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Global nonexistence of solutions to system of Klein-Gordon equations with degenerate damping and strong source terms in viscoelasticity

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Abstract. In this paper, we consider a system of nonlinear viscoelastic wave equations with degenerate damping and source terms. We prove, with positive initial energy, the global nonexistence of solution by concavity method.

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1. Introduction

In this paper, we consider a system of viscoelastic wave equations with degenerate damping and strong nonlinear source terms

$$\begin{cases} u_{tt} - \Delta u + m_1^2 \cdot u + \displaystyle\int_0^t g\left(t-s\right) \Delta u\left(x,s\right) ds + \left(a\left|u\right|^k + b\left|v\right|^l\right) |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + m_2^2 \cdot v + \displaystyle\int_0^t h\left(t-s\right) \Delta v\left(x,s\right) ds + \left(c\left|v\right|^\theta + d\left|u\right|^\varrho\right) |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{1.1}$$

where $m, r > 0, k, l, \theta, \varrho \geq 1$ and the functions $f_1\left(u,v\right), f_2\left(u,v\right)$ are defined by

$$\begin{aligned} f_1(\xi_1,\xi_2) &= a_1|\xi_1+\xi_2|^{2(\rho+1)}(\xi_1+\xi_2) + b_1|\xi_1|^\rho \xi_1 |\xi_2|^{(\rho+2)} \\ f_2(\xi_1,\xi_2) &= a_1|\xi_1+\xi_2|^{2(\rho+1)}(\xi_1+\xi_2) + b_1|\xi_1|^{(\rho+2)} |\xi_2|^\rho \xi_2, a_1, b_1 > 0, \end{aligned} \tag{1.2}$$

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where $\rho > -1$. In (1.1), $u = u(x,t), v = v(x,t)$, where $x \in \Omega$ is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$ and $t > 0$, $a, b, c, d, m_1, m_2 > 0$.
To above system (1.1), we add the initial conditions given by

$$(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), x \in \Omega \tag{1.3}$$

and boundary conditions given by

$$u(x) = v(x) = 0, x \in \partial\Omega. \tag{1.4}$$

This kind of problems arise in viscoelasticity. Dafermos was the first who study this type in [9], where the general decay was treated. In the last decades, problems related to system (1.1) had a lot of attention and many results appeared on the existence and long time behavior of solutions. See in this directions ([6, 3, 2, 4, 5, 8, 7, 11, 14, 17, 20, 19, 21, 27, 26]) and references therein.
In the absence of viscoelastic term, some special cases of the single wave equations with nonlinear damping and nonlinear source terms in the form

$$u_{tt} - \Delta u + a|u_t|^{m-1}u_t = b|u|^{p-1}u. \tag{1.5}$$

With nonlinear damping and source terms, it arises in the quantum-field and used to describe the movement of charged electromagnetic fields. Equation (1.5) equipped with initial and bounded conditions of Dirichlet type has been extensively studied and many results regarding existence, blow up and asymptotic behavior of solutions have been obtained. Many authors have studied the single wave equations in the presence of various mechanisms of dissipation, damping and non-linear sources. See ([1, 15, 18, 10, 12, 13, 24, 25, 28]) and references therein.
In [16], authors considered the nonlinear viscoelastic system

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + \displaystyle\int_0^t h(t-s)\Delta v(x,s)ds + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{1.6}$$

where

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \tag{1.7}$$

The global nonexistence theorem for some solutions with positive energy was proved using a method applied in [22].
In [23], the authors studied the nonlinear viscoelastic system in (1.6), where they obtained the decay of solutions for system. Under some restrictions on the nonlinearities of damping and source terms, they proved that, for some class of relaxation functions and some restrictions on the initial data, the rate of decay of relaxation functions affects the rate of decay of solution for system.
In this paper, we consider system (1.1)-(1.4) and proved a global nonexistence result of solutions. We extended to result in [16] and [27] to more general cases.

