3}$$

Now using Lemma 1.1 for some x such that $|x| \leq 1$, we have

$$|a_3 - a_2^2| = \left| \frac{1}{8}(c_1^2 - 2c_2) - \frac{1}{4}c_1^2 \right| = \frac{1}{8}|2c_2^2 + x(4 - c_1^2)|.$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_3 - a_2^2| &\leq \frac{1}{8}[2c^2 + \mu(4 - c^2)] \\ &= F_1(c, \mu). \end{aligned}$$

Let $\Omega = \{(c, \mu) : 0 \leq c \leq 2 \text{ and } 0 \leq \mu \leq 1\}$. Differentiating F_1 with respect to μ , we get

$$\frac{\partial F_1}{\partial \mu} = \frac{1}{8}(4 - c^2) \geq 0 \quad \text{for } 0 \leq \mu \leq 1.$$

Therefore $F_1(c, \mu)$ is a non-decreasing function of μ on the closed interval $[0, 1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0 \leq \mu \leq 1} F_1(c, \mu) = F_1(c, 1) = \frac{c^2 + 4}{8} = G_1(c).$$

We observe that $G_1(c)$ is an increasing function in $[0, 2]$, so it will attain maximum value at $c = 2$. Next, to find the critical point on the boundary of Ω , we examine all the four line segments of Ω . Along the line segment $c = 2$ with $0 \leq \mu \leq 1$, we have $F_1(c, \mu) = F_1(2, \mu) = 1$, which is a constant, thus every point on the line segment is the critical point. For the line segment $c = 0$ with $0 \leq \mu \leq 1$, we have $F_1(c, \mu) = F_1(0, \mu) = \mu/2$. For the line segment $\mu = 0$ with $0 \leq c \leq 2$, we have $F_1(c, \mu) = F_1(c, 0) = c^2/4$, which gives the critical point $(0, 0)$ and $F_1(0, 0) = 0$. Also, for the line segment $\mu = 1$ with $0 \leq c \leq 2$, we have $F_1(c, \mu) = F_1(c, 1) = (c^2 + 4)/8$, which gives another critical point $(0, 1)$ and $F_1(0, 1) = 1/2$.

Putting this all together we can conclude that the maximum of $F_1(c, \mu)$ lie at each point along the line segment $c = 2$ with $0 \leq \mu \leq 1$, which can also be verified

through the mathematica plot of $F_1(c, \mu)$ over the region Ω given below in the Figure 1. Hence

$$\max_{\Omega} F_1(c, \mu) = F_1(2, \mu) = 1.$$

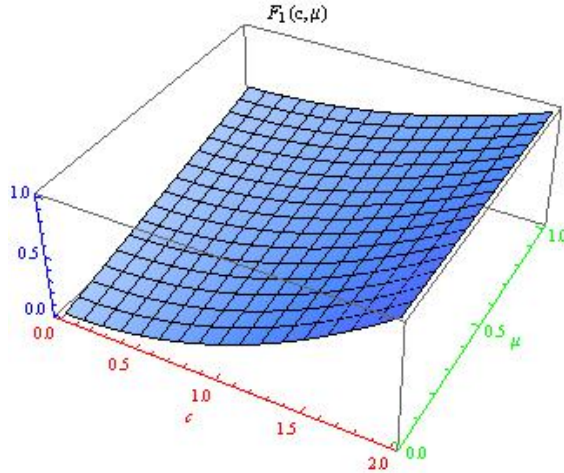


FIGURE 1. Mapping of $F_1(c, \mu)$ over Ω

To find the extremal function, setting $c_1 = 2$ and $x = 1$ in Lemma 1.1, we find that $c_2 = c_3 = 2$, using these values in (2.3), we get that $a_2 = -1$ and $a_3 = a_4 = 0$, therefore the extremal function would be $e_1(z) = z - z^2$. A simple calculation shows that $e_1(z) \in \mathcal{M}$. This complete the proof of Theorem 2.1. \square

Theorem 2.2. *Let the function $f \in \mathcal{M}$ be given by (1.1), then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}. \tag{2.4}$$

