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A note on the Laplace operator for holomorphic functions on complex Lie groups

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Abstract. In this note we obtain the local expression of the Laplace operator acting on holomorphic functions defined on a complex Lie group. Also, some applications to the theory of holomorphic last multipliers are given.

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1. Preliminaries on complex Lie groups

Let G be a complex Lie group of dimension n. Its Lie algebra, $\mathfrak{g}$, has as underlying vector space the holomorphic tangent space $T_e^{1,0}G$ at the identity $e \in G$. As known, an element $A \in T_e^{1,0}G$ determines a unique left invariant vector field which takes the value A at e; moreover, these vector fields are the elements of $\mathfrak{g}$.

Following the ideas from [1], let $\{E_\alpha\},\ \alpha = 1,\dots,n$, be a base of the Lie algebra $\mathfrak{g}$ and $\chi^\alpha,\ \alpha = 1,\dots,n$ the dual base for the 1-*forms of Maurer-Cartan*, that is, $\chi^\alpha(E_\beta) = \delta^\alpha_\beta,\ (\alpha,\beta = 1,\dots,n)$. It is known ([11], Lemma 1.6) that E_α are holomorphic vector fields (as they are left-invariant) and also χ^α are holomorphic left-invariant 1-forms.

A differential form η is said to be *left-invariant* if it is invariant by every left translation $L_a,\ (a \in G)$, that is, if $L_a^*\eta = \eta$ for every $a \in G$, where L_a^* is the holomorphic cotangent map of L_a. It follows that any left invariant form must be holomorphic. For an element $U \in \mathfrak{g}$ and an element η in the dual space $\mathfrak{g}^*$, $\eta(U)$ is constant on G. Since

$$\partial\eta(U,V) = U\eta(V) - V\eta(U) - \eta([U,V]),$$

where $d = \partial + \bar{\partial}$ is the usual decomposition of the exterior derivative, one obtains

$$\partial\eta(U,V) = -\eta([U,V]), \tag{1.1}$$

where U, V are elements of $\mathfrak{g}$ and η is any element of the dual space. By setting

$$[E_\beta, E_\gamma] = C_{\beta\,\gamma}^{\,\alpha} E_\alpha, \tag{1.2}$$

the relation (1.1) yields

$$\partial\chi^\alpha = -\frac{1}{2} C_{\beta\,\gamma}^{\,\alpha} \chi^\beta \wedge \chi^\gamma. \tag{1.3}$$

The complex constants $C_{\beta\,\gamma}^{\,\alpha}$ are called the *constants of structure* of $\mathfrak{g}$ with respect to the holomorphic base $\{E_1, \ldots, E_n\}$. These constants are not arbitrary since they must satisfy the relations

$$[E_\alpha, E_\beta] + [E_\beta, E_\alpha] = 0 \tag{1.4}$$

and

$$[E_\alpha, [E_\beta, E_\gamma]] + [E_\beta, [E_\gamma, E_\alpha]] + [E_\gamma, [E_\alpha, E_\beta]] = 0 \tag{1.5}$$

for all $\alpha, \beta, \gamma = 1, \ldots, n$, that is

$$C_{\beta\,\gamma}^{\,\alpha} + C_{\gamma\,\beta}^{\,\alpha} = 0 \tag{1.6}$$

and

$$C_{\alpha\,\beta}^{\,\rho} C_{\gamma\,\rho}^{\,\delta} + C_{\beta\,\gamma}^{\,\rho} C_{\alpha\,\rho}^{\,\delta} + C_{\gamma\,\alpha}^{\,\rho} C_{\beta\,\rho}^{\,\delta} = 0. \tag{1.7}$$

Equations (1.3) are called the *holomorphic Maurer-Cartan equations.*

Equation (1.2) indicates that the structure constants are the components of a holomorphic tensor on $T_e^{1,0}G$ of type $(1,2)$. A new holomorphic tensor on $T_e^{1,0}G$ can be defined by setting

$$C_{\alpha\beta} = C_{\alpha\,\sigma}^{\,\rho} C_{\rho\,\beta}^{\,\sigma} \tag{1.8}$$

with respect to the holomorphic left invariant base $\{E_\alpha\}$ $(\alpha = 1, \ldots, n)$ of $\mathfrak{g}$. It is easily verified that this holomorphic tensor is symmetric. Also, it can be shown that a necessary and sufficient condition for the complex Lie group G to be semi-simple is that the complex matrix $(C_{\alpha\beta})_{n\times n}$ is invertible.