2. Preliminaries

In this section, we present some notations and Lemmas.
We assume that the relaxation functions $g, h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\begin{cases} 1 - \int_0^\infty g(s)ds = l' > 0, \quad g(t) \geq 0, \quad g'(t) \leq 0, \\ 1 - \int_0^\infty h(s)ds = k' > 0, \quad h(t) \geq 0, \quad h'(t) \leq 0, \end{cases} \quad t \geq 0. \tag{2.1}$$

We introduce the "*modified*" energy functional E associated to our system

$$2E(t) = \|u_t\|_2^2 + \|v_t\|_2^2 + 2\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) + J(u,v) - 2\int_\Omega F(u,v)\,dx, \tag{2.2}$$

where $F(u,v)$ is defined for all $(u,v) \in \mathbb{R}^2$,

$$\begin{aligned} F(u,v) &= \frac{1}{2(\rho+2)}\left[uf_1(u,v) + vf_2(u,v)\right], \\ &= \frac{1}{2(\rho+2)}\left[|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}\right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v), \frac{\partial F}{\partial v} = f_2(u,v),$$

and

$$\begin{aligned} J(u,v) &= \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(1 - \int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\ &+ (g \circ \nabla u) + (h \circ \nabla v). \end{aligned} \tag{2.3}$$

Noting by

$$\begin{cases} (g \circ u)(t) = \int_0^t g(t-\tau)\|u(t) - u(\tau)\|_2^2\,d\tau, \\ (h \circ v)(t) = \int_0^t h(t-\tau)\|v(t) - v(\tau)\|_2^2\,d\tau. \end{cases} \tag{2.4}$$

We suppose that ρ satisfies

$$\begin{cases} -1 < \rho, & \text{if} \quad N = 1, 2, \\ -1 < \rho \leq \frac{4-N}{N-2} & \text{if} \quad N \geq 3. \end{cases} \tag{2.5}$$

Lemma 2.1. [22] *There exist two positive constants c_0 and c_1 with the end goal that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right).$$

Lemma 2.2. *Assume that (2.5) holds. There exists $\eta > 0$, such that for any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, the inequality*

$$2(\rho+2)\int_\Omega F(u,v)\,dx \leq \eta\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\rho+2} \tag{2.6}$$

holds.

Lemma 2.3. *Let $\nu > 0$, be a real positive number and let $L(t)$ be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{2.7}$$

defined in $[0, \infty)$.
If $L(0) > 0$, then the solution does not exist for $t \geq L(0)^{-\nu} \xi^{-\nu} \nu^{-1}$.

Proof. By simple integration of (2.7), we have

$$L^{-\nu}(0) - L^{-\nu}(t) \geq \xi \nu t.$$

Then, we obtain the following estimate

$$L^{\nu}(t) \geq [L^{-\nu}(0) - \xi \nu t]^{-1}. \tag{2.8}$$

Then, the RHS of (2.8) is unbounded for

$$\xi \nu t = L^{-\nu}(0).$$

The proof is completed. □

3. Blow up result

Lemma 3.1. *Assume that (2.5) holds. Let (u, v) be the solution of the system (1.1)–(1.4) then the energy functional is a non-increasing function, that is, for all $t \geq 0$,*

$$\begin{aligned} E'(t) &= -\int_{\Omega} \left(|u(t)|^{k} + |v(t)|^{l} \right) |u_t(t)|^{m+1} dx \\ &- \int_{\Omega} \left(|v(t)|^{\theta} + |u(t)|^{\varrho} \right) |v_t(t)|^{r+1} dx \\ &+ \frac{1}{2}(g' \circ \nabla u) + \frac{1}{2}(h' \circ \nabla v) - \frac{1}{2} g(s) \|\nabla u\|_2^2 - \frac{1}{2} h(s) \|\nabla v\|_2^2 . \end{aligned} \tag{3.1}$$

Lemma 3.2. *Suppose that (2.5) holds. Let (u, v) be the solution of the system (1.1)-(1.4), then the energy functional is a non-increasing function, that is, for all $t > 0$,*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\int_{\Omega} \left(|u(t)|^{k} + |v(t)|^{l} \right) |u_t(t)|^{m+1} dx \\ &- \int_{\Omega} \left(|v(t)|^{\theta} + |u(t)|^{\vartheta} \right) |v_t(t)|^{r+1} dx. \end{aligned} \tag{3.2}$$

The proof of Lemma 3.1 can be done by using a classical calculations.
Our main result reads as follows