The result (2.4) is sharp and equality is attained for the function

$$e_2(z) = z - \frac{1}{2}z^3 \text{ and } e_3(z) = z(1 - z^2)^{1/2}.$$

Proof. Using (2.3) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| -\frac{1}{96}c_1(6c_1c_2 - 8c_3 - c_1^3) - \frac{1}{64}(c_1^2 - 2c_2)^2 \right| \\ &= \frac{1}{192} \left| -3x^2(4 - c_1^2)^2 + 2c_1^2x(4 - c_1^2) - 4c_1^2x^2(4 - c_1^2) \right. \\ &\quad \left. + 8c_1(4 - c_1^2)(1 - |x|^2)z \right|. \end{aligned}$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Thus applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq \frac{1}{192} [(4 - c^2)\{3\mu^2(4 - c^2) + 2c^2\mu + 4\mu^2c^2 + 8c(1 - \mu^2)\}] \\ &= \frac{1}{192} [(4 - c^2)\{(12 - 8c + c^2)\mu^2 + 2c^2\mu + 8c\}] \\ &= F_2(c, \mu). \end{aligned}$$

Differentiating $F_2(c, \mu)$ in the above equation with respect to μ , we get

$$\frac{\partial F_2}{\partial \mu} = \frac{(4 - c^2)}{96} \{(12 - 8c + c^2)\mu + c^2\} \geq 0 \quad \text{for } 0 \leq \mu \leq 1.$$

Therefore $F_2(c, \mu)$ is a non-decreasing function of μ on closed interval $[0, 1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0 \leq \mu \leq 1} F_2(c, \mu) = F_2(c, 1) = \frac{16 - c^4}{64} = G_2(c).$$

We observe that $G_2(c)$ is a decreasing function in $[0, 2]$, so it will attain maximum value at $c = 0$. Next, to find the critical point on the boundary of Ω , we examine all the four line segments of Ω by the earlier method used in Theorem 2.1, and we are getting $(0, 0)$, $(2/\sqrt{3}, 0)$ and $(0, 1)$ are the critical points and $F_2(0, 0) = 0$, $F_2(2/\sqrt{3}, 0) = 2/9\sqrt{3}$ and $F_2(0, 1) = 1/4$. Therefore maximum value of $F_2(c, \mu)$ is obtained by putting $c = 0$ and $\mu = 1$, which can also be verified through the mathematica plot of $F_2(c, \mu)$ over Ω given below in Figure 2. Hence

$$\max_{\Omega} F_2(c, \mu) = F_2(0, 1) = \frac{1}{4}.$$

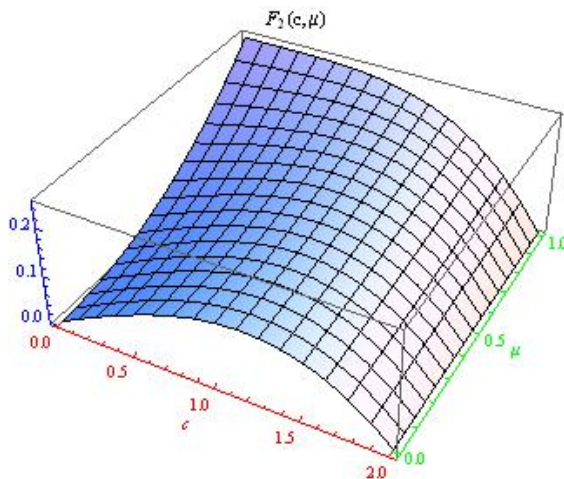


FIGURE 2. Mapping of $F_2(c, \mu)$ over Ω

Now, to find extremal function, set $c_1 = 0$ and selecting $x = 1$ in Lemma 1.1, we find that $c_2 = 2$ and $c_3 = 0$. Using these values in (2.3), we get $a_2 = a_4 = 0$ and $a_3 = 1/2$, therefore one of the extremal function of (2.4) would be $e_2(z) = z - \frac{1}{2}z^3$. We can also see that equality in (2.4) is attended for the function $e_3(z) = z(1 - z^2)^{1/2} \in \mathcal{M}$. A simple calculation shows that $e_2 \in \mathcal{M}$ and $e_3 \in \mathcal{M}$. This complete the proof of Theorem 2.2. \square

