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The combined Shepard operator of inverse quadratic and inverse multiquadric type

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Abstract. Starting with the classical, the modified and the iterative Shepard methods, we construct some new Shepard type operators, using the inverse quadratic and the inverse multiquadric radial basis functions. Given some sets of points, we compute some representative subsets of knot points following an algorithm described by J.R. McMahon in 1986.

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Keywords: Shepard operator, inverse quadratic, inverse multiquadric, knot points.

1. Preliminaries

Over the time Shepard method, introduced in 1968 in [21], has been improved in order to get better reproduction qualities, higher accuracy and lower computational cost (see, e.g.,[2]-[9], [22], [23]).

Let f be a real-valued function defined on $X \subset \mathbb{R}^2$, and $(x_i, y_i) \in X$, $i = 1, ..., N$ some distinct points. The bivariate Shepard operator is defined by

$$(S_\mu f)(x, y) = \sum_{i=1}^{N} A_{i,\mu}(x, y) f(x_i, y_i), \tag{1.1}$$

where

$$A_{i,\mu}(x, y) = \frac{\prod\limits_{\substack{j=1 \\ j\neq i}}^{N} r_j^\mu(x, y)}{\sum\limits_{k=1}^{N} \prod\limits_{\substack{j=1 \\ j\neq k}}^{N} r_j^\mu(x, y)}, \tag{1.2}$$

with the parameter $\mu > 0$ and $r_i(x, y)$ denoting the distances between a given point $(x, y) \in X$ and the points (x_i, y_i), $i = 1, ..., N$.

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In [11], Franke and Nielson introduced a method for improving the accuracy in reproducing a surface with the bivariate Shepard approximation. This method has been further improved in [10], [20], [19], and it is given by:

$$(Sf)(x,y) = \frac{\sum_{i=1}^{N} W_i(x,y)\, f(x_i,y_i)}{\sum_{i=1}^{N} W_i(x,y)}, \tag{1.3}$$

with

$$W_i(x,y) = \left[\frac{(R_w - r_i(x,y))_+}{R_w r_i(x,y)}\right]^2, \tag{1.4}$$

where R_w is a radius of influence about the node (x_i, y_i) and it is varying with i. R_w is taken as the distance from node i to the jth closest node to (x_i, y_i) for $j > N_w$ (N_w is a fixed value) and j as small as possible within the constraint that the jth closest node is significantly more distant than the $(j-1)$st closest node (see, e.g. [19]). As it is mentioned in [14], this modified Shepard method is one of the most powerful software tools for the multivariate approximation of large scattered data sets.

A.V. Masjukov and V.V. Masjukov introduced in [15] an iterative modification for the Shepard operator that requires no artificial parameter, such as a radius of influence or number of nodes. So, they defined the iterative Shepard operator as

$$u(x,y) = \sum_{k=0}^{K}\sum_{j=1}^{N}\left[u_j^{(k)} w\left((x-x_j, y-y_j)/\tau_k\right) / \sum_{p=1}^{N} w\left((x_p-x_j, y_p-y_j)/\tau_k\right)\right], \tag{1.5}$$

where w is the weight function, continuously differentiable, with the properties that

$$w(x,y) \geq 0,\ \forall (x,y) \in \mathbb{R}^2,\ w(0,0) > 0 \text{ and } w(x,y) = 0 \text{ if } \|(x,y)\| > 1,$$

and $u_j^{(k)}$ denotes the interpolation residuals at the kth step, with $u_j^{(0)} \equiv u_j$.

2. The Shepard operators combined with the inverse quadratic and inverse multiquadric radial basis functions

Let f be a real-valued function defined on $X \subset \mathbb{R}^2$. We denote by $\mathbf{x}$ the point $(x,y) \in X$ and we assume that $\mathbf{x_i} = (x_i, y_i) \in X$, $i = 1, ..., N'$, are some given interpolation nodes.

The radial basis functions (RBF) are some modern and very efficient tools for interpolating scattered data, thus they are intensively used (see, e.g., [1], [12] – [14], [18]). In the sequel we use two radial basis functions that are positive definite, the inverse quadratic RBF and the inverse multiquadric RBF.

