

# Majorization for certain classes of analytic functions defined by convolution structure

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**Abstract.** In this paper, we investigate majorization properties for certain classes of analytic functions defined by convolution structure. Also we point out some new and known consequences of our main result.

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## 1. Introduction

Let  $f(z)$  and  $g(z)$  be analytic in the open unit disc  $U = \{z \in \mathbb{C} : |z| < 1\}$ .

For analytic function  $f(z)$  and  $g(z)$  in  $U$ , we say that  $f(z)$  is majorized by  $g(z)$  in  $U$  (see [10]) and write

$$f(z) \ll g(z) \quad (z \in U), \quad (1.1)$$

if there exists a function  $\varphi(z)$ , analytic in  $U$  such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in U). \quad (1.2)$$

It may be noted that (1.1) is closely related to the concept of quasi-subordination between analytic functions.

If  $f(z)$  and  $g(z)$  are analytic functions in  $U$ , we say that  $f(z)$  is subordinate to  $g(z)$ , written symbolically as  $f(z) \prec g(z)$  if there exists a Schwarz function  $w$ , which (by definition) is analytic in  $U$  with  $w(0) = 0$  and  $|w(z)| < 1$  for all  $z \in U$ , such that  $f(z) = g(w(z))$ ,  $z \in U$ . Furthermore, if the function  $g(z)$  is univalent in  $U$ , then we have the following equivalence, (see [11, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let  $A(p)$  denote the class of functions  $f(z)$  of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N} = \{1, 2, \dots\}) \quad (1.3)$$

which are analytic and  $p$ -valent in the open unit disc. We note that  $A(1) = A$ . Let  $g(z) \in A(p)$ , be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of  $f(z)$  and  $g(z)$  is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.4}$$

For  $\lambda, \ell \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and  $f(z), g(z) \in A(p)$ , A. O. Mostafa, [12] defined the linear operator  $D_{\lambda, \ell, p}^m(f * g)$  as follows:

$$D_{p, \ell, \lambda}^m(f * g) = z^p + \sum_{k=p+1}^{\infty} \left[ \frac{p + \ell + \lambda(k - p)}{p + \ell} \right]^m a_k b_k z^k. \tag{1.5}$$

From (1.5), it is easy to verify that ( see [12]),

$$\lambda z (D_{\lambda, \ell, p}^m(f * g)(z))' = (\ell + p) D_{\lambda, \ell, p}^{m+1}(f * g)(z) - [p(1 - \lambda) + \ell] D_{\lambda, \ell, p}^m(f * g)(z). \tag{1.6}$$

We note that:

(i) For  $b_k = 1$  or  $g(z) = \frac{z^p}{1-z}$  we have  $D_{\lambda, \ell, p}^m f(z) = I_p^m(\lambda, \ell) f(z)$ , where the operator  $I_p^m(\lambda, \ell)$  was introduced and studied by Cătaş [4], which contains intern the operators  $D_p^m$ , (see [2] and [8]) and  $D_{\lambda}^m$  (see [1]).

(ii) For  $b_k = \frac{(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \dots (\beta_s)_{k-p} (1)_{k-p}}$ , the operator

$$D_{\lambda, \ell, p}^m(f * g)(z) = I_{p, q, r, \lambda}^{m, \ell}(\alpha_1, \beta_1) f(z),$$

where the operator  $I_{p, q, r, \lambda}^{m, \ell}(\alpha_1, \beta_1) f(z)$  was introduced and studied by El-Ashwah and Aouf [6],  $\alpha_1, \alpha_2, \dots, \alpha_q$  and  $\beta_1, \beta_2, \dots, \beta_s$  are real or complex number ( $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- = \{0, -1, -2, \dots\}; j = 1, \dots, s; (q \leq s + 1; q, s \in \mathbb{N}_0, p \in \mathbb{N}; z \in U)$  and

$$(\theta)_{\nu} = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \theta(\theta - 1) \dots (\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases}$$

Also, for many special operators of the operator  $I_{p, q, r, \lambda}^{m, \ell}(\alpha_1, \beta_1) f(z)$  (see [6]).

(iii) For  $m = 0, b_k = \frac{(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \dots (\beta_s)_{k-p} (1)_{k-p}}$ , the operator

$$D_{\lambda, \ell, p}^m(f * g)(z) = S_{p, q, s}^j(\gamma; \alpha_1) f(z),$$

where the operator  $S_{p, q, s}^j(\gamma; \alpha_1) f(z)$ , was introduced and studied by El-Ashwah [5].