Theorem 2.3. *Let the function $f \in \mathcal{M}$ be given by (1.1), then*

$$|a_2a_3 - a_4| \leq \frac{2\sqrt{3}}{9}. \tag{2.5}$$

Proof. Using (2.3) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned} |a_2a_3 - a_4| &= \left| \frac{1}{16}c_1(c_1^2 - 2c_2) + \frac{1}{48}(6c_1c_2 - 8c_3 - c_1^3) \right| \\ &= \frac{1}{24} |2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z|. \end{aligned}$$

As $|c_1| \leq 2$, letting $c_1 = c$, assume without restriction that $c \in [0, 2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{(4 - c^2)}{24} [2 + 2c\mu + (c - 2)\mu^2] \\ &= F_3(c, \mu). \end{aligned}$$

To find the maximum of F_3 over the region Ω , differentiating F_3 with respect to μ and c , we get

$$\frac{\partial F_3}{\partial \mu} = \frac{(4 - c^2)}{12} [c + (c - 2)\mu] \tag{2.6}$$

$$\frac{\partial F_3}{\partial c} = \frac{1}{24} [-4c + (8 - 6c^2)\mu + (4 + 4c - 3c^2)\mu^2]. \tag{2.7}$$

A critical point of $F_3(c, \mu)$ must satisfy $\frac{\partial F_3}{\partial \mu} = 0$ and $\frac{\partial F_3}{\partial c} = 0$. The condition $\frac{\partial F_3}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = -c/(c - 2)$. The interior point (c, μ) of Ω satisfying such condition in only $(0, 0)$, and at that point $(0, 0)$, we have

$$\left(\frac{\partial^2 F_3}{\partial \mu^2} \right) \left(\frac{\partial^2 F_3}{\partial c^2} \right) - \left(\frac{\partial^2 F_3}{\partial c \partial \mu} \right)^2 = 0.$$

Hence, it is not certain that at $(0, 0)$ function have maximum value in Ω . Since Ω is closed and bounded and F_3 is continuous, the maximum of F_3 shall be attained on the boundary of Ω . Along the line segment $c = 2$ with $0 \leq \mu \leq 1$, we have $F_3(c, \mu) = F_3(2, \mu) = 0$, which is a constant. For the line segment $c = 0$ with $0 \leq \mu \leq 1$, we have $F_3(c, \mu) = F_3(0, \mu) = (1 - \mu^2)/3$, which gives the same critical point $(0, 0)$ and $F_3(0, 0) = 1/3$. For the line segment $\mu = 0$ with $0 \leq c \leq 2$, we have $F_3(c, \mu) = F_3(c, 0) = (4 - c^2)/12$, which gives the same critical point $(0, 0)$. Also, for the line segment $\mu = 1$ with $0 \leq c \leq 2$, we have $F_3(c, \mu) = F_3(c, 1) = (4c - c^3)/8$,

which gives another critical point $(2/\sqrt{3}, 1)$ on this line and $F_3(2/\sqrt{3}, 1) = 2\sqrt{3}/9$. Therefore, the point $(0, 0)$ and $(2/\sqrt{3}, 1)$ are the only critical points of F_3 over Ω . Hence, the largest value of $F_3(c, \mu)$ over the region Ω lies at $(2/\sqrt{3}, 1)$ and

$$\max_{\Omega} F_3(c, \mu) = F_3(2/\sqrt{3}, 1) = \frac{2\sqrt{3}}{9}. \quad \square$$

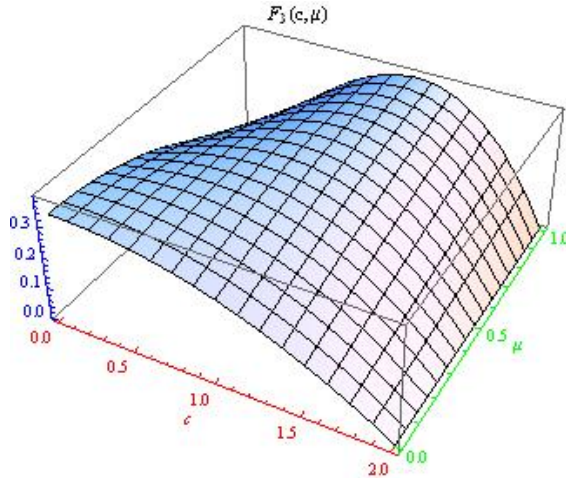


FIGURE 3. Mapping of $F_3(c, \mu)$ over Ω

Theorem 2.4. *Let the function $f \in \mathcal{M}$ be given by (1.1), then*

$$|H_{3,1}(f)| \leq \frac{81 + 16\sqrt{3}}{216}.$$

Proof. Using Lemma 1.3, Theorem 2.1, Theorem 2.2, Theorem 2.3 and the triangle inequality on $H_{3,1}(f)$, we get

