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On a new family of generalized Bernstein operators

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Abstract. In this paper we remark that α-Bernstein operators, introduced by X. Y. Chen et al., are combinations of two known operators (Stancu and Bernstein operators) and we establish the preservation of global smoothness properties by these linear operators, the global smoothness being expressed by a Lipschitz condition with a certain second order modulus of continuity.

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1. Introduction

X.Y. Chen et al. [5] introduced and studied a family of operators as follows. For a function $f : [0,1] \longrightarrow \mathbb{R}$, the α-Bernstein operators $T_{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ fixed, are defined by

$$T_{n,\alpha}(f,x) = \sum_{j=0}^{n} p_{n,j}^{(\alpha)}(x) f\left(\frac{j}{n}\right),\ x \in [0,1], \tag{1.1}$$

where $p_{1,0}^{(\alpha)}(x) = 1-x$, $p_{1,1}^{(\alpha)}(x) = x$ and for $n \geq 2$,

$$\begin{aligned} p_{n,j}^{(\alpha)}(x) \quad = \quad & \left[\binom{n-2}{j}(1-\alpha)x + \binom{n-2}{j-2}(1-\alpha)(1-x) + \binom{n}{j}\alpha x(1-x)\right] \cdot \\ & \cdot x^{j-1}(1-x)^{n-j-1} \end{aligned}$$

with the convention

$$\binom{k}{l} = \begin{cases} \dfrac{k!}{l!(k-l)!}, & \text{if } 0 \leq l \leq k, \\ 0, & \text{else.} \end{cases}$$

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It is obvious that for $\alpha = 1$ the class of Bernstein operators is obtained

$$T_{n,1}(f,x) = \sum_{j=0}^{n} p_{n,j}(x) f\left(\frac{j}{n}\right) = B_n(f,x),\ p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}.$$

We note that for the α-Bernstein operator another representation can be obtained as follows.

$$\begin{aligned} T_{n,\alpha}(f,x) &= (1-\alpha)\left[\sum_{j=0}^{n-2} p_{n-2,j}(x)(1-x) f\left(\frac{j}{n}\right) + \sum_{j=2}^{n} p_{n-2,j-2}(x) x f\left(\frac{j}{n}\right)\right] \\ &\quad + \alpha \sum_{j=0}^{n} p_{n,j}(x) f\left(\frac{j}{n}\right) \\ &= (1-\alpha)\left[\sum_{j=0}^{n-2} p_{n-2,j}(x)(1-x) f\left(\frac{j}{n}\right) + \sum_{j=0}^{n-2} p_{n-2,j}(x) x f\left(\frac{j+2}{n}\right)\right] \\ &\quad + \alpha B_n(f,x) \\ &= (1-\alpha)\sum_{j=0}^{n-2} p_{n-2,j}(x) \sum_{i=0}^{1} p_{1,i}(x) f\left(\frac{j+2i}{n}\right) + \alpha B_n(f,x) \end{aligned}$$

The following generalized Bernstein operators was introduced by D. D. Stancu (see [11])

$$S_{n,r,s}(f,x) = \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) f\left(\frac{j+ir}{n}\right), \tag{1.2}$$

$f \in C[0,1]$, $x \in [0,1]$, where $n \in \mathbb{N}$, $r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $rs < n$. Bernstein's operators are obtained for $s = 0$ or $s = 1$, $r = 0$ or $s = 1$, $r = 1$.

So the α-Bernstein operator can be expressed as

$$T_{n,\alpha}(f,x) = (1-\alpha) S_{n,2,1}(f,x) + \alpha B_n(f,x). \tag{1.3}$$

In [13] we introduced a two dimensional generalization of the Stancu operators (1.2) and established certain results related to the global smoothness preservation with respect to a second order modulus of continuity for functions defined on the 2-dimensional simplex. The corresponding results in the one-dimensional case are presented in the following section.

The preservation of global smoothness properties by the Bernstein operators was studied in [7], [8], [4], [2], [6], [3]. In [14], D.-X. Zhou showed that the Lipschitz classes with respect to the second order modulus

$$\omega_2(f,t) = \sup\left\{|f(x-h) - 2f(x) + f(x+h)| :\ x \pm h \in [0,1],\ 0 < h \le t\right\}$$

are not preserved by the Bernstein operators. He introduced the following second order modulus of smoothness

$$\begin{aligned} \widetilde{\omega}_2(f,t) &= \sup\{|f(x+h_1+h_2) - f(x+h_1) - f(x+h_2) + f(x)| : \\ &\qquad x, x+h_1+h_2 \in [0,1],\ h_1, h_2 > 0,\ h_1 + h_2 \le 2t\} \end{aligned}$$

and proved the following result:

Theorem A. *Let $f \in C[0,1]$, $n \in \mathbb{N}$, $M > 0$ and $0 < \mu \leq 1$.*

If $\widetilde{\omega}_2(f,t) \leq Mt^{\mu}$, $0 < t \leq \frac{1}{2}$, then $\widetilde{\omega}_2(B_n f,t) \leq Mt^{\mu}$, $0 < t \leq \frac{1}{2}$.