Consider the two radial basis functions as

$$\phi_i^\beta(x,y)=\sum_{j=1}^{i}\alpha_j\left[1+(\epsilon r_j)^2\right]^\beta+ax+by+c,\quad i=1,...,N',\tag{2.1}$$

with ϵ being a shape parameter and $r_j(x,y)=\sqrt{(x-x_j)^2+(y-y_j)^2}$.

For $\beta=-1$, $\phi_{\mathbf{i}}^{\mathbf{-1}}$ is the *inverse quadratic RBF* and for $\beta=-1/2$, $\phi_{\mathbf{i}}^{\mathbf{-1/2}}$ is the *inverse multiquadric RBF*.

The coefficients α_j, a, b, c are obtained as solutions of systems of the form

$$\begin{pmatrix}
1 & \left[1+(\epsilon r_{12})^2\right]^\beta & \cdots & \left[1+(\epsilon r_{1N'})^2\right]^\beta & x_1 & y_1 & 1\\
\left[1+(\epsilon r_{21})^2\right]^\beta & 1 & \cdots & \left[1+(\epsilon r_{2N'})^2\right]^\beta & x_2 & y_2 & 1\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\left[1+(\epsilon r_{N'1})^2\right]^\beta & \left[1+(\epsilon r_{N'2})^2\right]^\beta & \cdots & 1 & x_{N'} & y_{N'} & 1\\
x_1 & x_2 & \cdots & x_{N'} & 0 & 0 & 0\\
y_1 & y_2 & \cdots & y_{N'} & 0 & 0 & 0\\
1 & 1 & \cdots & 1 & 0 & 0 & 0
\end{pmatrix}\cdot\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{N'}\\ a\\ b\\ c\end{pmatrix}=\begin{pmatrix}f_1\\ f_2\\ \vdots\\ f_{N'}\\ 0\\ 0\\ 0\end{pmatrix}$$

with $r_{ij}=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}$ and $f_i=f(\mathbf{x}_i)$.

Shortly, this system can be written as

$$\begin{pmatrix}A & X^T\\ X & O_3\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{\alpha}\\ \mathbf{u}\end{pmatrix}=\begin{pmatrix}\mathbf{f}\\ \mathbf{0}\end{pmatrix},$$

considering the following notations:

- $A\in\mathcal{M}_{N'\times N'}(\mathbb{R})$, with the element on the entry (i,j) being $a_{ij}=\left[1+(\epsilon r_{ij})^2\right]^\beta$, where $r_{ij}=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}$, $i,j=1,...,N'$ and $\beta\in\{-1,\ -1/2\}$;
- $X\in\mathcal{M}_{3\times N'}(\mathbb{R})$, $X=\begin{pmatrix}x_1 & ... & x_{N'}\\ y_1 & ... & y_{N'}\\ 1 & ... & 1\end{pmatrix}$, O_3 is the zero square matrix of order 3;
- $\mathbf{u}=(a,\ b,\ c)^T$, $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_{N'})^T$, $\mathbf{0}=(0,\ 0,\ 0)^T$;
- $\mathbf{f}=(f_1,...,f_{N'})^T$, with $f_i=f(\mathbf{x}_i)$.

First, consider the classical Shepard operator given in (1.1).

Definition 2.1. The classical Shepard operator combined with the inverse quadratic and inverse multiquadric RBF is defined as

$$(S_\mu^\beta f)(\mathbf{x})=\sum_{i=1}^{N'}A_{i,\mu}(\mathbf{x})\phi_i^\beta(\mathbf{x}),\tag{2.2}$$

where $A_{i,\mu}$, $i=1,...,N'$, are defined by (1.2), for a given parameter $\mu>0$ and ϕ_i^β are given in (2.1), for $\beta\in\{-1,-1/2\}$ and $i=1,...,N'$.

Furthermore, we consider the improved form of the Shepard operator, given in (1.3).

Definition 2.2. We define the modified Shepard operator combined with the inverse quadratic and inverse multiquadric RBF as:

$$(S_W^{\beta} f)(\mathbf{x}) = \frac{\sum\limits_{i=1}^{N'} W_i(\mathbf{x})\, \phi_i^{\beta}(\mathbf{x})}{\sum\limits_{i=1}^{N'} W_i(\mathbf{x})}, \tag{2.3}$$

with W_i, $i = 1, ..., N'$, given by (1.4) and ϕ_i^{β} defined in (2.1), for $\beta \in \{-1, -1/2\}$ and $i = 1, ..., N'$.

