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A strong converse inequality for the iterated Boolean sums of the Bernstein operator

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Abstract. We establish a two-term strong converse estimate of the rate of approximation by the iterated Boolean sums of the Bernstein operator. The characterization is stated in terms of appropriate moduli of smoothness or K-functionals.

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1. Main results

The Bernstein operator is defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Its iterated Boolean sum $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$ is then defined by

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}$.

Gonska and Zhou [9] estimated the uniform norm of the approximation error for $\mathcal{B}_{r,n}$. They proved a neat direct inequality and a Stechkin-type converse inequality. The former states

$$\|\mathcal{B}_{r,n} f - f\| \le c \left(\omega_{\varphi}^{2r}(f, n^{-1/2}) + \frac{1}{n^r}\|f\|\right). \tag{1.1}$$

Above $\|\circ\|$ denotes the uniform norm on the interval $[0,1]$, c is a constant independent of the approximated function and the order of the operator (not necessarily the same

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at each occurrence), and $\omega_\varphi^r(f,t)$ denotes the Ditzian-Totik modulus of smoothness with $\varphi(x)=\sqrt{x(1-x)}$, which is given by (see [5, Chapter 1])

$$\omega_\varphi^r(f,t)=\sup_{0<h\le t}\|\Delta_{h\varphi}^r f\|,$$

where

$$\Delta_{h\varphi(x)}^r f(x)=\begin{cases}\displaystyle\sum_{k=0}^r(-1)^k\binom{r}{k}f\left(x+\left(\frac{r}{2}-k\right)h\varphi(x)\right), & x\pm rh\varphi(x)/2\in[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

Let us recall that the modulus $\omega_\varphi^r(f,t)$ is equivalent to the K-functional

$$K_{r,\varphi}(f,t^r)=\inf_{g\in AC_{loc}^{r-1}(0,1)}\left\{\|f-g\|+t^r\|\varphi^r g^{(r)}\|\right\}.$$

More precisely, we say that $\Phi(f,t)$ and $\Psi(f,t)$ are equivalent and write

$$\Phi(f,t)\sim\Psi(f,t)$$

if there exists a constant c such that $c^{-1}\Phi(f,t)\le\Psi(f,t)\le c\,\Phi(f,t)$ for all f and t under consideration. Thus there holds (see [5, Theorem 2.1.1])

$$K_{r,\varphi}(f,t^r)\sim\omega_\varphi^r(f,t),\quad 0<t\le t_0 \tag{1.2}$$

with some fixed $t_0>0$. It was shown in [11, Theorem 2.7] that we can take $t_0=2/r$. A smaller value of t_0 was given in [2, Chapter 6, Theorem 6.2].

Since the operator $\mathcal{B}_{r,n}$ preserves the algebraic polynomials of degree 1 and the modulus $\omega_\varphi^{2r}(f,n^{-1/2})$ is invariant to translation of f by such polynomials, we immediately deduce from (1.1) the estimate

$$\|\mathcal{B}_{r,n}f-f\|\le c\left(\omega_\varphi^{2r}(f,n^{-1/2})+\frac{1}{n^r}E_1(f)\right), \tag{1.3}$$

where $E_1(f)$ is the best approximation of f by algebraic polynomials of degree 1 in the uniform norm on $[0,1]$.

Later on Ding and Cao [3] characterized the error of the multivariate generalization of $\mathcal{B}_{r,n}$ on the simplex. In the univariate case, the direct inequality they proved is of the form

$$\|\mathcal{B}_{r,n}f-f\|\le c\,K_r(f,n^{-r}), \tag{1.4}$$

where

$$K_r(f,t)=\inf_{g\in C^{2r}[0,1]}\{\|f-g\|+t\|D^r g\|\},\quad Dg=\varphi^2g''.$$

They also proved a strong converse inequality of type D (in the terminology introduced in [4]), that is

$$K_r(f,n^{-r})\le c\max_{k\ge n}\|\mathcal{B}_{r,k}f-f\|. \tag{1.5}$$

As it was shown in [6, Theorem 5.1],

$$K_r(f,t)\sim K_{2r,\varphi}(f,t)+tE_1(f),\quad 0<t\le 1. \tag{1.6}$$

Therefore, taking also into account (1.2), we see that the function characteristics on the right side of (1.3) and (1.4) are equivalent.