(iv) For  $m = 0$  and  $b_k = \frac{\Gamma(p + \alpha + \beta) \Gamma(k + \beta)}{\Gamma(p + \beta) \Gamma(k + \alpha + \beta)}$ , the operator  $D_{p, \ell, \lambda}^m(f * g)(z) = Q_{p, \beta}^{\alpha}(f)$  ( $\alpha \geq 0, \beta > -1, p \in \mathbb{N}$ ), where the operator  $Q_{p, \beta}^{\alpha}$  was introduced by Liu and Owa [9].

For  $h(z)$  given by

$$h(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k$$

A function  $f(z) \in A(p)$  is said to be in the class  $S_{\lambda, \ell, p}^{m, j}(\gamma)$  of  $p$ -valent functions of complex order  $\gamma \neq 0$  in  $U$ , if and only if

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left( \frac{z(D_{\lambda, \ell, p}^m(f * h)(z))^{(j+1)}}{(D_{\lambda, \ell, p}^m(f * h)(z))^{(j)}} - p + j \right) \right\} > 0$$

$$(p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ell, \lambda \geq 0; \gamma \in \mathbb{C}^*; z \in U). \quad (1.7)$$

Clearly, we have the following relationships:

- (i)  $S_{\lambda, \ell, 1}^{0, 0}(\gamma) = S(\gamma) (\gamma \in \mathbb{C}^*)$ ,
- (ii)  $S_{\lambda, \ell, 1}^{0, 1}(\gamma) = \kappa(\gamma) (\gamma \in \mathbb{C}^*)$ ,
- (iii)  $S_{\lambda, \ell, 1}^{0, 0}(1 - \alpha) = S^*(\alpha) (0 \leq \alpha < 1)$ .

The classes  $S(\gamma)$  and  $\kappa(\gamma)$  are classes of starlike and convex functions of complex order  $\gamma \neq 0$  in  $U$  which were studied by Nasr and Aouf [13] and  $S^*(\alpha)$  is the class of starlike functions of order  $\alpha$  in  $U$ .

Also, for  $m = 0$  the operator  $S_p^j(h; \gamma)$  was introduced and studied by El-Ashwah and Aouf [7].

**Definition 1.1.** Let  $-1 \leq B < A \leq 1, p \in \mathbb{N}; j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*$ ,

$$|\gamma(A - B) + (p - j)B| < (p - j), f \in A(p).$$

Then  $f \in S_{\lambda, \ell, p}^{m, j}(\gamma; A, B)$ , the class of  $p$ -valent functions of complex order  $\gamma$  in  $U$  if and only if

$$\left\{ 1 + \frac{1}{\gamma} \left( \frac{z(D_{\lambda, \ell, p}^m(f * h)(z))^{(j+1)}}{(D_{\lambda, \ell, p}^m(f * h)(z))^{(j)}} - p + j \right) \right\} \prec \frac{1 + Az}{1 + Bz}. \quad (1.8)$$

A majorization problem for the subclasses of analytic function has recently been investigated by Altintas et al. [3] and MacGregor [11]. In this paper we investigate majorization problem for the class  $S_{\lambda, \ell, p}^{m, j}(\gamma; A, B)$  and some related subclasses.

## 2. Main results

Unless otherwise mentioned we shall assume throughout the paper that,  $-1 \leq B < A \leq 1, \gamma \in \mathbb{C}^*, \ell, \lambda \geq 0, p \in \mathbb{N}$  and  $m, j \in \mathbb{N}_0$ .

**Theorem 2.1.** Let the function  $f \in A(p)$  and suppose that  $g \in S_{\lambda, \ell, p}^{m, j}(\gamma; A, B)$ . If  $(D_{\lambda, \ell, p}^m(f * h)(z))^{(j)}$  is majorized by  $(D_{\lambda, \ell, p}^m(g * h)(z))^{(j)}$  in  $U$ , then

$$\left| (D_{\lambda, \ell, p}^{m+1}(f * h)(z))^{(j)} \right| \leq \left| (D_{\lambda, \ell, p}^{m+1}(g * h)(z))^{(j)} \right| \quad (|z| < r_1), \quad (2.1)$$

where  $r_1 = r_1(p, \gamma, \lambda, \ell, A, B)$  is the smallest positive root of the equation

$$\begin{aligned} & |\gamma\lambda(A - B) + (p + \ell)B| r^3 - [2\lambda|B| + (p + \ell)] r^2 - \\ & [|\gamma\lambda(A - B) + (p + \ell)B| + 2\lambda] r + (p + \ell) = 0. \end{aligned} \quad (2.2)$$