Theorem 3.3. *Suppose that (2.5) holds. Assume further that*

$$\rho > \max\left(\frac{k+m-3}{2}, \frac{l+m-3}{2}, \frac{\theta+r-3}{2}, \frac{\varrho+r-3}{2} \right), \tag{3.3}$$

and that there exists p *such that* $2 < p < 2(\rho+2)$ *, for which*

$$\max\left(\int_0^\infty g(s)\,ds, \int_0^\infty h(s)\,ds\right) < \frac{(p/2)-1}{(p/2)-1+1/(2p)}, \tag{3.4}$$

holds. Then any solution of problem (1.1)*–*(1.4)*, with initial data satisfying*

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \alpha_1^2, \text{ and } E(0) < E_2 \tag{3.5}$$

blows up in finite time, where the constants α_1 *and* E_2 *are defined in* (3.6)*.*

We take $a = b = c = d = 1, a_1 = b_1 = 1$ for convenience. We introduce the following constants

$$\begin{aligned} B &= \eta^{\frac{1}{2(\rho+2)}}, \qquad \alpha_1 = B^{-\frac{\rho+2}{\rho+1}}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \\ E_2 &= \left(\frac{1}{p} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \end{aligned} \tag{3.6}$$

where η is the optimal constant in (2.6).

Lemma 3.4. [22] *Suppose that* (2.5)*,* (3.3) *and* (3.4) *hold. Let* (u, v) *be a solutions of* (1.1)*–*(1.4)*. Assume further that* $E(0) < E_2$ *and*

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \alpha_1^2. \tag{3.7}$$

Then, there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$J(t) > \alpha_2^2, \tag{3.8}$$

and

$$2(\rho+2)\int_\Omega F(u,v)\,dx \geq (B\alpha_2)^{2(\rho+2)}, \ \forall\, t \geq 0. \tag{3.9}$$

Proof of Theorem 3.3*.* The proof is similar to one given in [14] with the necessary modification imposed by the nature of our problem. We assume that the solutions exists for all t and we get a contradiction. We set

$$H(t) = E_2 - E(t). \tag{3.10}$$

By using the definition of $H(t)$, we obtain

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t(t)|^{m+1}\,dx \\ &+ \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t(t)|^{r+1}\,dx \\ &- \frac{1}{2}(g'\circ\nabla u) - \frac{1}{2}(h'\circ\nabla v) + \frac{1}{2}g(s)\|\nabla u\|_2^2 + \frac{1}{2}h(s)\|\nabla v\|_2^2 \\ &\geq 0, \ \forall\, t \geq 0. \end{aligned} \tag{3.11}$$

Therefore,

$$H(0) = E_2 - E(0) > 0. \tag{3.12}$$

Then,

$$
\begin{aligned}
0 &< H(0) \leq H(t) \\
&= E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) - \frac{J(t)}{2} \\
&+ \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \qquad (3.13)
\end{aligned}
$$

Note that from (2.1) and (3.8), we get

$$
\begin{aligned}
E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{J(t)}{2} &< E_2 - \frac{1}{2}\alpha_2^2 \\
&< E_2 - \frac{1}{2}\alpha_1^2 \\
&< E_1 - \frac{1}{2}\alpha_1^2 \\
&= -\frac{1}{2(\rho+2)}\alpha_1^2 < 0, \ \forall\, t \geq 0. \qquad (3.14)
\end{aligned}
$$

Thus, by using (3.14) and Lemma 2.1, we get

$$
\begin{aligned}
0 &< H(0) \leq H(t) \leq \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\
&\leq \frac{c_1}{2(\rho+2)}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \ \forall\, t \geq 0. \qquad (3.15)
\end{aligned}
$$

We define the function M as

$$
M(t) = \frac{1}{2}\int_\Omega \left(u^2 + v^2\right)(x,t)\,dx, \qquad (3.16)
$$

and let

$$
L(t) = H^{1-\sigma}(t) + \varepsilon M'(t), \qquad (3.17)
$$

for ε small to be chosen later and

$$
\begin{aligned}
0 &< \sigma \leq \min\left\{\frac{1}{2}, \frac{2\rho+3-(k+m)}{2(m+1)(\rho+2)}, \frac{2\rho+3-(l+m)}{2(m+1)(\rho+2)},\right. \\
&\left.\frac{2\rho+3-(\varrho+r)}{2(r+1)(\rho+2)}, \frac{2\rho+3-(\theta+r)}{2(r+1)(\rho+2)}, \frac{2\rho+2}{4(\rho+2)}\right\}. \qquad (3.18)
\end{aligned}
$$