The holomorphic tensor defined by the equations (1.8) can now be used to raise and lower indices and, for this purpose, the inverse matrix $(C^{\alpha\beta})_{n\times n}$ will be considered.

In terms of a system of local complex coordinates $(u^1, \ldots, u^n)$ on G, the holomorphic vector fields E_α, $\alpha = 1, \ldots, n$, can be expressed as $E_\alpha = \chi_\alpha^i \dfrac{\partial}{\partial u^i}$. Since G is complex parallelizable (see [14]), the $n \times n$ matrix (χ_α^i) has rank n and so, by setting

$$g^{ij} = \chi_\alpha^i \chi_\beta^j C^{\alpha\beta}, \tag{1.9}$$

a positive definite and symmetric matrix $(g^{ij})_{n\times n}$ is obtained. Hence, a *holomorphic Riemannian metric* g on G can be defined by means of the complex quadratic form

$$ds^2 = g_{jk} du^j \otimes du^k, \tag{1.10}$$

where $(g_{jk})_{n\times n}$ denotes the matrix inverse to $(g^{jk})_{n\times n}$, that is, $g_{jk} = C_{\beta\gamma}\chi_j^\beta \chi_k^\gamma$.

Moreover, the holomorphic metric tensor g can be also used to raise and lower indices in the usual manner, and this holomorphic metric is completely determined by the complex Lie group G.

In the following, we define n holomorphic covariant vector fields χ^α ($\alpha = 1,\dots,n$) on G, with local components $\chi_i^\alpha (i = 1,\dots,n)$ given by

$$\chi_i^\alpha = C^{\alpha\beta}\chi_\beta^j g_{ij}. \tag{1.11}$$

It easily follows that

$$\chi_\alpha^i \chi_j^\alpha = \delta_j^i \quad \text{and} \quad \chi_\alpha^i \chi_i^\beta = \delta_\beta^\alpha. \tag{1.12}$$

Also, we consider the set of n^2 linear holomorphic 1-forms $\omega_j^i = \Gamma_{jk}^{\,i} du^k$ defined locally by setting

$$\Gamma_{jk}^{\,i} = \chi_\alpha^i \frac{\partial \chi_j^\alpha}{\partial u^k}. \tag{1.13}$$

By virtue of the equations (1.12), the holomorphic coefficients $\Gamma_{jk}^{\,i}$ can also be expressed as

$$\Gamma_{jk}^{\,i} = -\chi_j^\alpha \frac{\partial \chi_\alpha^i}{\partial u^k} \tag{1.14}$$

and they represent the local coefficients of a left invariant holomorphic connection ∇ on G, that is, ∇ is absolutely parallel with respect to every left-invariant holomorphic vector field $U = U^\alpha E_\alpha \in \mathfrak{g}$.

It is easily verified that in the overlap $U \cap U'$ of two local charts, the holomorphic 1-forms ω_j^i change by the rule

$$\frac{\partial u'^k}{\partial u^j}\omega_k'^i = \frac{\partial u'^i}{\partial u^k}\omega_j^k - \frac{\partial^2 u'^i}{\partial u^l \partial u^j} du^l.$$

The next natural step is to consider the torsion of this connection. As in the case of real Lie groups (see [1, 13]), the holomorphic torsion tensor will be written as

$$T_{jk}^{\,i} = \frac{1}{2}\chi_\alpha^i \left(\frac{\partial \chi_j^\alpha}{\partial u^k} - \frac{\partial \chi_k^\alpha}{\partial u^j} \right). \tag{1.15}$$

Since the equations (1.2) can be expressed in terms of the local coordinates (u^i) in the form

$$\chi_\beta^r \frac{\partial \chi_\gamma^i}{\partial u^r} - \chi_\gamma^r \frac{\partial \chi_\beta^i}{\partial u^r} = C_{\beta\ \gamma}^{\ \alpha} \chi_\alpha^i, \tag{1.16}$$

by using the holomorphic Maurer-Cartan equations (1.3) it easily follows that

$$T_{jk}^{\,i} = \frac{1}{2} C_{\beta\ \gamma}^{\ \alpha} \chi_\alpha^i \chi_j^\beta \chi_k^\gamma. \tag{1.17}$$