Finally, we follow the idea proposed in [15], which consists of using an iterative procedure that requires no artificial parameters.

Definition 2.3. The iterative Shepard operator combined with the inverse quadratic and inverse multiquadric RBF is defined as

$$u_{\phi^{\beta}}(\mathbf{x}) = \sum_{k=0}^{K} \sum_{j=1}^{N'} \left[u_{\phi_j^{\beta}}^{(k)} w\left((\mathbf{x} - \mathbf{x_j})/\tau_k\right) / \sum_{p=1}^{N'} w\left((\mathbf{x_p} - \mathbf{x_j})/\tau_k\right) \right], \tag{2.4}$$

with $\beta \in \{-1, -1/2\}$, where $u_{\phi_j^{\beta}}^{(k)}$ are the interpolation residuals at the kth step given by

$$u_{\phi_j^{\beta}}^{(0)} = \phi_j(\mathbf{x_j}),\ \mathbf{x_j} \in X,\ j = 1, ..., N'$$

and

$$u_{\phi_j^{\beta}}^{(k+1)} = u_{\phi_j^{\beta}}^{(k)} - \sum_{q=1}^{N'} \left[u_{\phi_q^{\beta}}^{(k)} w\left((\mathbf{x_j} - \mathbf{x_q})/\tau_k\right) / \sum_{p=1}^{N'} w\left((\mathbf{x_p} - \mathbf{x_q})/\tau_k\right) \right].$$

The functions ϕ_i^{β} are given in (2.1). We follow ideas from [15] for the parameters' choice. As an example, the sequence $\{\tau_k\}$ of scale factors is defined as

$$\tau_k = \tau_0 \gamma^k, \qquad 0 < \gamma < 1.$$

The setup parameter τ_k can be chosen such that it decreases from an initial value τ_0, which is given for instance as

$$\tau_0 > \sup_{(x,y) \in X} \max_{1 \le j \le N'} \|(\mathbf{x} - \mathbf{x_j})\|$$

to the final value τ_K such that

$$\tau_K < \min_{i \neq j} \|(\mathbf{x_i} - \mathbf{x_j})\|.$$

The behaviour of u_{ϕ}^{β} does not change very much for γ between 0.6 and 0.95, as shown in [15]. One can also choose smaller values for γ if the nodes are sparse and a decreased computational time is desired.

Finally, the weight function w is given by

$$w(\mathbf{x}) = w(x)w(y),$$

with

$$w(x) = \begin{cases} 5(1-|x|)^4 - 4(1-|x|)^5, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}.$$

We apply the three operators on two sets of points. For the first way, we consider a set of N initial interpolation nodes $\mathbf{x_i},\ i = 1, ..., N$, and for the second way, we consider a smaller set of $k \in \mathbb{N}^*$ knot points $\hat{\mathbf{x}}_\mathbf{j},\ j = 1, ..., k$, that will be representative for the original set. This set is obtained following the next steps (see, e.g., [16] and [17]):

Algorithm 2.4.
1. Consider the first subset of k knot points, $k < N$, randomly generated;
2. Using the Euclidean distance between two points, find the closest knot point for every point;
3. For the knot points with no point assigned, replace the knot by the nearest point;
4. Compute the arithmetic mean of all the points that are closest to the same knot and compute in this way the new subset of knot points;
5. Repeat steps 2-4 until the subset of knot points has not change for two consecutive iterations.

3. Numerical examples

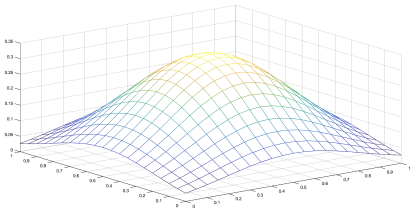
We consider the following test functions (see, e.g., [10], [20], [19]):

$$\begin{aligned} &\text{Gentle:} \quad f_1(x,y) = \exp[-\tfrac{81}{16}((x-0.5)^2 + (y-0.5)^2)]/3, \\ &\text{Saddle:} \quad f_2(x,y) = \frac{(1.25 + \cos 5.4y)}{6 + 6(3x-1)^2}, \\ &\text{Sphere:} \quad f_3(x,y) = \sqrt{64 - 81((x-0.5)^2 + (y-0.5)^2)}/9 - 0.5. \end{aligned} \tag{3.1}$$

