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An inequality of Ostrowski-Grüss type for double integrals

Hüseyin Budak and Mehmet Zeki Sarıkaya

Abstract. In this study, we establish Ostrowski-Grüss type involving functions of two independent variables for double integrals. Cubature formula is also provided.

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1. Introduction

In 1935, G. Grüss [7] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \tag{1.1}$$
$$\leq \frac{1}{4}(\Phi_1-\varphi_1)(\Phi_2-\varphi_2),$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi_1 \leq f(x) \leq \Phi_1 \text{ and } \varphi_2 \leq g(x) \leq \Phi_2 \text{ for all } x\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible.

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [9]:

Theorem 1.1 (Ostrowski inequality). *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to R$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.3}$$

for all $x \in [a,b]$*. The constant* $\frac{1}{4}$ *is the best possible.*

In 1882, P. L. Čebyšev [2] gave the following inequality:

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \left\|f'\right\|_\infty \left\|g'\right\|_\infty, \tag{1.4}$$

where $f,g : [a,b] \to \mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$\begin{aligned} & T(f,g) \\ = & \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \end{aligned} \tag{1.5}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]}|p(t)|$.

The following result of Grüss type was proved by Dragomir and Fedotov [4]:

Theorem 1.2. *Let* $f,u : [a,b] \to \mathbb{R}$ *be such that* u *is L-Lipshitzian on* $[a,b]$*, i.e,*

$$|u(x)-u(y)| \leq L|x-y| \quad \textit{for all } x \in [a,b], \tag{1.6}$$

f *is Riemann integrable on* $[a,b]$ *and there exist the real numbers* m, M *so that*

$$m \leq f(x) \leq M \quad \textit{for all } x \in [a,b]. \tag{1.7}$$

Then we have the inequality,

$$\left|\int_a^b f(x)du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2}L(M-m)(b-a).$$

From [8], if $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{1.8}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x \\ \frac{t-b}{b-a}, & x < t \leq b. \end{cases}$$

In [5], Dragomir and Wang proved following Ostrowski-Grüss type inequality using the inequality (1.1) and Montgomery identity (1.8):

Theorem 1.3. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differantiable mapping in* I° *and let* $a,b \in I^\circ$ *with* $a < b$*. If* $f \in L_1[a,b]$ *and*

$$\varphi_3 \leq f'(x) \leq \Phi_3, \ \forall x \in [a,b],$$

then we have the following inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \leq \frac{1}{4}(b-a)(\Phi_3 - \varphi_3), \tag{1.9}$$

for all $x \in [a,b]$.

Barnett and Dragomir established following Ostrowski inequality for double integrals in [1]:

Theorem 1.4. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be a continuous on* $[a,b] \times [c,d]$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a,b) \times (c,d)$, *and is bounded, i.e.,*

$$\|f_{xy}\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty$$

then we have the inequality

$$\begin{aligned} &\left| \int_a^b \int_c^d f(t,s)dsdt - \left[(b-a)\int_c^d f(x,s)ds \right.\right. \\ &\left.\left. + (d-c)\int_a^b f(t,y)dt - (b-a)(d-c)f(x,y) \right] \right| \\ \leq\ & \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2 \right] \|f_{xy}\|_\infty \end{aligned} \tag{1.10}$$

for all $(x,y) \in [a,b] \times [c,d]$.

In [1], the inequality (1.10) is established by the use of integral identity involving Peano kernels. In [10], Pachpatte obtained an inequality in the view (1.10) by using elementary analysis. The interested reader is also refered to ([1], [6], [10],[11],[13]-[15]) for Ostrowski type inequalities in several independent variables.

Recently, Sarikaya and Kiris have proved the following Grüss type inequality for double integrals in [12]:

Theorem 1.5. *Let* $f, g : [a,b] \times [c,d] \to \mathbb{R}$ *be two functions defined and integrable on* $[a,b] \times [c,d]$. *Then for*

$$\varphi \leq f(x,y) \leq \Phi \text{ and } \gamma \leq g(x,y) \leq \Gamma. \text{ for all } (x,y) \in [a,b] \times [c,d]$$

we have

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\right. \tag{1.11}$$
$$\left.-\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)dydx\right)\right|$$
$$\leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma).$$

Moreover, Cerone and Dragomir [3] extended Gruss type inequalities for Lebesgue integrals on measurable spaces. This includes domaind from the plane provided in [12].

In this work, using the inequality (1.11), we will obtain an Ostrowski-Grüss type inequality for functions of two independent variables.