Also, if we consider the local coefficients of the holomorphic Levi-Civita connection $\overset{\circ}{\nabla}$ with respect to the holomorphic metric $g = ds^2$ from (1.10) on G, they can be expressed as

$$\overset{\circ}{\Gamma}{}_{jk}^{\,i} = \frac{1}{2}\chi_\alpha^i \left(\frac{\partial \chi_j^\alpha}{\partial u^k} + \frac{\partial \chi_k^\alpha}{\partial u^j} \right), \tag{1.18}$$

from which follows that

$$\Gamma_{jk}^{\,i} = \overset{\circ}{\Gamma}{}_{jk}^{\,i} + T_{jk}^{\,i}. \tag{1.19}$$

We have

Lemma 1.1. *The elements of the Lie algebra $\mathfrak{g}$ of G define holomorphic translations in G.*

Proof. It follows in a similar manner to the case of real Lie groups, see [1]. □

2. Laplace operators for holomorphic functions on G

In this section, we introduce the Laplace operator acting on holomorphic functions on the complex Lie group G, depending on the given holomorphic metric tensor on G.

Denote by $\omega = \chi^1 \wedge \cdots \wedge \chi^n$, where $\chi^i, i = 1, \ldots, n$ are the elements of the base of holomorphic 1-forms defined in Section 1. Then ω is a nowhere vanishing holomorphic left-invariant n-form, called the *holomorphic volume element*, and it can be used to define the *divergence* of a holomorphic vector field $U = U^\alpha E_\alpha$ by setting

$$\operatorname{div}(U)\omega = \partial(i_U \omega). \tag{2.1}$$

Note that the divergence can also be defined by means of the Lie derivative L_U with respect with a left invariant holomorphic vector field U:

$$\operatorname{div}(U)\omega = L_U \omega, \tag{2.2}$$

where

$$L_U \eta = \left.\frac{d}{dt}\right|_{t=0} (\varphi_U^t)^* \eta$$

for an arbitrary holomorphic tensor η. The equivalence between definitions (2.1) and (2.2) is due to Cartan's formula $L_U \eta = \partial(i_U \eta) + i_U \partial \eta$ for $\eta = \omega$. The first definition is more convenient for computations, though. Another property of the divergence is

$$\operatorname{div}(fU) = Uf + f \operatorname{div} U$$

for a holomorphic vector field U and a holomorphic function f defined on G.

Also, for a given holomorphic vector field $U = U^\alpha E_\alpha$ on G, we have

$$\operatorname{div} U = E_\alpha(U^\alpha). \tag{2.3}$$

Let G be a semi-simple complex Lie group with the holomorphic Riemannian metric

$$g = g_{ij} du^i \otimes du^j, \quad g_{ij} = C_{\alpha\beta} \chi_i^\alpha \chi_j^\beta. \tag{2.4}$$

A simple computation gives $g(E_\alpha, E_\beta) = C_{\alpha\beta}$ and the holomorphic metric tensor g will now be used to define the *gradient* of a holomorphic function f on G. If $\operatorname{grad} f = V^\beta E_\beta$ is a holomorphic vector field defined in a local chart, then the classical definition

$$g(U, \operatorname{grad} f) = Uf$$

for $U = U^\alpha E_\alpha$ yields $V^\beta = C^{\beta\alpha}(E_\alpha f)$, hence

$$\operatorname{grad} f = C^{\beta\alpha}(E_\alpha f) E_\beta. \tag{2.5}$$

A *Laplace operator for holomorphic functions* on G can now be introduced by

$$\Delta f = (\operatorname{div} \circ \operatorname{grad}) f = C^{\beta\alpha} E_\beta(E_\alpha f). \tag{2.6}$$

In local coordinates, this reads

$$\begin{aligned}\Delta f &= C^{\beta\alpha}\chi_\beta^i \frac{\partial}{\partial u^i}\left(\chi_\alpha^j \frac{\partial f}{\partial u^j}\right)\\ &= C^{\beta\alpha}\chi_\beta^i \frac{\partial \chi_\alpha^j}{\partial u^i}\frac{\partial f}{\partial u^j} + C^{\beta\alpha}\chi_\beta^i\chi_\alpha^j \frac{\partial^2 f}{\partial u^i \partial u^j}.\end{aligned}$$

