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Ball convergence of a stable fourth-order family for solving nonlinear systems under weak conditions

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Abstract. We present a local convergence analysis of fourth-order methods in order to approximate a locally unique solution of a nonlinear equation in Banach space setting. Earlier studies have shown convergence using Taylor expansions and hypotheses reaching up to the fifth derivative although only the first derivative appears in these methods. We only show convergence using hypotheses on the first derivative. We also provide computable: error bounds, radii of convergence as well as uniqueness of the solution with results based on Lipschitz constants not given in earlier studies. The computational order of convergence is also used to determine the order of convergence. Finally, numerical examples are also provided to show that our results apply to solve equations in cases where earlier studies cannot apply.

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1. Introduction

Let B_1, B_2 be Banach spaces and D be a convex subset of B_1. Let also $L(B_1, B_2)$ denote the space of bounded linear operators from B_1 into B_2.

In the present paper, we deal with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq B_1 \to B_2$ is a Fréchet-differentiable operator.

Numerous problems can be written in the form of (1.1) using Mathematical Modelling [3, 5, 8, 9, 12, 13, 18, 19, 22, 26, 28, 29, 30]. Analytical methods for solving such problems are almost non-existent and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iterative procedure

[1-24]. In particular, we present the local convergence of the methods studied in [14] and defined for each $n = 0, 1, 2, 3, \ldots$ by

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - \dfrac{1}{\beta}F'(x_n)^{-1}F(y_n), \\ x_{n+1} = z_n - F'(x_n)^{-1}\left(\alpha F(y_n) + \beta F(z_n)\right), \end{cases} \tag{1.2}$$

where $\alpha = 2 - \frac{1}{\beta} - \beta,\ \beta \in \mathbb{R}\backslash\{0\}$ and $\alpha \in \mathbb{R}$.

Method (1.2) has fourth-order of convergence, except for $\beta = 1/5$. For this particular value, method attains fifth-order of convergence. The fourth order of convergence was based on Taylor expansions and hypotheses reaching up to the fifth derivative of function F although only the first derivative appears in these methods. Moreover, no computable error bounds on the distances $\|x_n - x^*\|$ or uniqueness results or compuatble radius of convergence were given. These problems reduce the applicability of these methods.

As a motivational example, define function F on $D = [\frac{-1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, the results in [14] cannot be used to solve the equation $F(x) = 0$, since function F''' is unbounded on D.

In the present study, we only use hypotheses on the first derivative and find error bounds, radii of convergence and uniqueness results based on Lipschitz constants. Moreover, since we avoid derivatives of order higher than one, we compute the computational order of convergence which does not require the knowledge of x^* or the existence of high order derivatives. This way we expand the applicability of these methods.

The rest of the paper is organized as follows: The local convergence of both methods is given in Section 2, whereas numerical examples are provided in the concluding Section 3.

2. Local convergence

We present the local convergence analysis of method (1.2) in this section.

The local convergence analysis is based on some scalar functions and parameters. Let $L_0 > 0,\ L > 0,\ M \geq 1,\ \beta \in \mathbb{R}\backslash\{0\}$ and $\alpha \in \mathbb{R}$ be given parameters. Define function

g_1, g_2, h_2, g_3 and h_3 on the interval $[0, \frac{1}{L_0})$ by

$$
\begin{aligned}
g_1(t) =& \frac{Lt}{2(1-L_0t)},\\
g_2(t) =& \Big(1+\frac{M}{|\beta|(1-L_0t)}\Big)g_1(t),\\
h_2(t) =& g_2(t)-1,\\
g_3(t) =& g_2(t)+\frac{M}{1-L_0t}\left(|\alpha|g_1(t)+|\beta|g_2(t)\right),\\
h_3(t) =& g_3(t)-1
\end{aligned}
$$

and parameter r_A by

$$r_A = \frac{2}{2L_0+L}.$$

We have that $g_1(r_A) = 1$ and $0 \leq g_1(t) < 1$ for each $t \in [0, r_A)$.

We also get that $h_2(0) = h_3(t) = -1 < 0$ and $h_2(t) \to +\infty,\ h_3(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. It follows from intermediate value theorem that functions h_2 and h_3 have zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_2 and r_3 the smallest such zeros.
Define the convergence radius r by

$$r = min\{r_A, r_2, r_3\}. \tag{2.1}$$

Then, we have that

$$0 < r \leq r_A \tag{2.2}$$

and

$$0 \leq g_i(t) < 1,\ i = 1, 2, 3. \tag{2.3}$$

Let $U(v, \rho)$ and $\bar{U}(v, \rho)$ stand, respectively for the open and closed balls in B_1 with center $v \in B_1$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1.2) using the preceding notation.