2. Main results

First, we give the following notations to simplify the presentation of some intervals.

$$\begin{aligned}\Delta_1 &= [a,x]\times[c,y],\ \Delta_2=[a,x]\times[y,d],\\ \Delta_3 &= [x,b]\times[c,y],\ \Delta_4=[x,b]\times[y,d].\end{aligned}$$

Theorem 2.1. *Let $f:\Delta:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous on Δ, $f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$ exists on Δ°. If f integrable and*

$$\varphi\leq f_{xy}(x,y)\leq\Phi,\ \forall(x,y)\in\Delta$$

then we have the following inequality

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt-\left[\frac{1}{(d-c)}\int_c^d f(x,s)ds\right.\right. \tag{2.1}$$
$$\left.+\frac{1}{(b-a)}\int_a^b f(t,y)dt-f(x,y)\right]$$
$$\left.-\frac{f(b,d)-f(b,c)-f(a,d)+f(a,c)}{(b-a)(d-c)}\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\right|$$
$$\leq \frac{1}{4}(P-p)(\Phi-\varphi)$$

where

$$P=\max\{(x-a)(y-c),(b-x)(d-y)\}$$

and

$$p = \min\left\{(x-a)(y-d), (x-b)(y-c)\right\}$$

for all $(x,y) \in \Delta$.

Proof. Define the kernel $p(x,t;y,s)$ by

$$p(x,t;y,s) := \begin{cases} (t-a)(s-c), & \text{if } (t,s) \in [a,x] \times [c,y] \\ (t-a)(s-d), & \text{if } (t,s) \in [a,x] \times (y,d] \\ (t-b)(s-c), & \text{if } (t,s) \in (x,b] \times [c,y] \\ (t-b)(s-d), & \text{if } (t,s) \in (x,b] \times (y,d]. \end{cases}$$

Then, we have

$$\int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)\,ds\,dt \tag{2.2}$$

$$= \int_a^x \int_c^y (t-a)(s-c) f_{ts}(t,s)\,ds\,dt + \int_a^x \int_y^d (t-a)(s-d) f_{ts}(t,s)\,ds\,dt$$

$$+ \int_x^b \int_c^y (t-b)(s-c) f_{ts}(t,s)\,ds\,dt + \int_x^b \int_y^d (t-b)(s-d) f_{ts}(t,s)\,ds\,dt$$

$$= I_1 + I_2 + I_3 + I_4.$$

Let us calculate the integrals I_1, I_2, I_3 and I_4. Firstly, we have the equality

$$I_1 = \int_a^x \int_c^y (t-a)(s-c) f_{ts}(t,s)\,ds\,dt \tag{2.3}$$

$$= \int_a^x (t-a) \left[(y-c) f_t(t,y) - \int_c^y f_t(t,s)\,ds \right] dt$$

$$= (y-c) \int_a^x (t-a) f_t(t,y)\,dt - \int_c^y \left(\int_a^x (t-a) f_t(t,s)\,dt \right) ds$$

$$= (y-c) \left[(x-a) f(x,y) - \int_a^x f(t,y)\,dt \right] - \int_c^y \left[(x-a) f(x,s) - \int_a^x f(t,s)\,dt \right] ds$$

$$= (x-a)(y-c) f(x,y) - (y-c) \int_a^x f(t,y)\,dt - (x-a) \int_c^y f(x,s)\,ds + \int_a^x \int_c^y f(t,s)\,ds\,dt.$$

Also, similar computations we have the equalities

$$I_2 = \int_a^x \int_y^d (t-a)(s-d) f_{ts}(t,s)\,ds\,dt \tag{2.4}$$

$$= (x-a)(d-y) f(x,y) - (d-y)\int_a^x f(t,y)\,dt - (x-a)\int_y^d f(x,s)\,ds + \int_a^x \int_y^d f(t,s)\,ds\,dt,$$

$$I_3 = \int_x^b \int_c^y (t-b)(s-c) f_{ts}(t,s)\,ds\,dt \tag{2.5}$$

$$= (b-x)(y-c) f(x,y) - (y-c)\int_x^b f(t,y)\,dt - (b-x)\int_c^y f(x,s)\,ds + \int_x^b \int_c^y f(t,s)\,ds\,dt,$$

and

$$I_4 = \int_x^b \int_y^d (t-b)(s-d) f_{ts}(t,s)\,ds\,dt \tag{2.6}$$

$$= (b-x)(d-y) f(x,y) - (d-y)\int_x^b f(t,y)\,dt - (b-x)\int_y^d f(x,s)\,ds + \int_x^b \int_y^d f(t,s)\,ds\,dt.$$