Quite recently, Cheng and Zhou [1] derived another converse inequality from the Stechkin-type converse inequality in [9]. It is similar to (1.5), though weaker than it.

Our main result improves (1.5). We will prove the following strong converse inequality of type B according to [4].

Theorem 1.1. *Let $r \in \mathbb{N}$. There exists $R \in \mathbb{N}$ such that for all $f \in C[0,1]$ and $k, n \in \mathbb{N}$ with $k \geq Rn$ there holds*

$$K_r(f, n^{-r}) \leq c \left(\frac{k}{n}\right)^r \left(\|\mathcal{B}_{r,n}f - f\| + \|\mathcal{B}_{r,k}f - f\|\right).$$

In particular,

$$K_r(f, n^{-r}) \leq c \left(\|\mathcal{B}_{r,n}f - f\| + \|\mathcal{B}_{r,Rn}f - f\|\right).$$

Let us recall that the assertion of the theorem for $r = 1$ was established in [4, Theorem 8.1] and then improved to a one-term converse inequality (i.e. $R = 1$) in [10, 12].

As we mentioned earlier in (1.6), the more complicated K-functional $K_r(f,t)$ can be replaced with the simpler function characteristics $K_{2r,\varphi}(f,t) + tE_1(f)$. In addition to this, we will establish also the following equivalence relation.

Theorem 1.2. *Let $r \in \mathbb{N}$. For all $f \in C[0,1]$ and $0 < t \leq 1$ we have*

$$K_r(f,t) \sim K_{2r,\varphi}(f,t) + K_{2,\varphi}(f,t).$$

Taking into account (1.2), we arrive at the following relation between $K_r(f,t)$ and the Ditzian-Totik modulus.

Corollary 1.3. *Let $r \in \mathbb{N}$. For all $f \in C[0,1]$ and $n \in \mathbb{N}$ such that $n \geq r^2$ we have*

$$K_r(f, n^{-r}) \sim \omega_\varphi^{2r}(f, n^{-1/2}) + \omega_\varphi^2(f, n^{-r/2}).$$

We establish Theorem 1.1 by means of the method given in [4]. To this end, we need a Voronovskaya-type inequality and several Bernstein-type inequalities, which relate the approximation operator $\mathcal{B}_{r,n}$ to the differential operator D^r. They are given in Section 2. Then, in the next section we prove Theorem 1.1. We present the short argument that verifies Theorem 1.2 in the last section.

2. Voronovskaya- and Bernstein-type inequalities for $\mathcal{B}_{r,n}$

We will use the following inequalities, which were obtained by Gonska and Zhou [9, (2) and (4)] for algebraic polynomials, but, as it is easy to see, the same considerations verify them for all functions in $C^{2r}[0,1]$.

Proposition 2.1. *For $g \in C^{2r}[0,1]$ there hold:*

(a) $\|\varphi^{2r} g^{(2r)}\| \leq c\,\|D^r g\|$;
(b) $\|D^j g\| \leq c\,\|D^r g\|, \quad j = 1, \ldots, r.$

We proceed to two Voronovskaya-type estimates (cf. [9, Lemma 4]).

Proposition 2.2. *Let $r \in \mathbb{N}$. For all $g \in C^{2r+2}[0,1]$ and all $n \in \mathbb{N}$ there hold*

$$\left\| \mathcal{B}_{r,n} g - g - \frac{(-1)^{r-1}}{(2n)^r} D^r g \right\| \le \frac{c}{n^{r+1}} \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right) \tag{2.1}$$

and

$$\left\| \mathcal{B}_{r,n} g - g - \frac{(-1)^{r-1}}{(2n)^r} D^r g \right\| \le \frac{c}{n^{r+1}} \|D^{r+1} g\|. \tag{2.2}$$

Proof. Assertion (2.1) for $r = 1$ follows from [8, Proposition 2.3].
Next, we set $J_{r,n} g = (I - B_n)^r g$ and

$$V_{r,n} g = \mathcal{B}_{r,n} g - g - \frac{(-1)^{r-1}}{(2n)^r} D^r g.$$

For $r \ge 2$ we use the relation

$$V_{r,n} g = V_{1,n} J_{r-1,n} g - \frac{1}{2n} D V_{r-1,n} g.$$

It implies

$$\|V_{r,n} g\| \le \|V_{1,n} J_{r-1,n} g\| + \frac{1}{n} \|\varphi^2 (V_{r-1,n} g)''\|. \tag{2.3}$$