But

$$\chi_\beta^i \frac{\partial \chi_\alpha^j}{\partial u^i} = -\chi_\beta^i \chi_\alpha^k \Gamma_{ki}^{\,j},$$

where $\Gamma_{jk}^{\,i} = -\chi_j^\beta \dfrac{\partial \chi_\beta^i}{\partial u^k}$ are the coefficients of the holomorphic connection written in the form (1.14), such that

$$\Delta f = g^{ij}\left(\frac{\partial^2 f}{\partial u^i \partial u^j} - \Gamma_{ji}^{\,k}\frac{\partial f}{\partial u^k}\right). \tag{2.7}$$

Since

$$\frac{\partial^2 f}{\partial u^i \partial u^j} - \Gamma_{ji}^{\,k}\frac{\partial f}{\partial u^k} = \nabla_i \nabla_j f,$$

where ∇_k is the covariant derivative with respect to the left invariant holomorphic connection ∇ defined in the previous section, this leads to the following formula for the Laplace operator of holomorphic functions on G:

$$\Delta f = g^{ij}\nabla_i \nabla_j f. \tag{2.8}$$

Remark 2.1. If G is not semi-simple then a holomorphic Riemannian metric on G can be defined by setting

$$h = h_{ij} du^i \otimes du^j, \quad h_{ij} = \delta_{\alpha\beta}\chi_i^\alpha \chi_j^\beta, \tag{2.9}$$

and similar computations as above lead to the following local expression of the Laplacian:

$$\Delta f = E_\alpha^2 f = h^{ij}\nabla_i \nabla_j f. \tag{2.10}$$

Let us compute the local expression of the Laplacian in two particular cases.

Example 2.2. Consider the standard 4-dimensional complex manifold $\mathbb{C}^4$ with the holomorphic coordinates (z^1, z^2, z^3, z^4) and the following multiplication rule:

$$\begin{aligned}&(z^1, z^2, z^3, z^4)\cdot(w^1, w^2, w^3, w^4) = \\ &\quad =(z^1 e^{\lambda w^3} + w^1, z^2 e^{-\lambda w^3} + w^2, z^3 + w^3, z^4 + w^4 - \lambda z^1 w^2 e^{\lambda w^3}),\end{aligned} \tag{2.11}$$

where λ is a nonzero complex parameter. The above multiplication rule endows $\mathbb{C}^4$ with a non-abelian complex Lie structure. For $\lambda = 0$, we obtain the usual abelian Lie group $\mathbb{C}^4$, therefore we will consider here $\lambda \neq 0$. We denote by G the non-abelian complex Lie group $\mathbb{C}^4$ endowed with the multiplication rule (2.11).

It is easy to see that the following left-invariant holomorphic vector fields given by

$$Z_1 = \frac{\partial}{\partial z^1}, \quad Z_2 = \frac{\partial}{\partial z^2} - \lambda z^1 \frac{\partial}{\partial z^4}, \quad Z_3 = \lambda z^1 \frac{\partial}{\partial z^1} - \lambda z^2 \frac{\partial}{\partial z^2} + \frac{\partial}{\partial z^3}, \quad Z_4 = \frac{\partial}{\partial z^4} \tag{2.12}$$

form a basis of the holomorphic Lie algebra $\mathfrak{g}$ of G. If we compute the Lie brackets of these holomorphic vector fields, we obtain

$$[Z_1, Z_2] = -\lambda Z_4, \quad [Z_1, Z_3] = \lambda Z_1, \quad [Z_2, Z_3] = -\lambda Z_2,$$
$$[Z_1, Z_4] = [Z_2, Z_4] = [Z_3, Z_4] = 0,$$

therefore, the components of the Lie brackets are constant. Hence, they are the structure constants of $\mathfrak{g}$ with respect to the basis $\{Z_1, Z_2, Z_3, Z_4\}$. We have $C_{\alpha\beta}^{\gamma} = 0$, $\alpha, \beta, \gamma = \overline{1,4}$ with the following exceptions:

$$C_{12}^{4} = -\lambda, \qquad C_{13}^{1} = \lambda, \qquad C_{23}^{2} = -\lambda.$$