$$\begin{aligned} |H_{3,1}(f)| &\leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \\ &\leq \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2\sqrt{3}}{9} + \frac{1}{4} \cdot 1 = \frac{81 + 16\sqrt{3}}{216}. \end{aligned}$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let the function $f \in \mathcal{N}$ be given by (1.1), then*

$$|a_2a_3 - a_4| \leq \frac{1}{12}. \tag{2.8}$$

The result (2.8) is sharp and equality in (2.8) is attained for the function e_4 where $e'_4(z) = (1 - z^3)^{1/3}$.

Proof. Let the function $f \in \mathcal{N}$ be given by (1.1), then by definitions it is clear that $f(z) \in \mathcal{N}$ if and only if $zf'(z) \in \mathcal{M}$, thus replacing a_n by na_n in (2.3), we get

$$a_2 = -\frac{1}{4}c_1, \quad a_3 = \frac{1}{24}(c_1^2 - 2c_2), \quad a_4 = \frac{1}{192}(6c_1c_2 - 8c_3 - c_1^3). \quad (2.9)$$

Now using (2.9) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned} |a_2a_3 - a_4| &= \left| -\frac{1}{96}c_1(c_1^2 - 2c_2) - \frac{1}{192}(6c_1c_2 - 8c_3 - c_1^3) \right| \\ &= \frac{1}{192} |3c_1x(4 - c_1^2) - 2c_1x^2(4 - c_1^2) + 4(4 - c_1^2)(1 - |x|^2)z|. \end{aligned}$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{(4 - c^2)}{192} [4 + 3c\mu + 2(c - 2)\mu^2] \\ &= F_4(c, \mu). \end{aligned}$$

Following the earlier method used in Theorem 2.3, we can show that the global maximum of $F_4(c, \mu)$ over the region Ω is achieved at $(0, 0)$ and $F_4(0, 0) = 1/12$. This can also be verified through the mathematica plot of $F_4(c, \mu)$ over Ω given below in Figure 4.

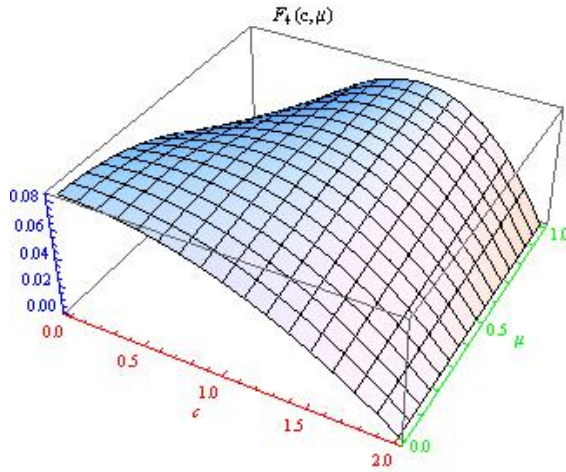


FIGURE 4. Mapping of $F_4(c, \mu)$ over Ω

Also observe that equality in (2.8) is attained for the function e_4 where

$$e_4'(z) = (1 - z^3)^{1/3}.$$

A computation shows that $e_4 \in \mathcal{N}$. Hence the result is obtained. □

Theorem 2.6. *Let the function $f \in \mathcal{N}$ be given by (1.1), then*

$$|a_2a_4 - a_3^2| \leq \frac{9}{320}. \tag{2.10}$$

Proof. Using (2.9) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{1}{192} \left| -\frac{1}{4}c_1(6c_1c_2 - 8c_3 - c_1^3) - \frac{1}{3}(c_1^2 - 2c_2)^2 \right| \\ &= \frac{1}{192} \left| \frac{1}{12}c_1^4 + \frac{1}{6}c_1^2c_2 + \frac{4}{3}c_2^2 - 2c_1c_3 \right| \\ &= \frac{1}{2304} |3xc_1^2(4 - c_1^2) - 6x^2c_1^2(4 - c_1^2) + 12zc_1(4 - c_1^2)(1 - |x|^2) \\ &\quad - 4(4 - c_1^2)^2x^2|. \end{aligned}$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Thus applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{(4 - c^2)}{2304} \{12c + 3c^2\mu + 2(8 - 6c + c^2)\mu^2\} \\ &= F_5(c, \mu). \end{aligned}$$