By differentiation of (3.17) with respect to time and using (1.1), we get

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &- \varepsilon \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\ &- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx \\ &+ \varepsilon \int_\Omega (u f_1(u,v) + v f_2(u,v))\, dx \\ &+ \varepsilon \int_\Omega \nabla u(t) \int_0^t g(t-s) \nabla u(\tau)\, dx ds \\ &+\varepsilon \int_\Omega \nabla v(t) \int_0^t h(t-s) \nabla v(\tau)\, dx ds. \end{aligned} \tag{3.19}$$

Then,

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\ &- \varepsilon \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\ &- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx \\ &+ \varepsilon \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\ &+ \varepsilon \left(\int_0^t g(s)\, ds\right) \|\nabla u\|_2^2 + \left(\int_0^t h(s)\, ds\right) \|\nabla v\|_2^2 \\ &+ \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) . [\nabla u(\tau) - \nabla u(t)]\, dx ds \\ &+\varepsilon \int_0^t h(t-s) \int_\Omega \nabla v(t) . [\nabla v(\tau) - \nabla v(t)]\, dx ds. \end{aligned} \tag{3.20}$$

By using Cauchy-Schwartz and Young's inequalities, we obtain the following estimate

$$\begin{aligned} &\int_0^t g(t-s) \int_\Omega \nabla u(t) . [\nabla u(\tau) - \nabla u(t)]\, dx ds \\ \leq &\int_0^t g(t-s) \|\nabla u\|_2 \|\nabla u(\tau) - \nabla u(t)\|_2\, d\tau \\ \leq &\ \lambda (g \circ \nabla u) + \frac{1}{4\lambda} \left(\int_0^t g(s)\, ds\right) \|\nabla u\|_2^2, \qquad \lambda > 0 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}&\int_0^t h(t-s)\int_\Omega \nabla v(t).\left[\nabla v(\tau)-\nabla v(t)\right]dxds\\ \leq\ & \lambda(h\circ\nabla v)+\frac{1}{4\lambda}\left(\int_0^t h(s)\,ds\right)\|\nabla v\|_2^2,\qquad \lambda>0.\end{aligned}\tag{3.22}$$

Adding $pE(t)$ and using the definition of $H(t), E_2$ leads to

$$\begin{aligned}L'(t)\ \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t)\\ +\ & \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right)\\ +\ & \varepsilon\left(\frac{p}{2}-\lambda\right)\left[(g\circ\nabla u)+(h\circ\nabla v)\right]+p\varepsilon H(t)-p\varepsilon E_2\\ -\ & \varepsilon\int_\Omega u\left(|u(t)|^k+|v(t)|^l\right)|u_t|^{m-1}u_t dx\\ -\ & \varepsilon\int_\Omega v\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v_t|^{r-1}v_t dx\\ +\ & \varepsilon\left(1-\frac{p}{2(\rho+2)}\right)\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right)\\ +\ & \varepsilon\left[\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty g(s)\,ds\right]\|\nabla u\|_2^2\\ +\ & \varepsilon\left[\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty h(s)\,ds\right]\|\nabla v\|_2^2,\end{aligned}\tag{3.23}$$

for some λ such that

$$a_1=\frac{p}{2}-\lambda>0,$$

and

$$a_2=\left[\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\lambda}\right)\max\left(\int_0^\infty g(s)\,ds,\ \int_0^\infty h(s)\,ds\right)\right]>0.$$

Then, (3.23) can be estimated as follows

$$\begin{aligned}L'(t)\ \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ +\ & \varepsilon a_1\left[(g\circ\nabla u)+(h\circ\nabla v)\right]+p\varepsilon H(t)-p\varepsilon E_2\\ -\ & \varepsilon\int_\Omega u\left(|u(t)|^k+|v(t)|^l\right)|u_t|^{m-1}u_t dx\\ -\ & \varepsilon\int_\Omega v\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v_t|^{r-1}v_t dx\\ +\ & \varepsilon\left(1-\frac{p}{2(\rho+2)}\right)\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right)\\ +\ & \varepsilon a_2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right).\end{aligned}\tag{3.24}$$