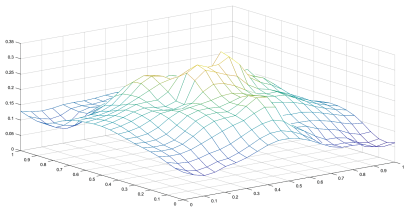
Tables 1 - 3 contain the maximum errors for approximating the functions (3.1) by the classical, the modified and the iterative Shepard operators given, respectively, by (1.1), (1.3) and (1.5), and the errors of approximating by the operators introduced in (2.2), (2.3) and (2.4). We construct the operators for both radial basis functions - the inverse quadratic and the inverse multiquadric. For each function we consider a set of $N = 100$ random points in $[0,1] \times [0,1]$, a subset of $k = 25$ representative knots, $\mu = 3$, $N_w = 19$, $K = 20$, $\tau_0 = 3$ and $\gamma = 0.66,\ 0.84,\ 0.91$.

In Figures 1 - 4 we plot the graphs of $f_1,\ f_2,\ f_3$ and of the corresponding Shepard operators $S_\mu^\beta f$, $S_W^\beta f$ and u_{ϕ^β}, combined with the inverse quadratic ($\beta = -1$) and the inverse multiquadric ($\beta = -1/2$) radial basis functions. We consider the sets of the $k = 25$ representative knot points.

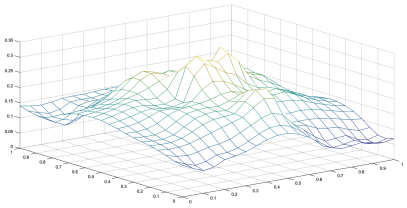
We remark that $S_W^\beta f$ and u_{ϕ^β} have better approximation properties than the classical Shepard operator $S_\mu^\beta f$, the results for u_{ϕ^β} depending on the values of γ. Also, we notice better approximation errors for the lower number of knots obtained using the Algorithm 2.4.



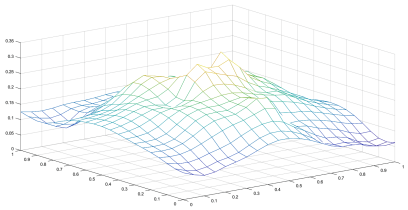
Function f_1.



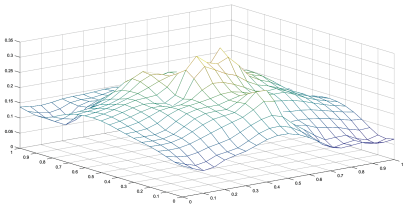
$S_\mu^{-1} f_1,\ \epsilon = 5.5.$



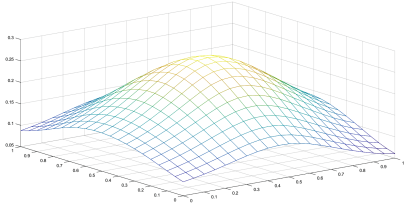
$S_\mu^{-1/2} f_1,\ \epsilon = 10.$



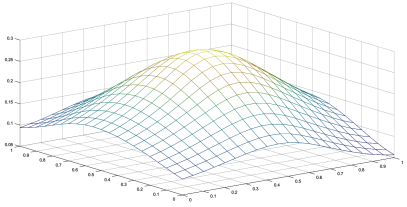
$S_W^{-1} f_1,\ \epsilon = 5.5.$



$S_W^{-1/2} f_1,\ \epsilon = 10.$

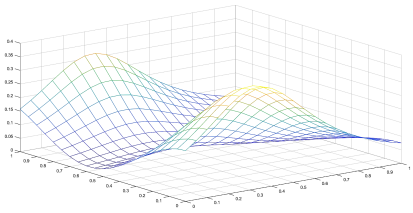


$u_{\phi^{-1}},\ \epsilon = 5.5,\ \gamma = 0.91.$

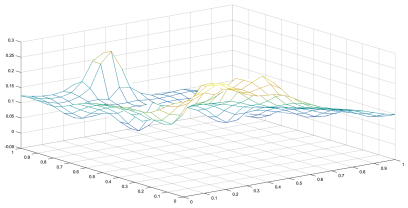


$u_{\phi^{-1/2}},\ \epsilon = 10,\ \gamma = 0.91.$

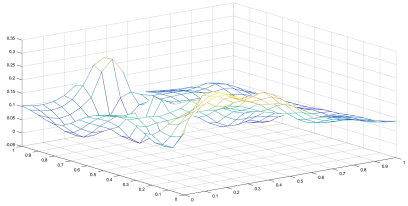
FIGURE 1. Graphs for f_1.