*Proof.* Since  $(g * h)(z) \in S_{\lambda, \ell, p}^{m, j}(\gamma; A, B)$ , we find from (1.8) that

$$1 + \frac{1}{\gamma} \left( \frac{z(D_{\lambda, \ell, p}^m(g * h)(z))^{(j+1)}}{(D_{\lambda, \ell, p}^m(g * h)(z))^{(j)}} - p + j \right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad (2.3)$$

where  $w$  is analytic in  $U$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in U$ ). From (2.3), we have

$$\frac{z(D_{\lambda,\ell,p}^m(g * h)(z))^{(j+1)}}{(D_{\lambda,\ell,p}^m(g * h)(z))^{(j)}} = \frac{(p - j) + [\gamma(A - B) + (p - j)B]w(z)}{1 + Bw(z)}. \tag{2.4}$$

In view of

$$\begin{aligned} \lambda z (D_{\lambda,\ell,p}^m(f * g)(z))^{(j+1)} &= (p + \ell) (D_{\lambda,\ell,p}^{m+1}(f * g)(z))^{(j)} \\ &\quad - [p(1 - \lambda) + \lambda j + \ell] (D_{\lambda,\ell,p}^m(f * g)(z))^{(j)} \end{aligned} \tag{2.5}$$

$0 \leq j \leq p; p \in \mathbb{N}, \lambda > 0; z \in U,$

(2.4) immediately yields the following inequality:

$$\left| (D_{\lambda,\ell,p}^m(g * h)(z))^{(j)} \right| \leq \frac{(p + \ell)(1 + |B||z|)}{(p + \ell) - |\gamma\lambda(A - B) + (p + \ell)B||z|} \left| (D_{\lambda,\ell,p}^{m+1}(g * h)(z))^{(j)} \right|. \tag{2.6}$$

Next, since  $(D_{\lambda,\ell,p}^m(f * h)(z))^{(j)}$  is majorized by  $(D_{\lambda,\ell,p}^m(g * h)(z))^{(j)}$  in  $U$ , from (1.2), we have

$$(D_{\lambda,\ell,p}^m(f * h)(z))^{(j)} = \varphi(z)(D_{\lambda,\ell,p}^m(g * h)(z))^{(j)}. \tag{2.7}$$

Differentiating (2.7) with respect to  $z$ , we have

$$z(D_{\lambda,\ell,p}^m(f * h)(z))^{(j+1)} = z\varphi'(z)(D_{\lambda,\ell,p}^m(g * h)(z))^{(j)} + z\varphi(z)(D_{\lambda,\ell,p}^m(g * h)(z))^{(j+1)}. \tag{2.8}$$

From (2.5) and (2.8), we have

$$(D_{\lambda,\ell,p}^{m+1}(f * h)(z))^{(j)} = \frac{\lambda z}{p + \ell} \varphi'(z)(D_{\lambda,\ell,p}^m(g * h)(z))^{(j)} + \varphi(z)(D_{\lambda,\ell,p}^{m+1}(g * h)(z))^{(j)}. \tag{2.9}$$

Thus, by noting that  $\varphi(z)$  satisfies the inequality (see [14]),

$$\left| \varphi'(z) \right| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in U),$$

we see that

$$\begin{aligned} &\left| (D_{\lambda,\ell,p}^{m+1}(f * h)(z))^{(j)} \right| \\ &\leq \left( |\varphi(z)| + \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \cdot \frac{\lambda |z| (1 + |B||z|)}{(p + \ell) - |\gamma\lambda(A - B) + (p + \ell)B||z|} \right) \left| (D_{\lambda,\ell,p}^{m+1}(g * h)(z))^{(j)} \right|, \end{aligned} \tag{2.10}$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),$$

leads us to the inequality

$$\begin{aligned} &\left| (D_{\lambda,\ell,p}^{m+1}(f * h)(z))^{(j)} \right| \\ &\leq \frac{\Theta(\rho)}{(1 - r^2)((p + \ell) - |\gamma\lambda(A - B) + (p + \ell)B|r)} \left| (D_{\lambda,\ell,p}^{m+1}(g * h)(z))^{(j)} \right|, \end{aligned}$$

where

$$\begin{aligned} \Theta(\rho) &= -r\lambda(1 + |B|r)\rho^2 + (1 - r^2)[(p + \ell) - |\gamma\lambda(A - B) + (p + \ell)B|r]\rho \\ &\quad + r\lambda(1 + |B|r), \end{aligned} \tag{2.11}$$

takes its maximum value at  $\rho = 1$ , with  $r_1 = r_1(p, \gamma, \lambda, \ell, A, B)$ , where  $r_1(p, \gamma, \lambda, \ell, A, B)$  is the smallest positive root of (2.2). Therefore the function  $\Phi(\rho)$  defined by

$$\begin{aligned} \Phi(\rho) = & -\sigma\lambda(1 + |B|\sigma)\rho^2 + (1 - \sigma^2)[(p + \ell) - |\gamma\lambda(A - B) + (p + \ell)B|\sigma]\rho \\ & + \sigma\lambda(1 + |B|\sigma) \end{aligned} \tag{2.12}$$

is an increasing function on the interval  $0 \leq \rho \leq 1$ , so that

$$\begin{aligned} \Phi(\rho) \leq \Phi(1) = & (1 - \sigma^2)[(p + \ell) - |\gamma(A - B) + (p + \ell)B|\sigma] \\ & (0 \leq \rho \leq 1; 0 \leq \sigma \leq r_0(p, \gamma, j, A, B)). \end{aligned} \tag{2.13}$$