By taking $c_3 = 1 - \frac{p}{\rho+2} - 2E_2 (B\alpha_2)^{-2(\rho+2)} > 0$, since $\alpha_2 > B^{-\frac{2(\rho+2)}{\rho+1}}$. Consequently, (3.24) takes the form

$$\begin{aligned}
L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) \\
&+ \varepsilon \left(1 + \frac{p}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\
&+ \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v)\right] \\
&+ \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + p\varepsilon H(t) \\
&+ \varepsilon c_3 \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\
&- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx. \qquad (3.25)
\end{aligned}$$

By using Young's inequality, we have

$$XY \leq \frac{\delta^\alpha X^\alpha}{\alpha} + \frac{\delta^{-\beta} Y^\beta}{\beta}, \qquad (3.26)$$

where $X, Y \geq 0$, $\delta > 0$ and $\alpha, \beta > 0$ such that $1/\alpha + 1/\beta = 1$, we obtain

$$\left| u |u_t|^{m-1} u_t \right| \leq \frac{\delta_1^{m+1}}{m+1} |u|^{m+1} + \frac{m}{m+1} \delta_1^{-(m+1)/m} |u_t|^{m+1}, \ \forall\, \delta_1 \geq 0 \qquad (3.27)$$

and

$$\begin{aligned}
&\int_\Omega \left(|u(t)|^k + |v(t)|^l\right) \left| u |u_t|^{m-1} u_t \right| dx \\
\leq\ & \frac{\delta_1^{m+1}}{m+1} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u|^{m+1} dx \\
&+ \frac{m}{m+1} \delta_1^{-(m+1)/m} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m+1} dx. \qquad (3.28)
\end{aligned}$$

Similarly, for any $\delta_2 > 0$,

$$\left| v |v_t|^{r-1} v_t \right| \leq \frac{\delta_2^{r+1}}{r+1} |v|^{r+1} + \frac{r}{r+1} \delta_2^{-(r+1)/r} |v_t|^{r+1}, \qquad (3.29)$$

which gives

$$\begin{aligned}
&\int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) \left| v |v_t|^{r-1} v_t \right| dx \\
\leq\ & \frac{\delta_2^{r+1}}{r+1} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v|^{r+1} dx \\
+\ & \frac{r}{r+1} \delta_2^{-(r+1)/r} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r+1} dx. \qquad (3.30)
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
L'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) \\
&+ \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon a_1\left[(g\circ\nabla u)+(h\circ\nabla v)\right] \\
&+ \varepsilon a_2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+p\varepsilon H(t) \\
&+ \varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&- \varepsilon\frac{\delta_1^{m+1}}{m+1}\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u|^{m+1}dx \\
&- \varepsilon\frac{m}{m+1}\delta_1^{-\frac{(m+1)}{m}}\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u_t|^{m+1}dx \\
&- \varepsilon\frac{\delta_2^{r+1}}{r+1}\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v|^{r+1}dx \\
&- \varepsilon\frac{r}{r+1}\delta_2^{-\frac{(r+1)}{r}}\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v_t|^{r+1}dx. \qquad (3.31)
\end{aligned}$$

Choosing δ_1 and δ_2 such that

$$\delta_1^{-\frac{(m+1)}{m}}=M_1H(t)^{-\sigma}, \delta_2^{-\frac{(r+1)}{r}}=M_2H(t)^{-\sigma}, \qquad (3.32)$$

for M_1 and M_2 large constants to be fixed later. Thus, by using (3.32), we obtain

$$\begin{aligned}
L'(t) &\geq ((1-\sigma)-M\varepsilon)H^{-\sigma}(t)H'(t) \\
&+ \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon a_1\left[(g\circ\nabla u)+(h\circ\nabla v)\right] \\
&+ \varepsilon a_2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+p\varepsilon H(t) \\
&+ \varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&- \varepsilon M_1^{-m}H^{\sigma m}(t)\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u|^{m+1}dx \\
&- \varepsilon\frac{m}{m+1}\delta_1^{-\frac{(m+1)}{m}}\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u_t|^{m+1}dx \\
&- \varepsilon M_2^{-r}H^{\sigma r}(t)\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v|^{r+1}dx \\
&- \varepsilon\frac{r}{r+1}\delta_2^{-\frac{(r+1)}{r}}\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v_t|^{r+1}dx, \qquad (3.33)
\end{aligned}$$

where $M=m/(m+1)M_1+r/(r+1)M_2$. Therefore, we have

$$\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u|^{m+1}dx=\|u\|_{k+m+1}^{k+m+1}+\int_\Omega|v|^l|u|^{m+1}dx, \qquad (3.34)$$

and

$$\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v|^{r+1}dx=\|v\|_{\theta+r+1}^{\theta+r+1}+\int_{\Omega}|u|^{\varrho}|v|^{r+1}dx. \tag{3.35}$$