For the Bernstein-type operators

$$L_n(f,x) = \sum_{j=0}^{n} p_{n,j}(x) F_{n,j}(f),\ f \in C[0,1],\ x \in [0,1], \tag{1.4}$$

where $F_{n,j} : C[0,1] \longrightarrow \mathbb{R}$, $j = \overline{1,n}$, are linear positive functionals, in [12] we studied simultaneous global smoothness preservation in terms of modulus of continuity ω_2^* introduced by R. Păltănea [9], [10] and independently by J. Adell and J. de la Cal [1], defined for $f \in \mathbf{C}[0,1]$ and $t > 0$ by

$$\begin{aligned} \omega_2^*(f,t) &= \sup\{|(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y, y))| : \\ &\qquad x, y \in [0,1],\ x < y,\ y - x \leq 2t,\ \lambda \in [0,1]\}. \end{aligned}$$

The preservation of global smoothness properties by α-Bernstein operators is obtained as a consequence of global smoothness preservation by Stancu operators.

2. Global smoothness preservation

Lemma 2.1. *For $f \in C[0,1]$, $0 \leq x < y \leq 1$, $\lambda \in [0,1]$ we have*

$$\begin{aligned} &S_{n,r,s}(f,(1-\lambda)x + \lambda y) \\ &= \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \cdot \\ &\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) f\left(\frac{k_2 + m_2 + r(k_1+m_1)}{n}\right), \end{aligned} \tag{2.1}$$

where

$$p_{n,k,l}(x,y) = \frac{n!}{k!l!\,(n-k-l)!} x^k y^l (1-x-y)^{n-k-l}.$$

Proof. Let $f \in C[0,1]$, $0 \leq x < y \leq 1$, $\lambda \in [0,1]$.

For the Bernstein type operator (1.4) in [12], proceeding similarly as in [14] (see also [4]), we obtained:

$$\begin{aligned} &L_n(f,(1-\lambda)x + \lambda y) \\ &= \sum_{k+l=0}^{n} p_{n,k,l}(x, y-x) \sum_{m=0}^{l} p_{l,m}(\lambda) F_{n,k+m}(f). \end{aligned} \tag{2.2}$$

Repeating the application of an adapted version of relation (2.2) yields

$$S_{n,r,s}(f,(1-\lambda)x + \lambda y) = \sum_{j=0}^{n-rs} p_{n-rs,j}((1-\lambda)x+\lambda y) \sum_{i=0}^{s} p_{s,i}((1-\lambda)x+\lambda y) f\left(\frac{j+ir}{n}\right)$$

$$= \sum_{j=0}^{n-rs} p_{n-rs,j}((1-\lambda)x+\lambda y) \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(x, y-x) \sum_{m_1=0}^{l_1} p_{l_1,m_1}(\lambda) f\left(\frac{j+r(k_1+m_1)}{n}\right)$$

$$= \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(x, y-x) \sum_{m_1=0}^{l_1} p_{l_1,m_1}(\lambda) \sum_{j=0}^{n-rs} p_{n-rs,j}((1-\lambda)x+\lambda y) f\left(\frac{j+r(k_1+m_1)}{n}\right)$$

$$= \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(x, y-x) \sum_{m_1=0}^{l_1} p_{l_1,m_1}(\lambda) \cdot$$

$$\cdot \sum_{k_2+l_2=0}^{n-rs} p_{n-rs,k_2,l_2}(x, y-x) \sum_{m_2=0}^{l_2} p_{l_2,m_2}(\lambda) f\left(\frac{k_2+m_2+r(k_1+m_1)}{n}\right).$$

□

Theorem 2.2. *Let $f \in C[0,1]$, $M > 0$ and $\mu \in (0,1]$. If*

$$\omega_1(f,t) \le Mt^\mu,\ t \in (0,1],$$

then

$$\omega_1(S_{n,r,s}f,t) \le Mt^\mu,\ t \in (0,1].$$

Proof. Let $x, y \in [0,1]$ be such that $|x-y| \le t$. We can assume that $x < y$.

$$|S_{n,r,s}(f,x) - S_{n,r,s}(f,y)|$$

$$\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x) \cdot$$

$$\cdot \left| f\left(\frac{k_2+rk_1}{n}\right) - f\left(\frac{k_2+l_2+r(k_1+l_1)}{n}\right)\right|$$

$$\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x) \omega_1\left(f, \frac{l_2+rl_1}{n}\right)$$

$$\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x) M\left(\frac{l_2+rl_1}{n}\right)^\mu$$

$$\le M\left(\sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x) \frac{l_2+rl_1}{n}\right)^\mu$$

$$= M\left(\sum_{k_2+l_2=0}^{n-rs} p_{n-rs,k_2,l_2}(x,y-x)\frac{l_2}{n} + \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(x,y-x)\frac{rl_1}{n}\right)^\mu$$

$$= M\left(\frac{n-rs}{n}(y-x) + \frac{rs}{n}(y-x)\right)^\mu \le Mt^\mu.$$

□

The next result relates to the global smoothness preservation by Stancu operators in terms of modulus of continuity ω_2^*.