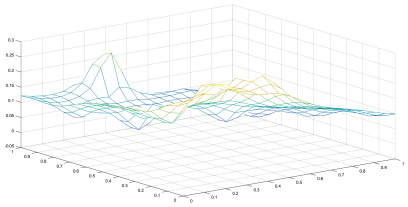
Function f_2.



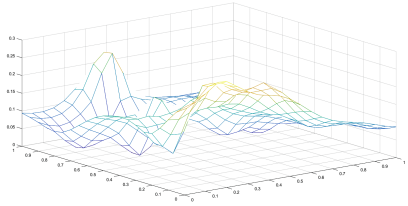
$S_\mu^{-1} f_2,\ \epsilon = 10.$



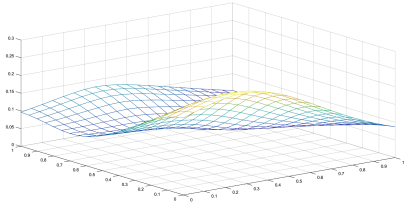
$S_\mu^{-1/2} f_2,\ \epsilon = 10.$



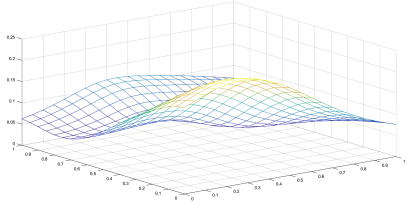
$S_W^{-1} f_2,\ \epsilon = 10.$



$S_W^{-1/2} f_2,\ \epsilon = 10.$

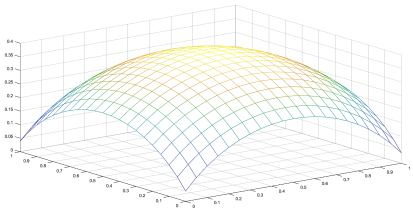


$u_{\phi^{-1}},\ \epsilon = 10,\ \gamma = 0.91.$

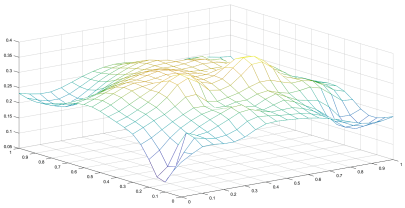


$u_{\phi^{-1/2}},\ \epsilon = 10,\ \gamma = 0.91.$

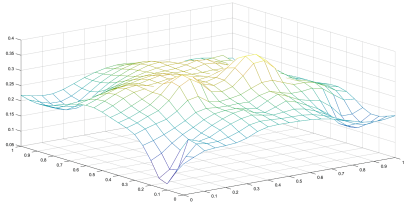
FIGURE 2. Graphs for f_2.



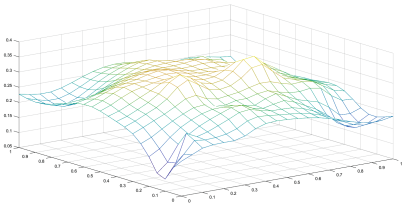
Function f_3.