Hence upon setting  $\rho = 1$  in (2.12), we conclude that (2.1) holds true for  $|z| \leq r_1 = r_1(p, \gamma, \lambda, \ell, A, B)$ , where  $r_1(p, \gamma, \lambda, \ell, A, B)$ , is the smallest positive root of (2.2). This completes the proof of Theorem 1.

Putting  $A = 1$  and  $B = -1$  in Theorem 1, we obtain the following result.

**Corollary 2.2.** *Let the function  $f \in A(p)$  and suppose that  $g \in S_{\lambda, \ell, p}^{m, j}(\gamma)$ .*

*If  $(D_{\lambda, \ell, p}^m(f * h)(z))^{(j)}$  is majorized by  $(D_{\lambda, \ell, p}^m(g * h)(z))^{(j)}$  in  $U$ , then*

$$\left| (D_{\lambda, \ell, p}^{m+1}(f * h)(z))^{(j)} \right| \leq \left| (D_{\lambda, \ell, p}^{m+1}(g * h)(z))^{(j)} \right| \quad (|z| < r_1),$$

where  $r_1 = r_1(p, \gamma, \lambda, \ell)$  is given by

$$r_1 = r_1(p, \gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4(p + \ell)|2\gamma\lambda - (p + \ell)|}}{2|2\gamma\lambda - (p + \ell)|}, \tag{2.14}$$

where  $k = 2\lambda + (p + \ell) + |2\gamma\lambda - (p + \ell)|$ .

Putting  $A = 1, B = -1$  and  $p = j = 1$  in Theorem 1, we obtain the following result.

**Corollary 2.3.** *Let the function  $f \in A$  and suppose that  $g \in S_{\lambda, \ell}^{m, 0}(\gamma)$ .*

*If  $(D_{\lambda, \ell}^m(f * h)(z))$  is majorized by  $(D_{\lambda, \ell}^m(g * h)(z))$  in  $U$ , then*

$$\left| (D_{\lambda, \ell}^{m+1}(f * h)(z)) \right| \leq \left| (D_{\lambda, \ell}^{m+1}(g * h)(z)) \right| \quad (|z| < r_2),$$

where  $r_2 = r_2(\gamma, \lambda, \ell)$  is given by

$$r_2 = r_2(\gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4(1 + \ell)|2\gamma\lambda - (1 + \ell)|}}{2|2\gamma\lambda - (1 + \ell)|}, \tag{2.15}$$

where  $k = 2\lambda + (1 + \ell) + |2\gamma\lambda - (1 + \ell)|$ .

Putting  $A = \lambda = 1, B = -1, m = \ell = 0$ , and  $h(z) = \frac{z^p}{1-z}$  (or  $c_{k+p} = 1$ ) in Theorem 1, we obtain the following result.

**Corollary 2.4.** *Let the function  $f \in A(p)$  and suppose that  $g \in S_p$ . If  $f(z)$  is majorized by  $g(z)$  in  $U$ , then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_3),$$

where  $r_3 = r_3(p, \gamma)$  is given by

$$r_3 = r_3(p; \gamma) = \frac{k - \sqrt{k^2 - 4p|2\gamma - p|}}{2|2\gamma - p|},$$

where  $k = 2 + p + |2\gamma - p|$ .

Putting  $\gamma = 1$  in Corollary 3, we obtain the following result.

**Corollary 2.5.** *Let the function  $f \in A(p)$  and suppose that  $g \in S_p(\gamma)$ . If  $f(z)$  is majorized by  $g(z)$  in  $U$ , then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_4),$$

where  $r_4$  is given by

$$r_4 = r_4(p) = \frac{k - \sqrt{k^2 - 4p|2-p|}}{2|2-p|},$$

where  $k = 2 + p + |2 - p|$

**Remarks 2.6.** (i) Putting  $p = 1$  in Corollary 3 we obtain the results obtained by Altintas et al. [3],

(ii) Putting  $p = 1$  in Corollary 4 we obtain the results obtained by MacGregor [10].

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