Theorem 2.3. *Let $f \in C[0,1]$, $M > 0$ and $\mu \in (0,1]$. If*

$$\omega_2^* (f,t) \leq Mt^{\mu},\ t \in \left(0, \frac{1}{2}\right],$$

then

$$\omega_2^* (S_{n,r,s}f,t) \leq Mt^{\mu},\ t \in \left(0, \frac{1}{2}\right].$$

Proof. Let $t \in \left(0, \frac{1}{2}\right]$, $x, y \in [0,1]$, $x < y$, $y - x \leq 2t$, $\lambda \in [0,1]$. By using the representation (2.1), we obtain:

$$\begin{aligned}
&|(1-\lambda)S_{n,r,s}f(x) + \lambda S_{n,r,s}f(y) - S_{n,r,s}f\left((1-\lambda)x - \lambda y\right)| \\
&\leq \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \cdot \\
&\cdot \Bigg| (1-\lambda) f\left(\frac{k_2 + rk_1}{n}\right) + \lambda f\left(\frac{k_2 + l_2 + r(k_1 + l_1)}{n}\right) \\
&- \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) f\left(\frac{k_2 + m_2 + r(k_1 + m_1)}{n}\right) \Bigg| \\
&\leq \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1 \neq 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \cdot \\
&\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) \cdot \Bigg| \left(1 - \frac{m_2 + rm_1}{l_2 + rl_1}\right) f\left(\frac{k_2 + rk_1}{n}\right) \\
&+ \frac{m_2 + rm_1}{l_2 + rl_1} f\left(\frac{k_2 + rk_1}{n} + \frac{l_2 + rl_1}{n}\right) - f\left(\frac{k_2 + rk_1}{n} + \frac{m_2 + rm_1}{n}\right) \Bigg| \\
&\leq \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1 \neq 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \cdot \\
&\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) \omega_2^* \left(f, \frac{l_2 + rl_1}{2n}\right) \\
&\leq M \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \left(\frac{l_2 + rl_1}{2n}\right)^{\mu} \\
&\leq M \left(\sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \frac{l_2 + rl_1}{2n}\right)^{\mu} \\
&= M \left(\frac{n - rs}{2n}(y-x) + \frac{rs}{2n}(y-x)\right)^{\mu} \\
&= M \left(\frac{y-x}{2}\right)^{\mu} \leq Mt^{\mu}.
\end{aligned}$$

Hence $\omega_2^*\left(S_{n,r,s}f,t\right) \le Mt^{\mu}$. □

For $n \in \mathbb{N}$, $\alpha \in [0,1]$, $r_1, s_1, r_2, s_2 \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ such that $r_1 s_1, r_2 s_2 < n$, we consider the operators

$$T_{n,\alpha}^{r_1,s_1,r_2,s_2}(f,x) = (1-\alpha)S_{n,r_1,s_1}(f,x) + \alpha S_{n,r_2,s_2}(f,x), \tag{2.3}$$

$f \in C[0,1]$, $x \in [0,1]$.
From Theorem 2.2, Theorem 2.3 and the inequalities

$$\omega_1\left(T_{n,\alpha}^{r_1,s_1,r_2,s_2}f,t\right) \le (1-\alpha)\omega_1\left(S_{n,r_1,s_1}f,t\right) + \alpha\omega_1\left(S_{n,r_2,s_2}f,t\right),$$
$$\omega_2^*\left(T_{n,\alpha}^{r_1,s_1,r_2,s_2}f,t\right) \le (1-\alpha)\omega_2^*\left(S_{n,r_1,s_1}f,t\right) + \alpha\omega_2^*\left(S_{n,r_2,s_2}f,t\right),$$

we obtain the final result:

Theorem 2.4. *Let $f \in C[0,1]$, $M > 0$ and $\mu \in (0,1]$.*

1. *If $\omega_1(f,t) \le Mt^{\mu}$, $t \in (0,1]$, then $\omega_1\left(T_{n,\alpha}^{r_1,s_1,r_2,s_2}f,t\right) \le Mt^{\mu}$, $t \in (0,1]$.*
2. *If $\omega_2^*(f,t) \le Mt^{\mu}$, $t \in \left(0,\frac{1}{2}\right]$, then $\omega_2^*\left(T_{n,\alpha}^{r_1,s_1,r_2,s_2}f,t\right) \le Mt^{\mu}$, $t \in \left(0,\frac{1}{2}\right]$.*

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