Also by using Young's inequality, we obtain

$$\begin{aligned}\int_{\Omega}|v|^{l}|u|^{m+1} &\leq \frac{l}{l+m+1}\delta_1^{(l+m+1)/l}\|v\|_{l+m+1}^{l+m+1}\\ &+ \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}\|u\|_{l+m+1}^{l+m+1},\\ \int_{\Omega}|u|^{\varrho}|v|^{r+1} &\leq \frac{\varrho}{\varrho+r+1}\delta_2^{(\varrho+r+1)/\varrho}\|u\|_{\varrho+r+1}^{\varrho+r+1}\\ &+ +\frac{r+1}{\varrho+r+1}\delta_2^{-(\varrho+r+1)/(r+1)}\|v\|_{\varrho+r+1}^{\varrho+r+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}&H^{\sigma m}(t)\int_{\Omega}\left(|u(t)|^{k}+|v(t)|^{l}\right)|u|^{m+1}dx\\ =\ & H^{\sigma m}(t)\|u\|_{k+m+1}^{k+m+1}+\frac{l}{l+m+1}\delta_1^{(l+m+1)/l}H^{\sigma m}(t)\|v\|_{l+m+1}^{l+m+1}\\ +\ & \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}H^{\sigma m}(t)\|u\|_{l+m+1}^{l+m+1},\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}&H^{\sigma r}(t)\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v|^{r+1}dx\\ =\ & H^{\sigma r}(t)\|v\|_{\theta+r+1}^{\theta+r+1}+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}H^{\sigma r}(t)\|u\|_{\varrho+r+1}^{\varrho+r+1}\\ +\ & \frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}H^{\sigma r}(t)\|v\|_{\varrho+r+1}^{\varrho+r+1}.\end{aligned} \tag{3.37}$$

Since (3.3) holds, we get by using (3.18)

$$\begin{cases} H^{\sigma m}(t)\|u\|_{k+m+1}^{k+m+1}\leq c_5\left(\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1}+\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{k+m+1}^{k+m+1}\right),\\ H^{\sigma r}(t)\|v\|_{\theta+r+1}^{\theta+r+1}\leq c_6\left(\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1}+\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{\theta+r+1}^{\theta+r+1}\right).\end{cases} \tag{3.38}$$

This implies

$$\begin{aligned}&\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}H^{\sigma m}(t)\|v\|_{l+m+1}^{l+m+1}\\ &\leq c_7\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}\left(\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l+m+1}+\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|v\|_{l+m+1}^{l+m+1}\right),\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}&\tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}H^{\sigma r}(t)\|u\|_{\varrho+r+1}^{\varrho+r+1}\\&\leq c_8\tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}\left(\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho+r+1}+\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|u\|_{\varrho+r+1}^{\varrho+r+1}\right).\end{aligned} \tag{3.40}$$

Using (3.18) and the algebraic inequality

$$z^{\nu}\leq(z+1)\leq\left(1+\frac{1}{a}\right)(z+a),\ \forall\ z\geq0,0<\nu\leq1,a>0, \tag{3.41}$$

we get, for all $t\geq0$,

$$\begin{cases}\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1}\leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H(0)\right)\leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\\\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1}\leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\ \forall\ t\geq0,\end{cases} \tag{3.42}$$

where $d=1+1/H(0)$. Similarly

$$\begin{cases}\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l(m+1)}\leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H(0)\right)\leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\\\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho(r+1)}\leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\ \forall\ t\geq0.\end{cases} \tag{3.43}$$