By virtue of (2.1) with $r = 1$,

$$\|V_{1,n} J_{r-1,n} g\| \le \frac{c}{n^2} \left(\|\varphi^2 (J_{r-1,n} g)^{(3)}\| + \|\varphi^4 (J_{r-1,n} g)^{(4)}\| \right). \tag{2.4}$$

Further, we estimate the first term on the right above by means of [7, Corollary 4.7] with $p = \infty$, $r - 1$ in place of r, $s = 3$ and $w = \varphi^2$ (i.e. $\gamma_0 = \gamma_1 = 1$). Thus we get

$$\|\varphi^2 (J_{r-1,n} g)^{(3)}\| \le \frac{c}{n^{r-1}} \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r} g^{(2r+1)}\| \right). \tag{2.5}$$

Similarly, again by [7, Corollary 4.7] with $p = \infty$ and $r - 1$ in place of r, but $s = 4$ and $w = \varphi^4$ (i.e. $\gamma_0 = \gamma_1 = 2$) we have for the other term

$$\|\varphi^4 (J_{r-1,n} g)^{(4)}\| \le \frac{c}{n^{r-1}} \left(\|\varphi^4 g^{(4)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right). \tag{2.6}$$

Next, by virtue of [7, Proposition 2.1] with $p = \infty$, $j = 1$, $m = 2r - 1$, $w_1 = \varphi^4$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = 2$), $w_2 = \varphi^{2r+2}$ (i.e. $\gamma_{2,0} = \gamma_{2,1} = r + 1$) and $g^{(3)}$ in place of g, we get

$$\|\varphi^4 g^{(4)}\| \le c \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right). \tag{2.7}$$

Likewise, by means of the same proposition with $p = \infty$, $m = 2r - 1$, $w_2 = \varphi^{2r+2}$ and $g^{(3)}$ in place of g, but with $j = 2r - 2$ and $w_1 = \varphi^{2r}$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = r$), we get

$$\|\varphi^{2r} g^{(2r+1)}\| \le c \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right). \tag{2.8}$$

Combining, (2.4)-(2.8), we get

$$\|V_{1,n} J_{r-1,n} g\| \le \frac{c}{n^{r+1}} \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right). \tag{2.9}$$

It remains to estimate the second term on the right side of (2.3). To this end, we apply [7, Corollary 4.11] with $p = \infty$, $r-1$ in place of r, $s = 2$, and $w = \varphi^2$ (i.e. $\gamma_0 = \gamma_1 = 1$) and get

$$\|\varphi^2 (V_{r-1,n} g)''\| \le \frac{c}{n^r} \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right). \tag{2.10}$$

Now, (2.3), (2.9) and (2.10) imply (2.1) for $r \ge 2$.
To prove the second assertion of the proposition, we observe that Proposition 2.1(a) with $r+1$ in place of r yields

$$\|\varphi^{2r+2} g^{(2r+2)}\| \le c \, \|D^{r+1} g\|. \tag{2.11}$$

Also, by virtue of [7, Proposition 2.1] with $p = \infty$, $j = 1$, $m = 2r$, $w_1 = \varphi^2$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = 1$), $w_2 = \varphi^{2r+2}$ (i.e. $\gamma_{2,0} = \gamma_{2,1} = r+1$) and $g^{(2)}$ in place of g, we get

$$\|\varphi^2 g^{(3)}\| \le c \left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right).$$

Taking into account (2.11) and Proposition 2.1(b) with $j = 1$ and $r+1$ in place of r, we arrive at

$$\|\varphi^2 g^{(3)}\| \le c \, \|D^{r+1} g\|. \tag{2.12}$$

Now, (2.2) follows from (2.1), (2.11) and (2.12). □

Next we shall establish several Bernstein-type inequalities.