The tensor field introduced by (1.8) will consequently vanish, i.e., $C_{\alpha\beta} = 0$ for all $\alpha, \beta = \overline{1,4}$, which means that G is not semi-simple. Then, according to (2.10), the Laplace operator Δ acting on holomorphic functions $f \in Hol(\mathbb{C}^4)$ is

$$\Delta f = \sum_{\alpha} Z_\alpha^2 f = Z_1^2 f + Z_2^2 f + Z_3^2 f + Z_4^2 f.$$

Now, a basic computation using (2.12) gives

$$\begin{aligned}\Delta f = &\big(1 + \lambda^2 (z^1)^2\big) \frac{\partial^2 f}{\partial (z^1)^2} + \big(1 + \lambda^2 (z^2)^2\big) \frac{\partial^2 f}{\partial (z^2)^2} + \frac{\partial^2 f}{\partial (z^3)^2} \\ &+ \big(1 + \lambda^2 (z^1)^2\big) \frac{\partial^2 f}{\partial (z^4)^2} - 2\lambda^2 z^1 z^2 \frac{\partial^2 f}{\partial z^1 \partial z^2} + 2\lambda z^1 \frac{\partial^2 f}{\partial z^1 \partial z^3} \\ &- 2\lambda z^2 \frac{\partial^2 f}{\partial z^2 \partial z^3} - 2\lambda z^1 \frac{\partial^2 f}{\partial z^2 \partial z^4} + \lambda^2 z^1 \frac{\partial f}{\partial z^1} + \lambda^2 z^2 \frac{\partial f}{\partial z^2}.\end{aligned}$$

Example 2.3. Let $G = \mathbb{C}^* \times \mathbb{C}$ with the multiplication

$$(z^1, z^2) \circ (w^1, w^2) = \big(z^1 w^1, \frac{1}{2} w z^1 w^2 + z^2 (w^1)^2\big)$$

and consider the vector fields $Z_1 = z^1 \frac{\partial}{\partial z^1} + 2z^2 \frac{\partial}{\partial z^2}$, $Z_2 = z^1 \frac{\partial}{\partial z^2}$. Then, $(G, \circ)$ is a complex Lie group with the holomorphic Lie algebra $\mathfrak{g} = \mathrm{span}\{Z_1, Z_2\}$. Moreover, G is not semi-simple, as it can be easily shown by computating the tensor $C_{\alpha\beta}$, as in the previous example. We therefore have $\Delta f = Z_1^2 f + Z_2^2 f$, which yields the Laplacian in the form

$$\begin{aligned}\Delta f = &(z^1)^2 \frac{\partial^2 f}{\partial (z^1)^2} + \big((z^1)^2 + 4(z^2)^2\big) \frac{\partial^2 f}{\partial (z^2)^2} \\ &+ 4z^1 z^2 \frac{\partial^2 f}{\partial z^1 \partial z^2} + z^1 \frac{\partial f}{\partial z^1} + 4z^2 \frac{\partial f}{\partial z^2}.\end{aligned}$$

We will use this example later for illustrating another property of the Laplace operator.

A straightforward computation gives an interesting property of the Laplacian introduced above in the general case.

Proposition 2.4. *The following identity holds:*

$$[\Delta, E_\alpha] = 2(h^{ij}\chi^k_\alpha - h^{ik}\chi^j_\alpha)\Gamma^{\,l}_{jk}\frac{\partial^2}{\partial u^i \partial u^l}, \tag{2.13}$$

where $h^{ij} = \delta^{\alpha\beta}\chi^i_\alpha\chi^j_\beta$ *and* $\Gamma^{\,l}_{jk}$ *are the local coefficients of the holomorphic connection* ∇.

Let us check the result in the case of the Lie group $G = \mathbb{C}^* \times \mathbb{C}$ from Example 2.3.