Differentiating $F_5(c, \mu)$ with respect to μ , we get

$$\frac{\partial F_5}{\partial \mu} = \frac{(4 - c^2)}{2304} \{4\mu(c^2 - 6c + 8) + 3c^2\} \geq 0 \quad \text{for } 0 \leq \mu \leq 1.$$

Therefore $F_5(c, \mu)$ is a non-decreasing function of μ on closed interval $[0, 1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0 \leq \mu \leq 1} F_5(c, \mu) = F_5(c, 1) = \frac{1}{2304} (64 + 4c^2 - 5c^4) = G_5(c).$$

We can see that $G_5(c)$ is an increasing function in $[0, \sqrt{2/5}]$, so $G_5(c)$ attains maximum value at $c = \sqrt{2/5}$. Next, to find the critical points on the boundary of Ω , we examine all the four line segments of Ω by the earlier method used in Theorem 2.1 and 2.3, and we get $(0, 0)$, $(2/\sqrt{3}, 0)$ and $(0, 1)$ are the critical points and $F_5(0, 0) = 0$, $F_5(2/\sqrt{3}, 0) = 1/36\sqrt{3}$ and $F_5(0, 1) = 1/36$. Therefore $F_5(c, \mu)$ have maximum value at $\mu = 1$ and $c = \sqrt{2/5}$ in the region Ω . Thus

$$\max_{\Omega} F_5(c, \mu) = F_5(\sqrt{2/5}, 1) = \frac{9}{320}.$$

This completes the proof of Theorem 2.6. □

Remark 2.7. For $f \in \mathcal{S}$, Thomas [27, p. 166] conjectured that

$$|H_{2,n}(f)| = |a_n a_{n+2} - a_{n+1}^2| \leq 1, \quad n = 2, 3, 4, \dots$$

Subsequently, Li and Srivastava [15, p. 1040] shown that this conjecture is not valid for $n \geq 4$, i.e. conjecture is valid only for $n = 2, 3$. From Theorem 2.6, we found that, if function f is member of class \mathcal{N} and having form (1.1), then $|H_{2,2}(f)| \leq 9/320$.

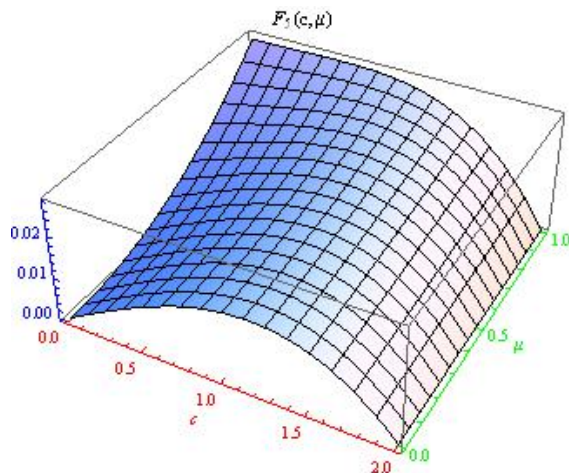


FIGURE 5. Mapping of $F_5(c, \mu)$ over Ω

Since all functions in \mathcal{N} are univalent in \mathbb{D} . Therefore, Theorem 2.6 validates the Thomas conjecture when $n = 2$ for the function belonging to the classes \mathcal{N} .

Theorem 2.8. *Let the function $f \in \mathcal{N}$ be given by (1.1), then*

$$|H_{3,1}(f)| \leq \frac{139}{5760}.$$

Proof. Using Lemma 1.2, Lemma 1.4, Theorem 2.5, Theorem 2.6 and the triangle inequality on $H_{3,1}(f)$, we get

$$\begin{aligned} |H_{3,1}(f)| &\leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \\ &\leq \frac{1}{6} \frac{9}{320} + \frac{1}{12} \frac{1}{12} + \frac{1}{20} \frac{1}{4} = \frac{139}{5760}. \end{aligned}$$

This completes the proof of Theorem 2.8. □

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