Theorem 2.1. *Let $F : D \subseteq B_1 \to B_2$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$ and $L_0 > 0$ such that for each $x \in D$*

$$F(x^*) = 0,\ F'(x^*)^{-1} \in L(B_2, B_1), \tag{2.4}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|. \tag{2.5}$$

Moreover, suppose that there exist constants $L > 0$ and $M \geq 1$ such that for each $x, y \in D_0 := D \cap U\left(x^*, \frac{1}{L_0}\right)$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \tag{2.6}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M \tag{2.7}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.8}$$

where the radius of convergence r is defined by (2.1). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r)$-$\{x^*\}$ by method (1.2) is well defined, remains in $U(x^*, r)$ and *converges to the solution x^* of equation $F(x) = 0$. Moreover, the following estimates hold*

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \tag{2.9}$$

$$\|z_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| \tag{2.10}$$

and

$$\|x_{n+1} - x^*\| \le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{2.11}$$

where the "g" functions are defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$, the limit point x^* is the only solution of $F(x) = 0$ in $D_1 := U(x^*, T) \cap D$.

Proof. We shall show estimates (2.9)–(2.11) using mathematical induction. By hypothesis $x_0 \in U(x^*, r)$-$\{x^*\}$, (2.1), (2.4) and (2.5), we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.12}$$

It follows from (2.12) and the Banach lemma on invertible functions [7, 26, 28, 30] that $F'(x_0)^{-1} \in L(B_2, B_1)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.13}$$

Hence, y_0, z_0, x_1 are well defined by method (1.2) for $n = 0$. We can have that

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0). \tag{2.14}$$

Using (2.1), (2.2), (2.3) (for $i = 1$), (2.6), (2.13) and (2.14), we obtain in turn that

$$\begin{aligned}
\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \le \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \|\int_0^1 F'(x^*)^{-1}\Big(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\Big)(x_0 - x^*)d\theta\| \\
&\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < r,
\end{aligned} \tag{2.15}$$

which shows (2.9) for $n = 0$ and $y_0 \in U(x^*, r)$. We also have that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.16}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Then, by (2.7) and (2.16), we get that

$$\|F'(x^*)^{-1}F(x_0)\| \le M\|x_0 - x^*\|. \tag{2.17}$$

In view of (2.1), (2.2), (2.3) (for $i = 2$), (2.13), (2.15) and (2.17) (for $x_0 = y_0$), we get that

$$\begin{aligned}
\|z_0 - x^*\| &\le \|y_0 - x^*\| + \frac{M\|y_0 - x^*\|}{|\beta|(1 - L_0\|x_0 - x^*\|)} \\
&\le \Big(1 + \frac{M}{|\beta|(1 - L_0\|x_0 - x^*\|)}\Big)g_1(\|x_0 - x^*\|))\|x_0 - x^*\| \\
&= g_2(\|x_0 - x^*\|))\|x_0 - x^*\| \le \|x_0 - x^*\| < r,
\end{aligned} \tag{2.18}$$

which shows (2.10) for $n = 0$ and $z_0 \in U(x^*, r)$. By (2.1), (2.2), (2.3) (for $i = 3$), (2.13), (2.15) and (2.17) (for $x_0 = y_0$), we get that

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|z_0 - x^*\| + \frac{M}{1 - L_0(\|x_0 - x^*\|)}\big(|\alpha|\|y_0 - x^*\| + |\beta|\|z_0 - x^*\|\big) \\
&\leq \Big[g_2(\|x_0 - x^*\|) + \frac{M}{1 - L_0(\|x_0 - x^*\|)}\big(|\alpha| g_1(\|x_0 - x^*\|) \\
&\quad + |\beta| g_2(\|x_0 - x^*\|)\big)\Big]\|x_0 - x^*\| \\
&= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned} \tag{2.19}$$

which shows (2.11) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, x_1 by x_n, y_n, x_{n+1} in the preceding estimates, we complete the induction for estimates (2.9)–(2.11). Then, in view of the estimate

$$\|x_{n+1} - x^*\| \leq c\|x_n - x^*\| < r, \qquad c = g_3(\|x_0 - x^*\|) \in [0, 1),$$

we deduce that $\lim_{n\to\infty} x_n = x^*$ and $x_{n+1} \in U(x^*, r)$. Finally, to show the uniqueness part, let $y^* \in D_1$ with $F(y^*) = 0$. Define

$$Q = \int_0^1 F'(x^* + \theta(y^* - x^*)d\theta.$$

Using (2.5), we get that

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \frac{L_0}{2}\|x^* - y^*\| \leq \frac{L_0}{2}T < 1. \tag{2.20}$$

Hence, $Q^{-1} \in L(B_2, B_1)$. Then, by the identity $0 = F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. $\square$

Remark 2.2. 1. The condition (2.7) can be dropped, since this condition follows from (2.5), if we set

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

2. The results obtained here can also be used for operators F satisfying autonomous differential equations [5, 7] of the form:

$$F'(x) = P(F(x)),$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$.