If we substitute the equalities (2.3)-(2.6) in (2.2), then we have

$$\int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)\,ds\,dt \tag{2.7}$$

$$= (b-a)\,(d-c)\,f(x,y) - (b-a)\int_c^d f(x,s)\,ds - (d-c)\int_a^b f(t,y)\,dt + \int_a^b \int_c^d f(t,s)\,ds\,dt.$$

Applying Theorem 1.5 to mappings $p(x,.;y,.)$ and $f_{ts}(.,.)$, we establish

$$\begin{aligned} &\left| \frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)\,ds\,dt \right. \\ &- \left(\frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d p(x,t;y,s)\,ds\,dt \right) \\ &\left. \times \left(\frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d f_{ts}(t,s)\,ds\,dt \right) \right| \\ \leq\ & \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma). \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
\Gamma &= \sup_{(t,s)\in\Delta} p(x,t;y,s) && (2.9)\\
&= \max\Bigg\{ \sup_{(t,s)\in\Delta_1} (t-a)(s-c), \sup_{(t,s)\in\Delta_2} (t-a)(s-d),\\
&\quad \sup_{(t,s)\in\Delta_3} (t-b)(s-c), \sup_{(t,s)\in\Delta_4} (t-b)(s-d) \Bigg\}\\
&= \max\{(x-a)(y-c),(b-x)(d-y)\} = P,
\end{aligned}$$

and

$$\begin{aligned}
\gamma &= \inf_{(t,s)\in\Delta} p(x,t;y,s) && (2.10)\\
&= \min\Bigg\{ \inf_{(t,s)\in\Delta_1} (t-a)(s-c), \inf_{(t,s)\in\Delta_2} (t-a)(s-d),\\
&\quad \inf_{(t,s)\in\Delta_3} (t-b)(s-c), \inf_{(t,s)\in\Delta_4} (t-b)(s-d) \Bigg\}\\
&= \min\{(x-a)(y-d),(x-b)(y-c)\} = p.
\end{aligned}$$

Also, we have the equalities

$$\begin{aligned}
&\int_a^b\int_c^d p(x,t;y,s)\,dsdt && (2.11)\\
&= \int_a^x\int_c^y (t-a)(s-c)dsdt + \int_a^x\int_y^d (t-a)(s-d)dsdt\\
&\quad + \int_x^b\int_c^y (t-b)(s-c)dsdt + \int_x^b\int_y^d (t-b)(s-d)dsdt\\
&= \frac{(x-a)^2(y-c)^2}{4} - \frac{(x-a)^2(d-y)^2}{4}\\
&\quad - \frac{(b-x)^2(y-c)^2}{4} + \frac{(b-x)^2(d-y)^2}{4}\\
&= \frac{\left[(x-a)^2-(b-x)^2\right]\left[(y-c)^2-(d-y)^2\right]}{4}\\
&= (b-a)(d-c)\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)
\end{aligned}$$

and

$$\int_a^b \int_c^d f_{ts}(t,s)dsdt = f(b,d) - f(b,c) - f(a,d) + f(a,c). \tag{2.12}$$

If we put the equalities (2.7) and (2.9)-(2.12) in (2.8), then we obtain the desired inequality (2.1). □

Corollary 2.2. *With the assumptions in Theorem* 2.1*, if* $|f_{xy}(x,y)| \leq M$ *for all* $(x,y) \in [a,b] \times [c,d]$ *and some positive constant* M*, then we have*

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right. \\ &- \left[\frac{1}{(d-c)} \int_c^d f(x,s)ds + \frac{1}{(b-a)} \int_a^b f(t,y)dt - f(x,y) \right] \\ &\left. - \frac{f(b,d) - f(b,c) - f(a,d) + f(a,c)}{(b-a)(d-c)} \left(x - \frac{a+b}{2} \right) \left(y - \frac{c+d}{2} \right) \right| \\ \leq\ & \frac{1}{2} (P-p) M \end{aligned}$$

where

$$P = \max \{ (x-a)(y-c), (b-x)(d-y) \}$$

and

$$p = \min \{ (x-a)(y-d), (x-b)(y-c) \}$$

for all $(x,y) \in [a,b] \times [c,d]$.