Example 2.5. First, we compute

$$\begin{aligned}[]
[\Delta, Z_1]f &= 2(z^1)^2\frac{\partial^2 f}{\partial (z^2)^2}; \\
[\Delta, Z_2]f &= -2(z^1)^2\frac{\partial^2 f}{\partial z^1 \partial z^2} - 4z^1z^2\frac{\partial^2 f}{\partial (z^2)^2}.
\end{aligned} \tag{2.14}$$

Then, from $Z_\alpha = \chi^i_\alpha \dfrac{\partial}{\partial z^i}$ we get

$$\chi^1_1 = z^1, \quad \chi^2_1 = 2z^2, \quad \chi^1_2 = 0, \quad \chi^2_2 = z^1,$$

such that, using (1.13), we can compute the coefficients $\Gamma^{\,l}_{jk}$:

$$\Gamma^{\,1}_{11} = -\frac{1}{z^1}, \quad \Gamma^{\,1}_{12} = \Gamma^{\,1}_{21} = \Gamma^{\,1}_{22} = 0,$$

$$\Gamma^{\,2}_{11} = \frac{2z^2}{(z^1)^2}, \quad \Gamma^{\,2}_{12} = -\frac{2}{z^1}, \quad \Gamma^{\,2}_{21} = -\frac{1}{z^1}, \quad \Gamma^{\,2}_{22} = 0.$$

We also need $h^{ij} = \delta^{\alpha\beta}\chi^i_\alpha\chi^j_\beta$, that is,

$$h^{11} = (z^1)^2, \quad h^{12} = h^{21} = 2z^1z^2, \quad h^{22} = (z^1)^2 + 4(z^2)^2.$$

Hence, replacing the nonzero terms in the left-hand side of the first identity (2.14) and doing a straightforward computation yields

$$\begin{aligned}[]
[\Delta, Z_1]f &= 2(h^{ij}\chi^k_1 - h^{ik}\chi^j_1)\Gamma^{\,l}_{jk}\frac{\partial^2}{\partial z^i \partial z^l} \\
&= 2\big[(h^{11}\chi^2_1 - h^{12}\chi^1_1)\Gamma^{\,2}_{12} + (h^{12}\chi^1_1 - h^{11}\chi^2_1)\Gamma^{\,2}_{21}\big]\frac{\partial^2 f}{\partial z^1 \partial z^2} \\
&\quad + 2\big[(h^{21}\chi^2_1 - h^{22}\chi^1_1)\Gamma^{\,2}_{12} + (h^{22}\chi^1_1 - h^{21}\chi^2_1)\Gamma^{\,2}_{21}\big]\frac{\partial^2 f}{\partial (z^2)^2} \\
&= 2(z^1)^2\frac{\partial^2 f}{\partial (z^2)^2},
\end{aligned}$$

since the first term vanishes. The second identity from (2.14) follows analogously.

We shall also illustrate the property from Proposition 2.4 in the case of the complex Lie group $GL(n, \mathbb{C})$.

Example 2.6. As $\dim(GL(n,\mathbb{C})) = n^2$, all the indices from the general case will be replaced by pairs of indices, for instance α becomes $\binom{\alpha}{\beta}$, i becomes $\binom{i}{m}$, etc. As a convention, these pairs will be rewritten in a manner that should be clear from the text below.

First, let $u \in GL(n, \mathbb{C})$ be a complex matrix with elements $\{A_i^\alpha\}$, such that a left-invariant holomorphic vector field will be denoted by

$$E_\alpha^\beta := E_{\binom{\alpha}{\beta}} = \chi_{\binom{\alpha}{\beta}}^{\binom{i}{m}} \frac{\partial}{\partial u^{\binom{i}{m}}} =: \chi_{\alpha m}^{i\beta} \frac{\partial}{\partial u_m^i},$$

where $\chi_{\alpha m}^{i\beta} \dfrac{\partial}{\partial u_m^i} = \delta_\alpha^i A_m^\beta$ (see [7] for more details). The holomorphic Riemannian metric is

$$h^{\binom{i}{m}\binom{j}{n}} =: h_{mn}^{ij} = \delta^{\alpha\beta} \delta_{\nu\mu} \chi_{\alpha m}^{i\nu} \chi_{\beta n}^{j\mu}$$

(the group $GL(n, \mathbb{C})$ is *not* semi-simple). The local coefficients of the holomorphic connection defined in Section 1 are

$$\Gamma_{\binom{j}{n}\,\binom{k}{p}}^{\binom{l}{q}} =: \Gamma_{jkq}^{lnp} = \chi_{j\tau}^{\varepsilon n} \frac{\partial \chi_{\varepsilon q}^{l\tau}}{\partial u_p^k}.$$