Also, since

$$(X+Y)^{s}\leq C\left(X^{s}+Y^{s}\right),\qquad X,Y\geq0,s>0, \tag{3.44}$$

by using (3.18) and (3.41) we have

$$\begin{aligned}\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{k+m+1}^{k+m+1}&\leq c_9\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{k+m+1}^{2(\rho+2)}\right)\\&\leq c_{10}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{2(\rho+2)}^{2(\rho+2)}\right),\end{aligned} \tag{3.45}$$

similarly

$$\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{\theta+r+1}^{\theta+r+1}\leq c_{11}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \tag{3.46}$$

$$\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|v\|_{l+m+1}^{l+m+1}\leq c_{12}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right) \tag{3.47}$$

and

$$\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|u\|_{\varrho+r+1}^{\varrho+r+1}\leq c_{13}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{2(\rho+2)}^{2(\rho+2)}\right). \tag{3.48}$$

Taking into account (3.36)-(3.48), then (3.33) written as

$$\begin{aligned}
L'(t) &\geq ((1-\sigma)-M\varepsilon) H^{-\sigma}(t) H'(t) \\
&+ 2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon\left[2-CM_1^{-m}\left(1+\frac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\frac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right. \\
&- \left. CM_2^{-r}\left(1+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}+\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}\right)\right] H(t) \\
&+ \varepsilon\left[c_4-CM_1^{-m}\left(1+\frac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\frac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right. \\
&- \left. CM_2^{-r}\left(1+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}+\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}\right)\right] \\
&\times \left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \qquad (3.49)
\end{aligned}$$

At this point and for large values of M_1 and M_2, we can find positive constants Λ_1 and Λ_2 such that (3.49) becomes

$$\begin{aligned}
L'(t) &\geq ((1-\sigma)-M\varepsilon) H^{-\sigma}(t) H'(t) \\
&+ 2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon\Lambda_1\left(\|u(t)\|_{2(\rho+2)}^{2(\rho+2)}+\|v(t)\|_{2(\rho+2)}^{2(\rho+2)}\right)+\varepsilon\Lambda_2 H(t). \qquad (3.50)
\end{aligned}$$

Once M_1 and M_2 are fixed (hence Λ_1 and Λ_2), we choose ε small enough so that $((1-\sigma)-M\varepsilon)\geq 0$ and

$$L(0)=H^{1-\sigma}(0)+\varepsilon\int_\Omega [u_0.u_1+v_0.v_1]\,dx>0. \qquad (3.51)$$

Therefore, there exists $\Gamma>0$ such that (3.50) can be written as

$$L'(t)\geq\varepsilon\Gamma\left(H(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \qquad (3.52)$$

Then, we have $L(t)\geq L(0)>0$, for all $t\geq 0$. Next, by using Holder's and Young's inequalities, we have the estimate

$$\begin{aligned}
&\left(\int_\Omega u.u_t(x,t)\,dx+\int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\
\leq\ & C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}}+\|u_t\|_2^{\frac{s}{1-\sigma}}+\|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}}+\|v_t\|_2^{\frac{s}{1-\sigma}}\right), \qquad (3.53)
\end{aligned}$$

for $1/\tau+1/s=1$. We takes $s=2(1-\sigma)$, to get $\dfrac{\tau}{1-\sigma}=\dfrac{2}{1-2\sigma}$. From (3.10) and (3.41), we have

$$\|u\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}}\leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right), \qquad (3.54)$$

and

$$\|v\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}}\leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\ \forall\, t\geq 0. \qquad (3.55)$$

Consequently, (3.53) can be written as

$$\begin{aligned} & \left(\int_{\Omega} uu_t(x,t)\,dx + \int_{\Omega} vv_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ \leq \;& c_{14}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2 + \|v_t\|_2^2\right) \\ + \;& c_{14}\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 + H(t)\right), \ \forall\, t \geq 0. \end{aligned}$$

Also, we have

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \varepsilon\int_{\Omega}(u.u_t + v.v_t)(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\ &\leq c_{15}\left(H(t) + \left|\int_{\Omega}(u.u_t(x,t) + v.v_t(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \\ &\leq c_{16}\left[H(t) + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2\right] \\ &+ c_{16}\left[\|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right], \ \forall\, t \geq 0, \end{aligned} \tag{3.56}$$

from (3.56) and (3.52), we get

$$L'(t) \geq a_0 L^{\frac{1}{1-\sigma}}(t), \ \forall\, t \geq 0. \tag{3.57}$$

Finally, a simple integration of (3.57) gives the desired result. □

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