3. The radius $\bar{r_A} = \frac{2}{2L_0 + L_1}$ was shown by us to be the convergence radius of Newton's method [5]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \text{ for each } n = 0, 1, 2, \ldots \tag{2.21}$$

provided the conditions (2.4)–(2.6) hold on D. Let L_1 be the corresponding to L constant. It follows from the definition of r that the convergence radius r of the method (1.2) cannot be larger than the convergence radius $\bar{r_A}$ of the second order Newton's method (3.3). As already noted in [5], $\bar{r_A}$ is at least as large as the convergence ball given by Rheinboldt [28]

$$r_R = \frac{2}{3L_1}.$$

In particular, for $L_0 < L_1$, we have that

$$r_R < r_1$$

and

$$\frac{r_R}{\bar{r_A}} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is our convergence ball $\bar{r_A}$ is atmost three times larger than Rheinboldt's. The same value of r_R was given by Traub [30]. Notice that $L \leq L_1$, since $D_0 \subseteq D$. Therefore, $\bar{r_A} \leq r_A$.

4. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of stronger conditions used in [14]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi^* = \sup \frac{\ln\left(\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}\right)}{\ln\left(\frac{\|x_n-x^*\|}{\|x_{n-1}-x^*\|}\right)},$$

or the approximate computational order of convergence (ACOC) defined by

$$\xi = \sup \frac{\ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right)}{\ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right)}.$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F. Notice also that the computation of ξ does not require knowledge of x^*.

3. Numerical examples

We present numerical examples in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet derivative is given by

$$F'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have that $L_0 = e - 1$, $L = e^{\frac{1}{L_0}} = 1.789572397$, $M = e^{\frac{1}{L_0}} = 1.7896$ and $L_1 = e$. The parameters using method (1.2) are:

$$r_A = 0.382692,\ r_2 = 0.145318,\ r_3 = 0.0826175,\ r = 0.0826175,\ \bar{r_A} = 0.324947.$$

Example 3.2. Let $B_1 = B_2 = C[0, 1]$, the space of continuous functions defined on $[0, 1]$ and be equipped with the max norm. Let $D = \bar{U}(0, 1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\phi)(x) = \phi(x) - 5 \int_0^1 x\theta\phi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\phi(\xi))(x) = \xi(x) - 15 \int_0^1 x\theta\phi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D. \tag{3.2}$$

Then, we get that $x^* = 0, L_0 = 7.5, L_1 = 15, L = 15, M = 2$. The parameters using method (1.2) are:

$$r_A = 0.0666667,\ r_2 = 0.0198959,\ r_3 = 0.0101189,\ r = 0.0101189,\ \bar{r_A} = 0.0666667.$$

Example 3.3. Let $B_1 = B_2 = \mathbb{R}$, $D = \bar{U}(0, 1)$. Define F on D by

$$F(x) = e^x - 1.$$

Then, $F'(x) = e^x$ and $\xi = 0$. We get that $L_0 = e - 1 < L = e^{\frac{1}{L_0}} < L_1 = e$ and $M = 2$. Then, for method (1.2) the parameters are:

$$r_A = 0.382692,\ r_2 = 0.13708,\ r_3 = 0.0742433,$$

$$r = 0.0742433,\ \bar{r_A} = 0.324947,\ \xi = 3.8732.$$

Example 3.4. Let $B_1 = B_2 = \mathbb{R}$ and define function F on $D = \mathbb{R}$ by

$$F(x) = \beta x - \gamma \sin(x) - \delta, \tag{3.3}$$

where β, γ, δ are given real numbers. Suppose that there exists a solution ξ of $F(x) = 0$ with $F'(\xi) \neq 0$. Then, we have

$$L_1 = L_0 = L = \frac{|\gamma|}{|\beta - \gamma \cos \xi|},\ M = \frac{|\gamma| + |\beta|}{|\beta - \gamma \cos \xi|}.$$

Then one can find the convergence radii for different values of β, γ and δ. As a specific example, let us consider Kepler's equation (3.3) with $\beta = 1$, $0 \leq \gamma < 1$ and $0 \leq \delta \leq \pi$. A numerical study was presented in [15] for different values of γ and δ. Let us take $\gamma = 0.9$ and $\delta = 0.1$. Then the solution is given by $x^* = 0.6308435$.
Hence, for method (1.2) the parameters are:

$$r_A = 0.202387,\ r_2 = 0.032669,\ r_3 = 0.00804637,$$

$$r = 0.00804637,\ \bar{r_A} = 0.202387,\ \xi = 4.0398.$$

Example 3.5. Returning back to the motivational example at the introduction of this paper, we have that $L = L_0 = 146.6629073$, $M = 2$, $L_1 = L$. The parameters using method (1.2) are:

$$r_A = 0.00689682,\ r_2 = 0.0033639187,\ r_3 = 0.00230533728667086,$$
$$r = 0.00230533728667086,\ \bar{r_A} = 0.00689682 \text{ and } \xi = 3.4324.$$

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