Proposition 2.3. *Let $r \in \mathbb{N}$. Then for all $f \in C[0,1]$ and $n \in \mathbb{N}$ there holds*

$$\|D^r \mathcal{B}_{r,n} f\| \le c \, n^r \|f\|.$$

Proof. It is established by induction on r that (cf. [9, p. 24])

$$D^r g = \varphi^2 \sum_{i=2}^{r+1} q_{r,i-2} \, g^{(i)} + \sum_{i=2}^{r} \varphi^{2i} \, \tilde{q}_{r,r-i} \, g^{(i+r)},$$

where $q_{r,j}$ and $\tilde{q}_{r,j}$ are algebraic polynomials of degree at most j.
Therefore

$$\|D^r g\| \le c \left(\sum_{i=2}^{r+1} \|\varphi^2 g^{(i)}\| + \sum_{i=2}^{r} \|\varphi^{2i} g^{(i+r)}\| \right). \tag{2.13}$$

Let $r \ge 2$. We apply [7, Proposition 2.1] with $p = \infty$, $j = i-2$, where $i \in \{2, \dots, r+1\}$, $m = 2r-2$, $w_1 = \varphi^2$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = 1$), $w_2 = \varphi^{2r}$ (i.e. $\gamma_{2,0} = \gamma_{2,1} = r$) and $g^{(2)}$ in place of g to get

$$\|\varphi^2 g^{(i)}\| \le c \left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r} g^{(2r)}\| \right), \quad i = 2, \dots, r+1. \tag{2.14}$$

Also, this trivially holds for $r = 1$.

Let $r \ge 3$. Similarly, [7, Proposition 2.1] with $p = \infty$, $j = i + r - 2$, where $i \in \{2, \dots, r-1\}$, $m = 2r-2$, $w_1 = \varphi^{2i}$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = i$), where $i \in \{2, \dots, r\}$, $w_2 = \varphi^{2r}$ (i.e. $\gamma_{2,0} = \gamma_{2,1} = r$) and $g^{(2)}$ in place of g to get

$$\|\varphi^{2i} g^{(i+r)}\| \le c \left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r} g^{(2r)}\| \right), \quad i = 2, \dots, r-1. \tag{2.15}$$

The above estimate trivially holds for $i = r$, $r \ge 2$, as well.

The inequalities (2.13)-(2.15) yield

$$\|D^r g\| \le c\left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r} g^{(2r)}\|\right), \quad r \in \mathbb{N}. \tag{2.16}$$

Setting $g = \mathcal{B}_{r,n} f$ we get

$$\|D^r \mathcal{B}_{r,n} f\| \le c\left(\|\varphi^2 (\mathcal{B}_{r,n} f)^{(2)}\| + \|\varphi^{2r} (\mathcal{B}_{r,n} f)^{(2r)}\|\right). \tag{2.17}$$

Then we take into account that the operator $\mathcal{B}_{r,n}$ is a linear combination of iterates of B_n and also that (see [5, (9.3.7)])

$$\|\varphi^{2\ell} (B_n g)^{(2\ell)}\| \le c\, \|\varphi^{2\ell} g^{(2\ell)}\|, \quad g \in C^{2\ell}[0,1], \tag{2.18}$$

to derive from (2.17) the estimate

$$\|D^r \mathcal{B}_{r,n} f\| \le c\left(\|\varphi^2 (B_n f)^{(2)}\| + \|\varphi^{2r} (B_n f)^{(2r)}\|\right).$$

Now, the assertion of the proposition follows from

$$\|\varphi^{2\ell} (B_n f)^{(2\ell)}\| \le c\, n^{\ell} \|f\|, \quad \ell \in \mathbb{N},$$

which was established in [5, Theorem 9.4.1]. □

Proposition 2.4. *Let $r \in \mathbb{N}$. Then for all $g \in C^{2r}[0,1]$ and $n \in \mathbb{N}$ there holds*

$$\|D^{r+1} \mathcal{B}_{r,n} g\| \le c\, n \|D^r g\|.$$

Proof. We make use of (2.16) with $r+1$ in place of r and $\mathcal{B}_{r,n} g$ in place of g, then apply (2.18), [7, Proposition 4.13(a)] with $p = \infty$, $w = \varphi^{2r}$ (i.e. $\gamma_0 = \gamma_1 = r$), $\ell = 1$, $s = 2r$, and, finally, Proposition 2.1 with $j = 1$, to arrive at

$$\begin{aligned}
\|D^{r+1} \mathcal{B}_{r,n} g\| &\le c\left(\|\varphi^2 (\mathcal{B}_{r,n} g)^{(2)}\| + \|\varphi^{2r+2} (\mathcal{B}_{r,n} g)^{(2r+2)}\|\right)\\
&\le c\left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r+2} (B_n g)^{(2r+2)}\|\right)\\
&\le c\left(\|\varphi^2 g^{(2)}\| + n\, \|\varphi^{2r} g^{(2r)}\|\right)\\
&\le c\, n \|D^r g\|.
\end{aligned}$$

Thus the proposition is verified. □

3. A proof of the converse inequalities

Equipped with the estimates established in the previous section, we are now ready to verify Theorem 1.1.