Corollary 2.3. *Under assumptions of Theorem* 2.1 *with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$*, we have the following inequality*

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt - \left[\frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s \right) ds \right. \right. \\ &\left. \left. + \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2} \right) dt - f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right] \right| \\ \leq\ & \frac{1}{8} (b-a)(d-c)(\Phi - \varphi). \end{aligned}$$

Corollary 2.4. *Under assumption of Theorem* 2.1 *with* $x = b$ *and* $y = d$, *we get the inequality*

$$\begin{aligned} & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right. \\ & - \left[\frac{1}{(d-c)} \int_c^d f(b,s)ds + \frac{1}{(b-a)} \int_a^b f(t,d)dt - f(b,d) \right] \\ & \left. - \frac{f(b,d) - f(b,c) - f(a,d) + f(a,c)}{4} \right| \\ \leq \; & \frac{1}{4}(b-a)(d-c)(\Phi - \varphi). \end{aligned}$$

3. Applications for cubature formulae

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$, and $l_j := y_{j+1} - y_j$ $(j = 0, ..., m-1)$,

$$\begin{aligned} \upsilon(h) &:= \max\{h_i | \; i = 0, ..., n-1\}, \\ \mu(l) &:= \max\{l_j | \; j = 0, ..., m-1\}. \end{aligned}$$

Then, the following theorem holds.

Theorem 3.1. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be as in Theorem* 2.1 *and* $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ *be intermediate points. Then we have the cubature formula:*

$$\begin{aligned} & \int_a^b \int_c^d f(t,s)dsdt \qquad (3.1) \\ = \; & \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, s)ds + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(t, \eta_j)dt \\ & - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j) \\ & + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j) \right] \\ & \times \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right) \\ & + R(\xi, \eta, I_n, J_m, f). \end{aligned}$$

where the remainer term $R(\xi,\eta,I_n,J_m,f)$ satisfies the estimation

$$|R(\xi,\eta,I_n,J_m,f)| \leq \frac{1}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\left(\Phi-\varphi\right) \tag{3.2}$$

where

$$P_{ij}=\max\left\{\left(\xi_i-x_i\right)\left(\eta_j-y_j\right),\left(x_{i+1}-\xi_i\right)\left(y_{j+1}-\eta_j\right)\right\},$$

and

$$p_{ij}=\min\left\{\left(\xi_i-x_i\right)\left(\eta_j-y_{j+1}\right),\left(\xi_i-x_{i+1}\right)\left(\eta_j-y_j\right)\right\}.$$

Proof. Aplying Theorem 2.1 on the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we get

$$\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt\right. \\
&-\left[h_i\int_{y_j}^{y_{j+1}} f(\xi_i,s)ds+l_j\int_{x_i}^{x_{i+1}} f(t,\eta_j)dt-h_il_jf(\xi_i,\eta_j)\right] \\
&-\left[f\left(x_{i+1},y_{j+1}\right)-f\left(x_{i+1},y_j\right)-f\left(x_i,y_{j+1}\right)+f\left(x_i,y_j\right)\right] \\
&\left.\times\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)\right| \\
\leq\ & \frac{1}{4}h_il_j\left(P_{ij}-p_{ij}\right)\left(\Phi_{ij}-\varphi_{ij}\right)
\end{aligned} \tag{3.3}$$

where

$$\Phi_{ij}:=\sup_{(t,s)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]}|f_{ts}(t,s)|,\quad \varphi_{ij}:=\inf_{(t,s)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]}|f_{ts}(t,s)|$$

for all $i=0,1,...,n-1$; $j=0,1,...,m-1$.

Summing the inequality (3.3) over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality, we get

$$\begin{aligned}
|R(\xi,\eta,I_n,J_m,f)| &\leq \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}h_il_j\left(P_{ij}-p_{ij}\right)\left(\Phi_{ij}-\varphi_{ij}\right) \\
&\leq \frac{1}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\max_{ij}\left(\Phi_{ij}-\varphi_{ij}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}1 \\
&= \frac{nm}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\left(\Phi-\varphi\right).
\end{aligned}$$

This completes the proof. □

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Hüseyin Budak
Düzce University, Faculty of Science and Arts
Department of Mathematics
Konuralp Campus, Düzce, Turkey
e-mail: `hsyn.budak@gmail.com`

Mehmet Zeki Sarıkaya
Düzce University, Faculty of Science and Arts
Department of Mathematics
Konuralp Campus, Düzce, Turkey
e-mail: `sarikayamz@gmail.com`