These yield

$$\begin{aligned}
[\Delta, E_\gamma] f &= 2\big(h_{mn}^{ij} \chi_{\gamma p}^{k\sigma} - h_{mp}^{ik} \chi_{\gamma n}^{j\sigma}\big) \Gamma_{jkq}^{lnp} \frac{\partial^2 f}{\partial u_m^i \partial u_q^l} \\
&= -2\big(\delta^{\alpha\beta} \delta_{\nu\mu} \chi_{\alpha m}^{i\nu} \chi_{\beta n}^{j\mu} \chi_{\gamma p}^{k\sigma} - \delta^{\alpha\beta} \delta_{\nu\mu} \chi_{\alpha m}^{i\nu} \chi_{\beta p}^{k\mu} \chi_{\gamma n}^{j\sigma}\big) \chi_{j\tau}^{\varepsilon n} \frac{\partial \chi_{\varepsilon q}^{l\tau}}{\partial u_p^k} \frac{\partial^2 f}{\partial u_m^i \partial u_q^l} \\
&= -2\Bigg(\delta^{\alpha\beta} \delta_{\nu\mu} \delta_\alpha^i A_m^\nu \delta_\beta^j A_n^\mu \delta_\gamma^k A_p^\sigma \delta_j^\varepsilon A_\tau^n \delta_\varepsilon^l \frac{\partial A_q^\tau}{\partial u_p^k} \\
&\qquad - \delta^{\alpha\beta} \delta_{\nu\mu} \delta_\alpha^i A_m^\nu \delta_\beta^k A_p^\mu \delta_\gamma^j A_n^\sigma \delta_j^\varepsilon A_\tau^n \delta_\varepsilon^l \frac{\partial A_q^\tau}{\partial u_p^k} \Bigg) \frac{\partial^2 f}{\partial u_m^i \partial u_q^l} \\
&= 2\Bigg(\delta_{\nu\mu} A_m^\nu A_p^\sigma \frac{\partial A_q^\mu}{\partial u_p^k} - \delta_{\nu\mu} A_m^\nu A_p^\mu \frac{\partial A_q^\sigma}{\partial u_p^k} \Bigg) \frac{\partial^2 f}{\partial u_m^i \partial u_q^l}.
\end{aligned}$$

Remark 2.7. Denoting by $\overset{\circ}{\nabla}$ the covariant derivative with respect to the Levi-Civita connection, the substitution of (1.19) in (2.7) yields

$$\begin{aligned}
\Delta f &= g^{ij} \left(\frac{\partial^2 f}{\partial u^i \partial u^j} - \overset{\circ}{\Gamma}{}_{ji}^{k} \frac{\partial f}{\partial u^k} \right) - T_{ji}^k \frac{\partial f}{\partial u^k} \\
&= g^{ij} \, \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j \, f - T_{ji}^k \frac{\partial f}{\partial u^k},
\end{aligned}$$

such that a harmonic holomorphic function f on G must satisfy the identity

$$T_{ji}^k \frac{\partial f}{\partial u^k} = g^{ij} \, \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j \, f. \tag{2.15}$$

Note that T_{jk}^i is the holomorphic torsion tensor of the holomorphic connection from (1.13).

3. Holomorphic last multipliers for holomorphic vector fields on G

The holomorphic volume element ω on G defined in Section 2 will now be used to introduce the notion of holomorphic last multipliers. The computations are similar to the case of smooth manifolds, [2, 3, 4, 5], or complex manifolds [6]. More precisely, consider a holomorphic vector field of the form $U = U^i \dfrac{\partial}{\partial u^i}$, $\theta = i_U\omega$ and let

$$\frac{du^i}{dt} = U^i(u^1(t), \dots, u^n(t)),\ 1 \le i \le n,\ t \in \mathbb{R}$$

be a complex ODE system on G defined by the holomorphic vector field U. The classical definition of a last multiplier function for a vector field on smooth manifolds, [2, 3], can now be applied to the case of the complex Lie group G.

Definition 3.1. A holomorphic function μ on G is called a holomorphic last multiplier of the complex ODE system generated by U (or holomorphic last multiplier for U) if

$$\partial(\mu\theta) := \partial\mu \wedge \theta + \mu \cdot \partial\theta = 0. \tag{3.1}$$

Note that for every holomorphic function μ on G, $\partial\mu\wedge\omega = 0$, such that for every holomorphic vector field U on G we have

$$0 = i_U(\partial\mu \wedge \omega) = (i_U\partial\mu) \cdot \omega - \partial\mu \wedge (i_U\omega)$$

or, equivalently,

$$U(\mu) \cdot \omega = \partial\mu \wedge (i_U\omega) = \partial\mu \wedge \theta.$$

Now, definitions (2.1) and (3.1) yield the following result.