Proof of Theorem 1.1. We apply [4, Theorem 3.2] with the operator $Q_n = \mathcal{B}_{r,n}$ and the spaces $X = C[0,1]$ (with the uniform norm on $[0,1]$), $Y = C^{2r}[0,1]$ and $Z = C^{2r+2}[0,1]$.
As is known,

$$\|B_n f\| \le \|f\|.$$

Therefore, since $\mathcal{B}_{r,n}$ is linear combination of iterates of B_n, we have

$$\|\mathcal{B}_{r,n} f\| \le c\, \|f\|, \quad f \in C[0,1],\ n \in \mathbb{N}.$$

Thus [4, (3.3)] is satisfied.

By virtue of the Voronovskaya-type inequality (2.2), we have [4, (3.4)] with $(-1)^{r-1}D^r$ in place of D, $\Phi(f) = \|D^{r+1}f\|$, $\lambda(n) = (2n)^{-r}$ and $\lambda_1(\alpha) = c\,n^{-r-1}$, where the constant c is the one in (2.2).

Next, Proposition 2.4 with $g = \mathcal{B}_{r,n}f$ implies [4, (3.5)] with $\ell = 1$ and $m = 2$.

Finally, Proposition 2.3 yields [4, (3.6)]. □

Let us note that [4, Theorems 10.4 and 10.5] are not applicable because condition (c) there is not satisfied.

4. Relations between K-functionals

Proof of Theorem 1.2. In view of (1.6), it is sufficient to show that

$$K_{2r,\varphi}(f,t) + tE_1(f) \sim K_{2r,\varphi}(f,t) + K_{2,\varphi}(f,t), \quad 0 < t \le 1. \tag{4.1}$$

Trivially, for any $g \in C[0,1]$ such that $g \in AC^1_{loc}(0,1)$ and $\varphi^2 g'' \in L_\infty[0,1]$, and any $t \in (0,1]$ we have the estimates

$$tE_1(f) \le \|f-g\| + t\,\|g - B_1 g\| \le \|f-g\| + ct\|\varphi^2 g''\|;$$

hence

$$tE_1(f) \le c\,K_{2,\varphi}(f,t), \quad 0 < t \le 1.$$

Above we used the inequality

$$\|g - B_1 g\| \le \|\varphi^2 g''\|,$$

which is directly established by Taylor's formula (see e.g. [4, p. 87]).
To complete the proof of (4.1), it remains to show that

$$K_{2,\varphi}(f,t) \le c\,(K_{2r,\varphi}(f,t) + tE_1(f)), \quad 0 < t \le 1. \tag{4.2}$$

Let $g \in C[0,1]$ be such that $g \in AC^{2r-1}_{loc}(0,1)$ and $\varphi^{2r} g^{(2r)} \in L_\infty[0,1]$. Then, by e.g. [7, (2.9)] with $p = \infty$, $w = 1$, $j = 1$ and $m = r$, we deduce that $\varphi^2 g^{(2)} \in L_\infty[0,1]$ too, as, moreover,

$$\|\varphi^2 g^{(2)}\| \le c\left(\|g\| + \|\varphi^{2r} g^{(2r)}\|\right).$$

Consequently, we have for $t \in (0,1]$

$$\begin{aligned} K_{2,\varphi}(f,t) &\le \|f-g\| + t\|\varphi^2 g^{(2)}\| \\ &\le c\left(\|f-g\| + t\|\varphi^{2r} g^{(2r)}\|\right) + ct\|f\|. \end{aligned}$$

Taking the infimum on g, we arrive at

$$K_{2,\varphi}(f,t) \le c\,(K_{2r,\varphi}(f,t) + t\|f\|)\,.$$

Finally, we replace f with $f - p_1$, where p_1 is the algebraic polynomial of degree 1 of best approximation in $C[0,1]$ to f, to get (4.2). □

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