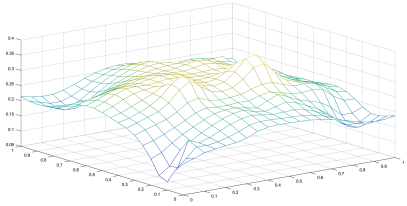
$S_\mu^{-1} f_3,\ \epsilon = 5.5.$



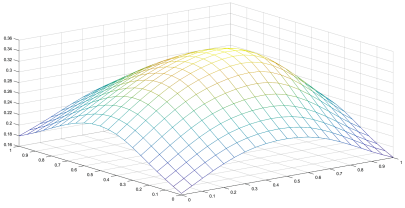
$S_\mu^{-1/2} f_3,\ \epsilon = 9.$



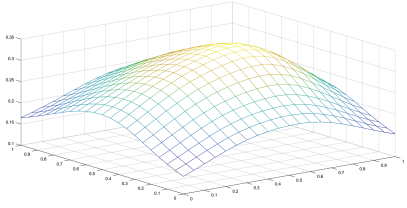
$S_W^{-1} f_3,\ \epsilon = 5.5.$



$S_W^{-1/2} f_3,\ \epsilon = 9.$



$u_{\phi^{-1}},\ \epsilon = 5.5,\ \gamma = 0.91.$



$u_{\phi^{-1/2}},\ \epsilon = 9,\ \gamma = 0.91.$

FIGURE 4. Graphs for f_3.

TABLE 1. Maximum approximation errors for the **Gentle** function.

	ϵ	Classical S_μ		Modified S_W		Iterative u_ϕ		
		k=25	N=100	k=25	N=100	γ (input)	k=25	N=100
f_1	–	0.0864	0.0855	0.0725	0.0644	0.66	0.0967	0.1158
						0.84	0.0757	0.1159
						0.91	0.0528	0.1105
ϕ^{-1}	5.5	0.1023	0.5564	0.0994	0.5543	0.66	0.1061	0.2866
						0.84	0.0847	0.2644
						0.91	0.0627	0.2396
	10	0.1313	0.1876	0.1293	0.1681	0.66	0.1026	0.1488
						0.84	0.0772	0.1251
						0.91	0.0579	0.1123
$\phi^{-1/2}$	9	0.1098	0.2402	0.1063	0.2219	0.66	0.1002	0.2155
						0.84	0.0866	0.1985
						0.91	0.0686	0.1887
	10	0.1129	0.2292	0.1096	0.2094	0.66	0.0994	0.1936
						0.84	0.0854	0.1750
						0.91	0.0673	0.1653

TABLE 2. Maximum approximation errors for the **Saddle** function.

	ϵ	Classical S_μ		Modified S_W		Iterative u_ϕ		
		k=25	N=100	k=25	N=100	γ (input)	k=25	N=100
f_2	–	0.1096	0.1152	0.0970	0.1033	0.66	0.2083	0.2051
						0.84	0.1902	0.1828
						0.91	0.1633	0.1567
ϕ^{-1}	7	0.1669	0.9372	0.1575	0.8615	0.66	0.2198	0.3754
						0.84	0.2103	0.4007
						0.91	0.1938	0.4456
	10	0.1813	0.1693	0.1828	0.1697	0.66	0.2175	0.1909
						0.84	0.2045	0.1797
						0.91	0.1825	0.1626
$\phi^{-1/2}$	9	0.1677	0.5409	0.1639	0.4933	0.66	0.2301	0.3125
						0.84	0.2222	0.3202
						0.91	0.2077	0.3344
	10	0.1582	0.2952	0.1630	0.2659	0.66	0.2292	0.2000
						0.84	0.2195	0.2020
						0.91	0.2029	0.2028

TABLE 3. Maximum approximation errors for the **Sphere** function.

	ϵ	Classical S_μ		Modified S_W		Iterative u_ϕ		
		k=25	N=100	k=25	N=100	γ (input)	k=25	N=100
f_3	–	0.2011	0.2156	0.1934	0.1744	0.66	0.1837	0.1850
						0.84	0.1730	0.1743
						0.91	0.1593	0.1645
ϕ^{-1}	5	0.1849	1.3107	0.1806	1.1997	0.66	0.1576	0.2703
						0.84	0.1488	0.4361
						0.91	0.1390	0.5255
	5.5	0.1926	0.9074	0.1898	0.8297	0.66	0.1637	0.1925
						0.84	0.1533	0.2901
						0.91	0.1456	0.3494
$\phi^{-1/2}$	7	0.1584	0.8948	0.1526	0.8150	0.66	0.1401	0.2258
						0.84	0.1291	0.3072
						0.91	0.1183	0.3464
	9	0.1796	0.3682	0.1779	0.3341	0.66	0.1537	0.1772
						0.84	0.1417	0.2091
						0.91	0.1344	0.2216

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