Proposition 3.2. *A holomorphic function μ on G is a holomorphic last multiplier for the holomorphic vector field U if and only if*

$$U(\mu) + \mu \cdot \operatorname{div} U = 0. \tag{3.2}$$

Remark 3.3. Relation (3.2) indicates that if ν is a holomorphic non-zero function on G which satisfies the equation

$$L_U(\nu) := U(\nu) = (\operatorname{div} U) \cdot \nu, \tag{3.3}$$

then $1/\nu$ is a holomorphic last multiplier for U and the holomorphic function ν which satisfies (3.3) will be called an inverse holomorphic multiplier for U.

Proposition 3.4. *Let μ be a holomorphic function on G. The set of holomorphic vector fields for which μ is a holomorphic last multiplier is a Lie subalgebra in the algebra of holomorphic vector fields on G.*

Proof. The proof follows as in [6]. □

It is now interesting to search for a holomorphic last multiplier for a holomorphic vector field U of divergence type, that is, $\mu = \operatorname{div} V$ for some holomorphic vector field V on G. From (3.2),

$$U(\operatorname{div} V) + \operatorname{div} V \cdot \operatorname{div} U = 0. \tag{3.4}$$

Multiplying (3.4) by ω gives

$$L_U(\operatorname{div} V) \cdot \omega + \operatorname{div} V \cdot L_U\omega = 0$$

or, equivalently,

$$L_U(\operatorname{div} V \cdot \omega) = L_U L_V \omega = 0.$$

Hence, we have

Proposition 3.5. *If V is a holomorphic vector field which satisfies $L_U L_V \omega = 0$, then $\mu = \operatorname{div} V$ is a holomorphic last multiplier for the holomorphic vector field U.*

The next step is to study holomorphic last multipliers for holomorphic gradient vector fields on the complex Lie group G endowed with a holomorphic Riemannian metric (for instance g or h from Section 2). Such a metric g defines a *holomorphic metric volume form* ω_g (see [10]), as a holomorphic n-form on G such that

$$\omega_g(U_1, \ldots, U_n) = \pm 1,$$

where $\{U_i\}$, $i = 1, \ldots, n$, is an orthonormal holomorphic frame on (G, g), that is, $g(U_j, U_k) = \delta_{jk}$, $j, k = 1, \ldots, n$. As a complex manifold, if (G, g) admits such a volume element, it admits precisely two of them.

If f is a holomorphic function on G, $U = \operatorname{grad} f$ is the gradient vector field of f defined in Section 2 and α is a holomorphic last multiplier for U, then relation (3.2) becomes

$$g(\operatorname{grad} f, \operatorname{grad} \mu) + \mu \Delta f = 0. \tag{3.5}$$

A straightforward computation in local complex coordinates on G yields a similar identity to the case of holomorphic Riemannian manifolds, [6]:

$$g(\operatorname{grad} f, \operatorname{grad} \mu) = \frac{1}{2}\big(\Delta(f\mu) - f\Delta\mu - \mu\Delta f\big). \tag{3.6}$$

Hence,

$$\Delta(f\alpha) + \mu\Delta f = f\Delta\alpha, \tag{3.7}$$

which leads to the following result.

Proposition 3.6. *Let G be a complex Lie group endowed with a holomorphic metric g. If f, μ are holomorphic functions on G such that f is a holomorphic last multiplier for* $\operatorname{grad} \mu$ *and μ is a holomorphic last multiplier for* $\operatorname{grad} f$, *then $f\alpha$ is a holomorphic harmonic function on G.*

Corollary 3.7. *If G is a complex Lie group endowed with a holomorphic metric g and f is a holomorphic function on G, then μ is a holomorphic last multiplier for $U = \operatorname{grad} \mu$ if and only if μ^2 is a holomorphic harmonic function on G.*

Corollary 3.8. *If G is a complex Lie group endowed with a holomorphic metric g and f is a holomorphic function on G, then μ^2 is a holomorphic harmonic function on G if and only if*

$$\mu\Delta\mu + g(\operatorname{grad} \mu, \operatorname{grad} \mu) = 0.$$

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